An application of the Grönwall lemma avoiding exponential of the final time: a posteriori error estimates for the Stefan and Richards problems

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Inria Paris & Ecole des Ponts

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Outline

The heat equation ($f \in L^2(0,T;L^2(\Omega))$, $u_0 \in L^2(\Omega)$)

The heat equation

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\partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0,T),\\
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$$X := L^2(0,T;H_0^1(\Omega)), \|v\|_X^2 := \int_0^T \|\nabla v\|^2 \, \mathrm{d}t,$$

$$Y := L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \|v\|_Y^2 := \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \, \mathrm{d}t + \|v(T)\|^2$$

Y norm error characterization, $u_{h\tau} \in Y$

$$\|u - u_{h\tau}\|_Y^2 = \underbrace{\sup_{v \in X, \|v\|_X = 1} \left[\int_0^T (f,v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v) \, \mathrm{d}t \right]^2}_{\text{dual norm of the residual}} + \underbrace{\|u_0 - u_{h\tau}(0)\|^2}_{\text{initial condition}}$$

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Guaranteed **error upper bound** (reliability) ($u_{h\tau}$ FE in space, DG in time approx.)

$$\underbrace{|||u - u_{h\tau}|||}_{\text{unknown error}} \le \underbrace{\eta(u_{h\tau})}_{\text{computable estimator}}$$

- C_{eff} a generic constant independent of Ω, u, $u_{h\tau}$, h, p, τ, q,

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Time dependency, nonsymmetry

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- Details in Ern, Smears, and Vohralík, SINUM (2017)

Outline

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$$\xi(t) \leq \alpha(t) + \int_0^t \xi(s)\mathrm{d}s \Longrightarrow \xi(t) \leq e^t \alpha(t)$$

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- gives rise to **time-integrated** and **exponentially weighted** norms

Outline

Modelling problems with evolving interfaces and phase changes

The Stefan equation

Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u - \Delta \beta(u) &= f \quad \text{in } \Omega \times (0, T), \\ \beta(u) &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(0) &= u_0 \quad \text{in } \Omega. \end{aligned}$$

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Setting

- u: enthalpy
- $\beta(u)$: temperature
- $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, open polytope with Lipschitz boundary $\partial\Omega$
- T: final time
- source term $f \in L^2(0, T; L^2(\Omega))$, initial enthalpy $u_0 \in L^2(\Omega)$
- **nonlinear (degenerate) function** β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0, 1)$, strictly increasing otherwise

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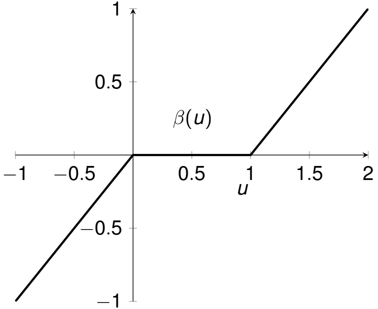

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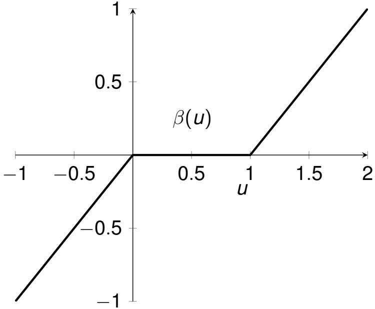

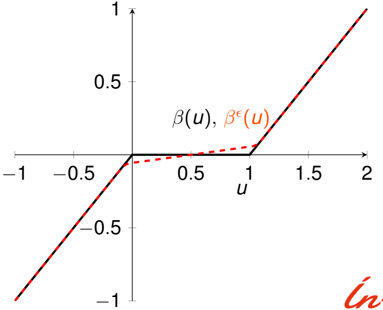

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Residual $\mathcal{R}(u_{h\tau}) \in X'$, for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

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3. The Stefan equation
   - A posteriori error estimates
   - Numerical experiments
   - Wrap up
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   - Wrap up
5. Two-phase porous media flows
6. Conclusions

Duality estimate

Lemma (Duality estimate)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$. Then, for a.e. $t \in (0, T)$,

$$\frac{2}{L_\beta}\|\beta(u) - \beta(u_{h\tau})\|^2_{Q_t} + \|(u - u_{h\tau})(t)\|^2_{H^{-1}(\Omega)}$$
$$\leq \|(u - u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'_t} + \|u - u_{h\tau}\|^2_{X'_t}.$$

- $W(t) \in H^1_0(\Omega)$ the solution to:

$$(\nabla W(t), \nabla\psi) = ((u - u_{h\tau})(t), \psi) \qquad \forall \psi \in H^1_0(\Omega)$$

- duality:

$$\|\nabla W(t)\| = \|(u - u_{h\tau})(t)\|_{H^{-1}(\Omega)}$$

- there holds

$$\langle \mathcal{R}(u_{h\tau}), W\rangle_{X'_t, X_t} \leq \frac{1}{2}\|\mathcal{R}(u_{h\tau})\|^2_{X'_t} + \frac{1}{2}\|u - u_{h\tau}\|^2_{X'_t}$$

Duality estimate

Lemma (Duality estimate)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$. Then, for a.e. $t \in (0, T)$,

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$$\leq \|(u - u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'_t} + \|u - u_{h\tau}\|^2_{X'_t}.$$

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Duality estimate

- definition of the residual:

$$\langle \mathcal{R}(u_{h\tau}), W\rangle_{X_t', X_t} = \underbrace{\int_0^t \langle \partial_t(u - u_{h\tau}), W\rangle(s)\mathrm{d}s}_{\mathfrak{R}_1} + \underbrace{\int_0^t (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla W)(s)\mathrm{d}s}_{\mathfrak{R}_2}$$

- definition of W:

$$\mathfrak{R}_1 = \int_0^t (\partial_t \nabla W, \nabla W)(s)\mathrm{d}s = \frac{1}{2}\left(\|\nabla W(t)\|^2_{L^2(\Omega)} - \|\nabla W(0)\|^2_{L^2(\Omega)}\right)$$
$$= \frac{1}{2}\left(\|(u - u_{h\tau})(t)\|^2_{H^{-1}(\Omega)} - \|u_0 - u_{h\tau}(0)\|^2_{H^{-1}(\Omega)}\right)$$

- definition of W and Lipschitz continuity of β:

$$\mathfrak{R}_2 = \int_0^t (u - u_{h\tau}, \beta(u) - \beta(u_{h\tau}))(s)\mathrm{d}s$$
$$\geq \frac{1}{L_\beta}\int_0^t (\beta(u) - \beta(u_{h\tau}), \beta(u) - \beta(u_{h\tau}))(s)\mathrm{d}s = \frac{1}{L_\beta}\|\beta(u) - \beta(u_{h\tau})\|^2_{Q_t}$$

Relation error – residual **featuring e^T**, 1st component

Recall

$$\frac{2}{L_\beta}\|\beta(u)-\beta(u_{h\tau})\|_{Q_t}^2+\|(u-u_{h\tau})(t)\|_{H^{-1}(\Omega)}^2$$
$$\leq\|(u-u_{h\tau})(0)\|_{H^{-1}(\Omega)}^2+\|\mathcal{R}(u_{h\tau})\|_{X_t'}^2+\int_0^t\|u-u_{h\tau}\|_{X_s'}^2\mathrm{d}s$$

Relation error – residual featuring e^T, 1st component

Recall

$$\cancel{\frac{2}{L_\beta}\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_t}} + \underbrace{\|(u-u_{h\tau})(t)\|^2_{H^{-1}(\Omega)}}_{\xi(t)}$$

$$\leq \underbrace{\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'_t}}_{\alpha(t)} + \int_0^t \underbrace{\|u-u_{h\tau}\|^2_{X'_s}}_{\xi(s)}\,\mathrm{d}s$$

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The Grönwall lemma (common form, $\alpha \geq 0$ nondecreasing)

$$\xi(t) \leq \alpha(t) + \int_0^t \xi(s)\mathrm{d}s \Longrightarrow \xi(t) \leq e^t\alpha(t)$$

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Thus

$$\|(u-u_{h\tau})(t)\|^2_{H^{-1}(\Omega)} \leq e^t\left(\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'_s}\right)$$

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$$\Longrightarrow \|u-u_{h\tau}\|^2_{X'} \leq (e^T-1)\left(\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'}\right)$$

Relation error – residual featuring e^T, 2nd component

Recall

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Relation error – residual featuring e^T, 2nd component

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$$\xi(t)\leq \alpha(t)+\int_0^t \xi(s)\mathrm{d}s \Longrightarrow \xi(t)\leq e^t\alpha(t)$$

Thus

$$\frac{2}{L_\beta}\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_T}+\|(u-u_{h\tau})(T)\|^2_{H^{-1}(\Omega)}\leq e^T\left(\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)}+\|\mathcal{R}(u_{h\tau})\|^2_{X'}\right)$$

Relation error – residual **featuring e^T**, altogether

Lemma (Relation error – residual featuring e^T)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$. Then

$$\frac{L_\beta}{2}\|u - u_{h\tau}\|_{X'}^2 + \frac{L_\beta}{2}\|(u - u_{h\tau})(T)\|_{H^{-1}(\Omega)}^2 + \|\beta(u) - \beta(u_{h\tau})\|_{Q_T}^2$$
$$\leq \frac{L_\beta}{2}(2e^T - 1)\left(\|(u - u_{h\tau})(0)\|_{H^{-1}(\Omega)}^2 + \|\mathcal{R}(u_{h\tau})\|_{X'}^2\right)$$

Relation error – residual **without e^T** by the **sharp Grönwall**

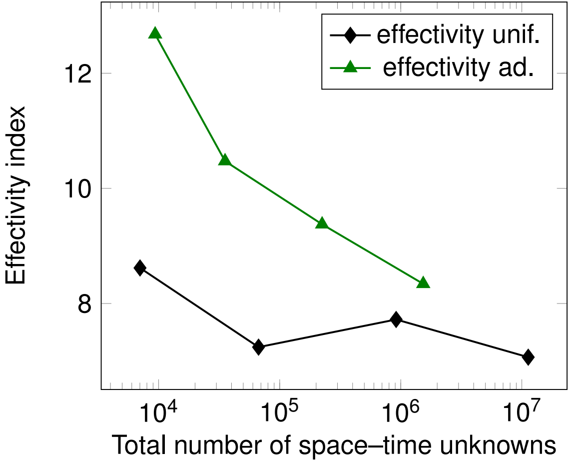
Lemma (Relation error – residual without e^T)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$. Then

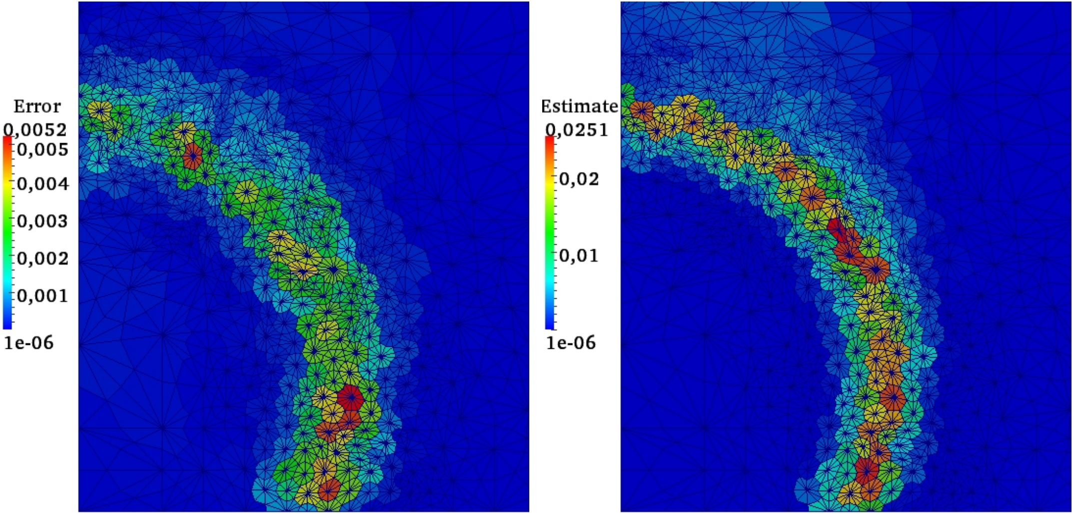
$$
\begin{aligned}
&\frac{L_\beta}{2}\|u - u_{h\tau}\|_{X'}^2 + \frac{L_\beta}{2}\|(u - u_{h\tau})(\cdot, T)\|_{H^{-1}(\Omega)}^2 + \|\beta(u) - \beta(u_{h\tau})\|_{Q_T}^2 \\
&+ 2\int_0^T \left(\|\beta(u) - \beta(u_{h\tau})\|_{Q_t}^2 + \int_0^t \|\beta(u) - \beta(u_{h\tau})\|_{Q_s}^2 e^{t-s} \,\mathrm{d}s \right) \mathrm{d}t \\
\leq\ & \frac{L_\beta}{2}\Bigg\{ (2e^T - 1)\|(u - u_{h\tau})(\cdot, 0)\|_{H^{-1}(\Omega)}^2 + \|\mathcal{R}(u_{h\tau})\|_{X'}^2 \\
&+ 2\int_0^T \left(\|\mathcal{R}(u_{h\tau})\|_{X_t'}^2 + \int_0^t \|\mathcal{R}(u_{h\tau})\|_{X_s'}^2 e^{t-s} \,\mathrm{d}s \right) \mathrm{d}t \Bigg\}.
\end{aligned}
$$

Outline

How **large** is the error? (Effectivity indices)



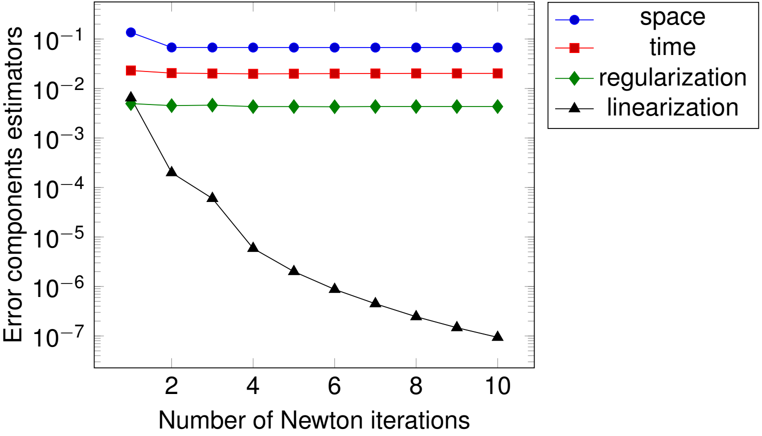
Where (in space and time) is the error **localized**?



How large are the **error components**? (Linearization)

Linearization stopping criterion

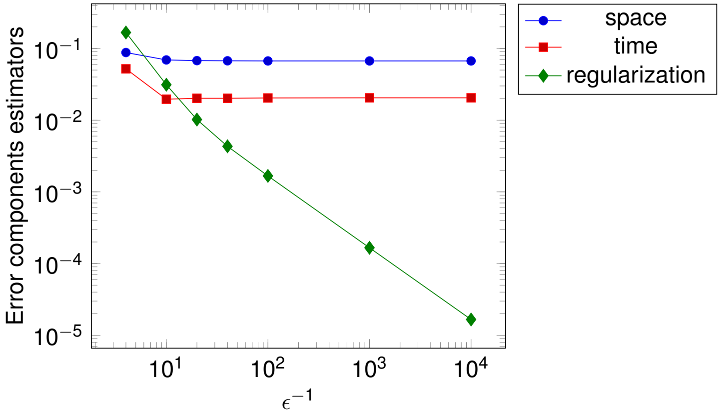
$$\eta_{\mathrm{lin}}^{n,\epsilon,k} \leq \Gamma_{\mathrm{lin}}\left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k}\right)$$



How large are the **error components**? (Regularization)

Regularization stopping criterion

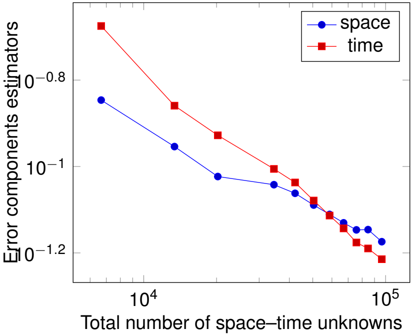
$$\eta_{\mathrm{reg}}^{n,\epsilon,k_n} \leq \Gamma_{\mathrm{reg}}\left(\eta_{\mathrm{sp}}^{n,\epsilon,k_n} + \eta_{\mathrm{tm}}^{n,\epsilon,k_n}\right)$$

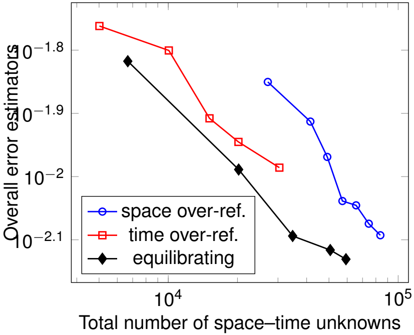


How large are the **error components**? (Time and space)

Equilibrating time and space errors

$$\gamma_{\mathrm{tm}} \eta_{\mathrm{sp}}^{n,\epsilon_n,k_n} \leq \eta_{\mathrm{tm}}^{n,\epsilon_n,k_n} \leq \Gamma_{\mathrm{tm}} \eta_{\mathrm{sp}}^{n,\epsilon_n,k_n}$$





Outline

A posteriori error estimates for the Stefan equation

Time dependency, nonsymmetry, nonlinearity, degeneracy

- treatment of time-dependent nonlinearity: **sharp Grönwall lemma** not neglecting the integral terms
- **avoids** the appearance of e^T but gives rise to **time-integrated** and **exponentially-weighted** norms

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- Details in D. Pietro, M. Vohralík, and S. Yousef, Math. Comp. (2015)

Outline

Modelling flow of water and air through soil

The Richards equation

Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$
\begin{aligned}
\partial_t S(u) - \nabla \cdot [\boldsymbol{K} \kappa(S(u))(\nabla u + \boldsymbol{g})] &= f(S(u)) \quad \text{in } \Omega \times (0, T), \\
u &= 0 \quad \text{on } \partial\Omega \times (0, T), \\
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Setting

- u: pressure
- $S(u)$: saturation
- $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, open polytope with Lipschitz boundary $\partial\Omega$
- T: final time
- diffusion tensor $\boldsymbol{K}$, source term $f \in C^1([0, 1] \times \Omega \times \mathbb{R})$, gravity $\boldsymbol{g}$, initial saturation $s_0 \in L^\infty(\Omega)$, $0 \le s_0 \le 1$
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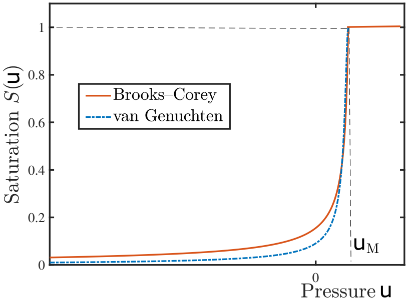
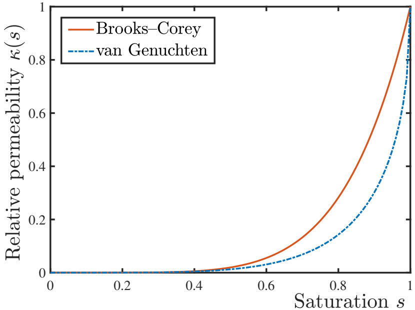
Nonlinear (degenerate) functions S and κ





Weak formulation

Spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Total pressure (Kirchhoff transform)

$$\mathcal{K}(p) := \begin{cases} \int_0^p \kappa(S(\varrho))\,\mathrm{d}\varrho & \text{for } p \le p_{\mathrm{M}}, \\ P_{\mathrm{M}} + \kappa(1)(p - p_{\mathrm{M}}) & \text{for } p > p_{\mathrm{M}}, \end{cases} \qquad \theta \circ \mathcal{K} = S$$

Weak formulation

$$\Psi \in X \qquad \text{with } s := \theta(\Psi) \in Z, \qquad s(0) = s_0 \quad \text{in } \Omega,$$

$$\int_0^T \langle \partial_t \theta(\Psi), v\rangle + \int_0^T (\boldsymbol{K}(\nabla\Psi + \boldsymbol{g}\kappa(\theta(\Psi))), \nabla v) = \int_0^T (f(\theta(\Psi)), v) \qquad \forall v \in X$$

Residual $\mathcal{R}(\Psi_{h\tau}) \in X'$, for $\Psi_{h\tau} \in X$ such that $s_{h\tau} := \theta(\Psi_{h\tau}) \in Z$

$$\langle \mathcal{R}(\Psi_{h\tau}), v\rangle_{X',X} := \int_0^T \{(f(\theta(\Psi_{h\tau})), v) - \langle \partial_t\theta(\Psi_{h\tau}), v\rangle - (\boldsymbol{K}(\nabla\Psi_{h\tau} + \boldsymbol{g}\kappa(\theta(\Psi_{h\tau}))), \nabla v)\}(s)\mathrm{d}s$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v\rangle_{X',X}$$

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Dual norm of the residual

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Weak formulation

Spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

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$$\mathcal{K}(p) := \begin{cases} \int_0^p \kappa(S(\varrho))\,\mathrm{d}\varrho & \text{for } p \le p_{\mathrm{M}}, \\ P_{\mathrm{M}} + \kappa(1)(p - p_{\mathrm{M}}) & \text{for } p > p_{\mathrm{M}}, \end{cases} \qquad \theta \circ \mathcal{K} = S$$

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Outline

Time-integration functionals based on the **sharp Grönwall** lemma

Time-integration functionals, $\alpha : [0, T] \to [0, \infty)$

$$\mathcal{J}_\alpha : L^2([0,T]) \to [0,\infty),$$

$$\mathcal{J}_\alpha(\varrho) := \left[\exp\left(-\int_0^T \alpha\right) \int_0^T \left(\varrho^2(t) + \alpha(t) \exp\left(\int_t^T \alpha\right) \int_0^t \varrho^2 \right) \mathrm{d}t \right]^{\frac{1}{2}}$$

- define norm on $L^2([0,T])$
- actually equivalent to the $L^2([0,T])$ norm

$$\exp\left(-\frac{1}{2}\int_0^T \alpha\right) \|\varrho\|_{L^2([0,T])} \le \mathcal{J}_\alpha(\varrho) \le \|\varrho\|_{L^2([0,T])}$$

- yield an almost constant value of error independent of $T \ge 1$ in applications

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Relation error – residual **without e^T** by the **sharp Grönwall** lemma

Lemma (Relation error – residual without e^T)

Let $\Psi_{h\tau} \in X$ such that $s_{h\tau} := \theta(\Psi_{h\tau}) \in Z$. Then

$$e^{-\int_0^T(\lambda+\mathfrak{C}_1)}\|(s-s_{h\tau})(T)\|^2_{H^{-1}(\Omega)} + \mathcal{J}_{\lambda+\mathfrak{C}_1}\left(\theta_{\partial,\mathrm{M}}^{-\frac{1}{2}}\|s-s_{h\tau}\|\right)^2$$

$$\leq \|s_0 - s_{h\tau}(0)\|^2_{H^{-1}(\Omega)} + \mathcal{J}_{\lambda+\mathfrak{C}_1}(\lambda^{-\frac{1}{2}}\|\mathcal{R}(\Psi_{h\tau})\|_{H^{-1}(\Omega)})^2,$$

$$e^{-\int_0^T\mathfrak{C}_2}\|(s-s_{h\tau})(T)\|^2 + \frac{1}{2}\mathcal{J}_{\mathfrak{C}_2}\left(\left\|D(s)^{-\frac{1}{2}}\mathcal{K}^{\frac{1}{2}}\nabla(\Psi-\Psi_{h\tau})\right\|\right)^2$$

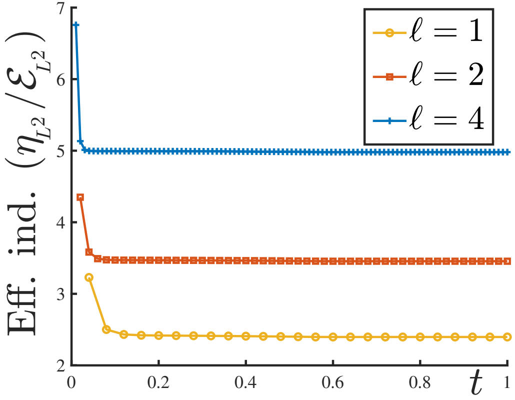
$$\leq \|s_0 - s_{h\tau}(0)\|^2 + \mathcal{J}_{\mathfrak{C}_2}\left(\eta^{\mathrm{deg}}\right)^2 + 4\,\mathcal{J}_{\mathfrak{C}_2}\left(D_{\mathrm{m}}^{-\frac{1}{2}}\|\mathcal{R}(\Psi_{h\tau})\|_{H^{-1}(\Omega)}\right)^2,$$

$$\mathcal{J}_\lambda(\|\partial_t(s-s_{h\tau})\|_{H^{-1}(\Omega)})^2$$

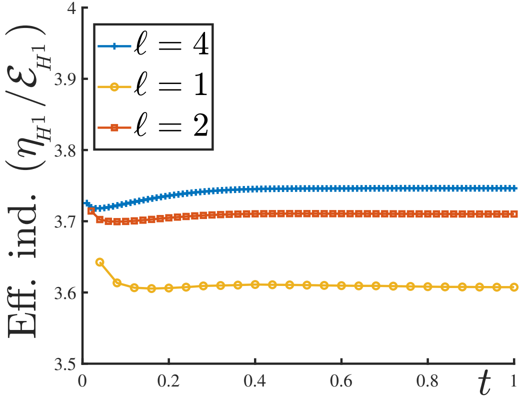
$$\leq 3\left[\mathcal{J}_\lambda(\|\Psi-\Psi_{h\tau}\|_{H^{-1}(\Omega)})^2 + \mathfrak{C}_3(T)\,\mathcal{J}_\lambda\left(\|s-s_{h\tau}\|\right)^2 + \mathcal{J}_\lambda(\|\mathcal{R}(\Psi_{h\tau})\|_{H^{-1}(\Omega)})^2\right]$$

Outline

How **large** is the error? **Robustness** wrt the final time (known sol.)



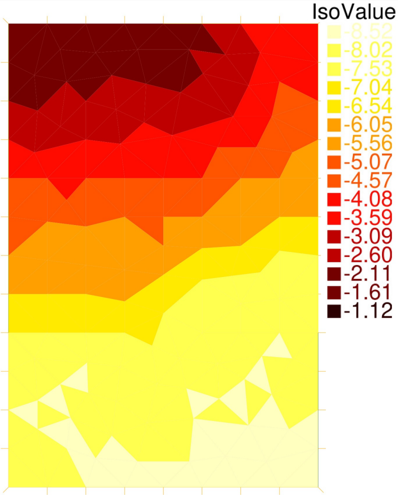
Integrated L^2 effectivity indices



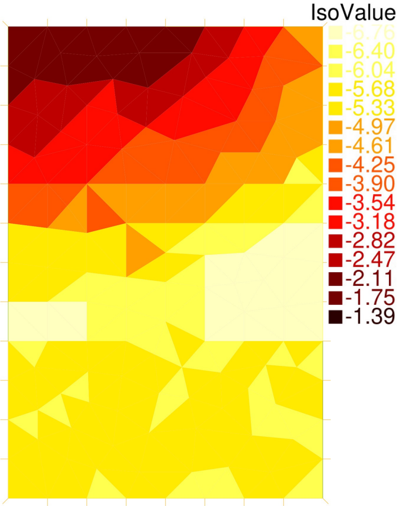
Integrated H^1 effectivity indices

K. Mitra, M. Vohralík, preprint (2022)

Where (in space and time) is the error **localized**? (benchmark case)



Estimated local error



Exact local error

K. Mitra, M. Vohralík, preprint (2022)

Realistic case

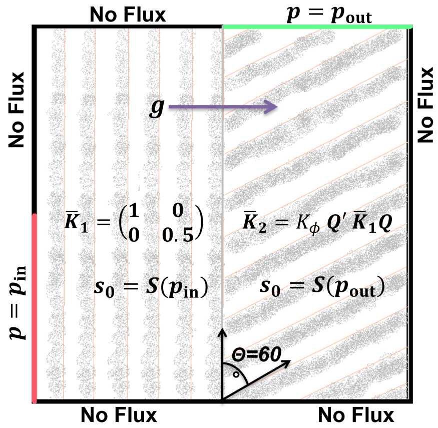
Setting

- unit square $\Omega = (0,1)^2$
- $T = 1$
- $f(\boldsymbol{x}, u) = 0$, heterogeneous and anisotropic $\boldsymbol{K}$, $\boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- **Brooks–Corey**-type **saturation** and **permeability** laws

$$S(u) := \begin{cases} \frac{1}{(2-u)^{\frac{1}{3}}} & \text{if } u < 1, \\ 1 & \text{if } u \geq 1 \end{cases}, \qquad \kappa(s) := s^3$$

- $(h, \tau) = (h_0, \tau_0)/\ell$ with $\ell \in \{1, 2, 4\}$, $h_0 = 0.2$, and $\tau_0 = 0.04$

Realistic case



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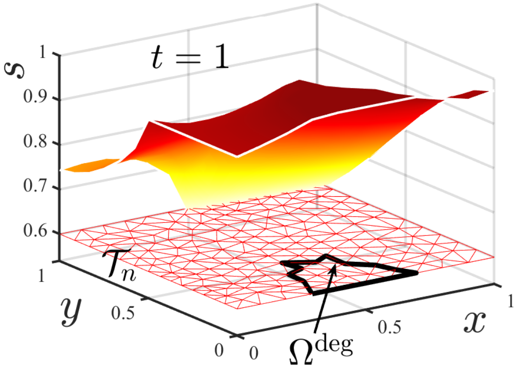
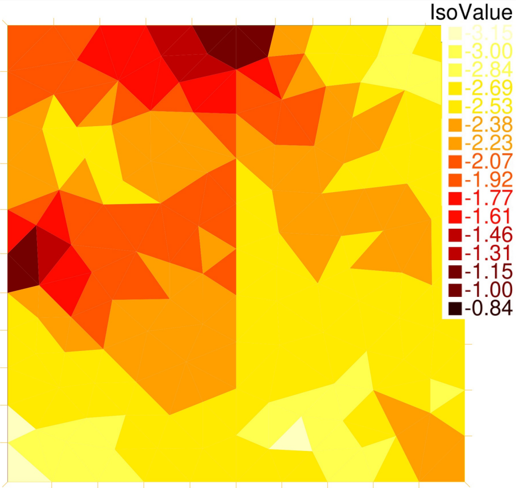


Numerical saturation for $\ell = 2$ at $t = 1$

Where (in space and time) is the error **localized**? (realistic test case)



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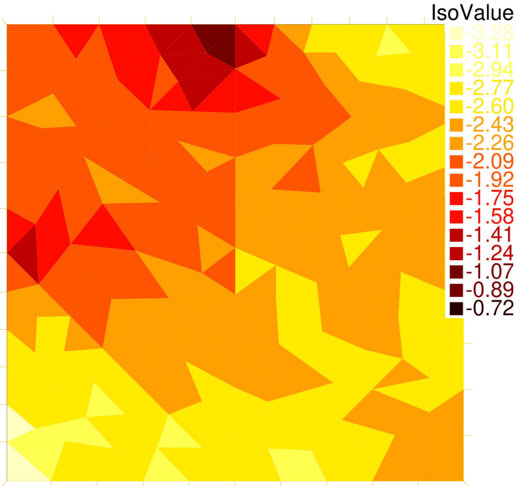

Exact local error

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Outline

A posteriori error estimates for the Richards equation

Time dependency, nonsymmetry, nonlinearity, double degeneracy

- treatment of time-dependent nonlinearity: combined **energy** & **negative** norms together with **weighted time-integration functionals** ($\sim$ sharp Grönwall lemma)

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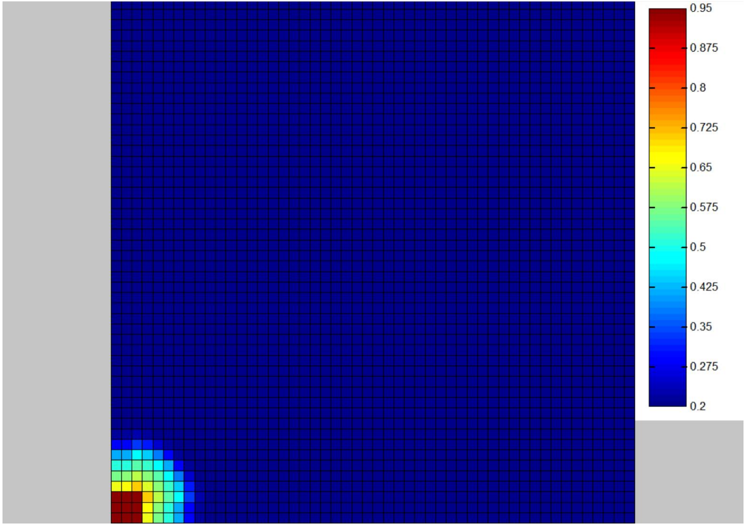
- Details in K. Mitra, M. Vohralík, Math. Comp. (2024)

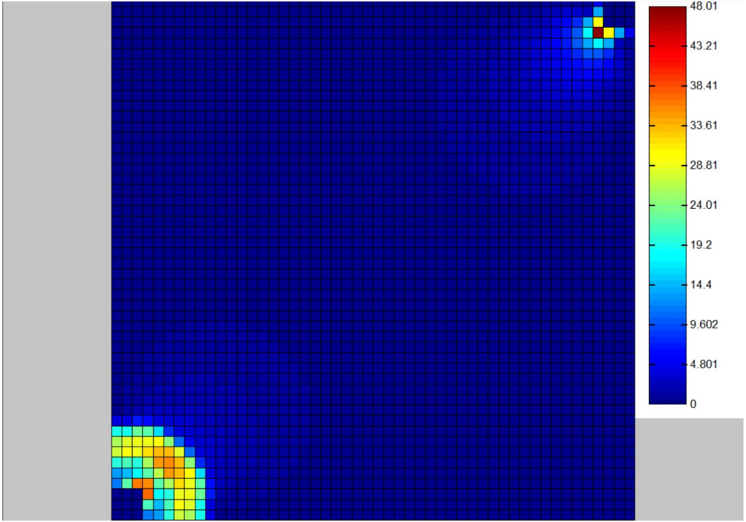
Outline

Two-phase flow, water saturation



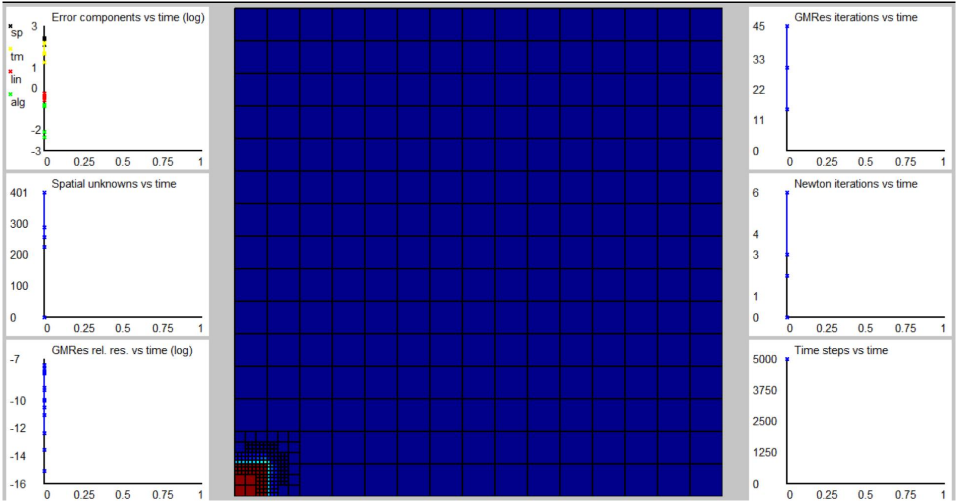
M. Vohralík, M. Wheeler, Computational Geosciences (2013)

Where (in space and time) is the error **localized**? (two-phase flow)



M. Vohralík, M. Wheeler, Computational Geosciences (2013)

All error components (two-phase flow)



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Conclusions

- a posteriori **error certification** for unsteady, nonlinear, & degenerate pbs

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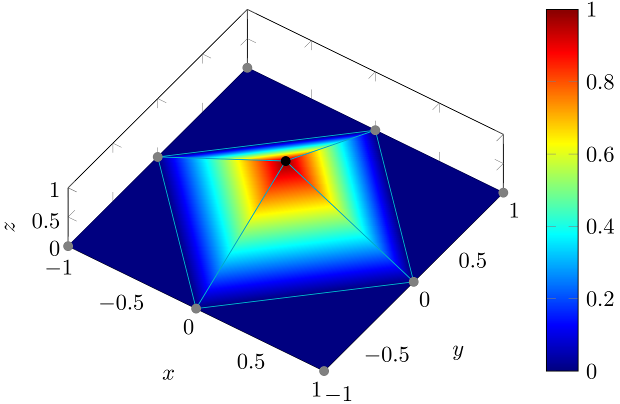
Thank you for your attention!

Outline

Partition of unity

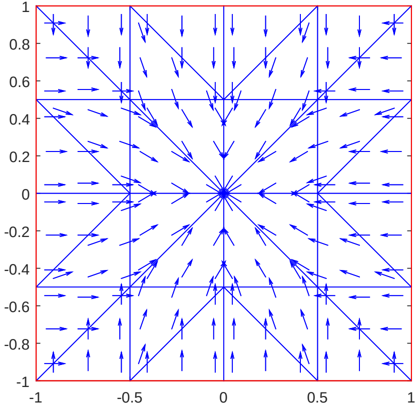
$$\boxed{\sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \psi^{\boldsymbol{a}} = 1}$$



Hat basis function $\psi^{\boldsymbol{a}}$

Equilibrated flux reconstruction

Destuynder and Métivet (1998), Braess & Schöberl (2008), Ern & Vohralík (2013)



Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$ (e.g. FE flux $-\nabla u_\ell$)

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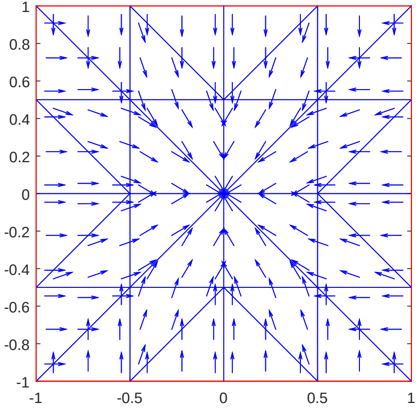
Destuynder and Métivet (1998), Braess & Schöberl (2008), Ern & Vohralík (2013)



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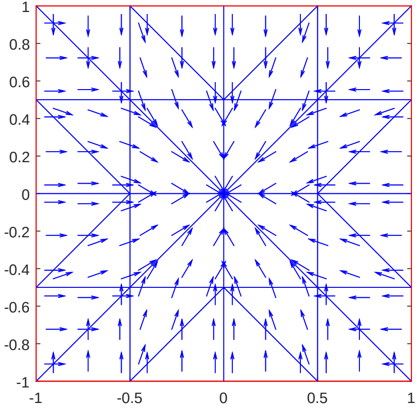

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$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell), f \in \mathcal{P}_p(\mathcal{T}_\ell)}$$

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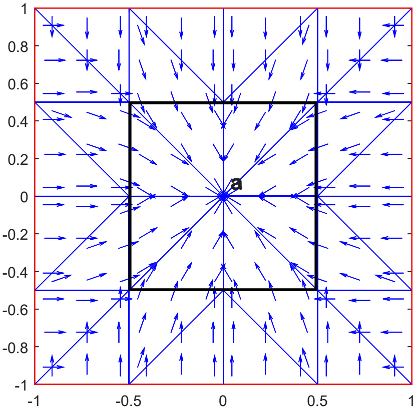

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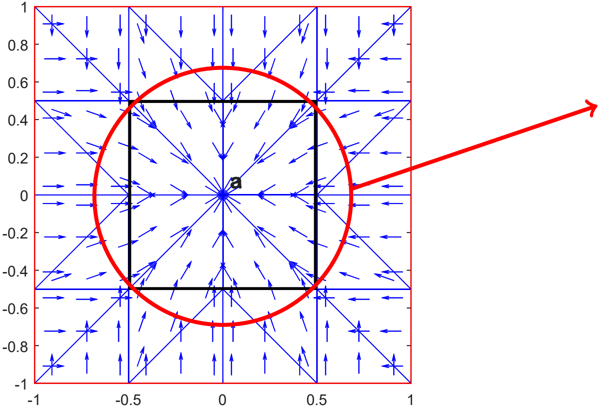
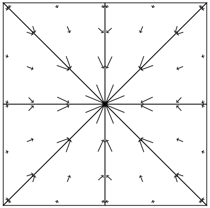
$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_\ell,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_\ell^{\mathrm{int}}}$$

Equilibrated flux reconstruction

Destuynder and Métivet (1998), Braess & Schöberl (2008), Ern & Vohralík (2013)





$\psi^{\boldsymbol{a}} \boldsymbol{\iota}_\ell$

Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$, $\nabla \cdot \boldsymbol{\iota}_\ell \neq f$

$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell), f \in \mathcal{P}_p(\mathcal{T}_\ell)}$$

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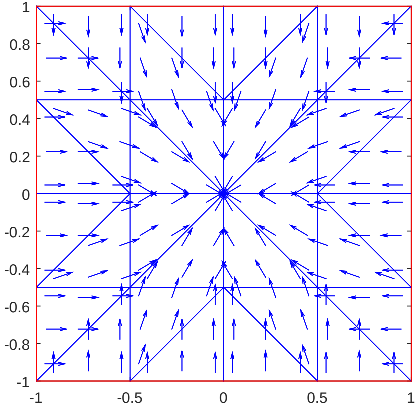

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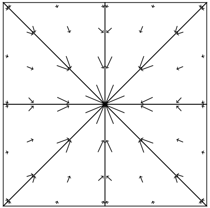
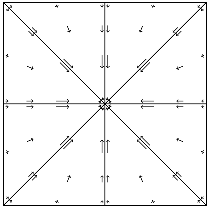


$\psi^{\boldsymbol{a}}\boldsymbol{\iota}_\ell$



$\boldsymbol{\sigma}_\ell^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\iota}_\ell \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)} \qquad \boldsymbol{\sigma}_\ell^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_\ell \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_\ell = f\psi^{\boldsymbol{a}} + \boldsymbol{\iota}_\ell\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_\ell - \boldsymbol{v}_\ell\|_{\omega_{\boldsymbol{a}}}^2$$

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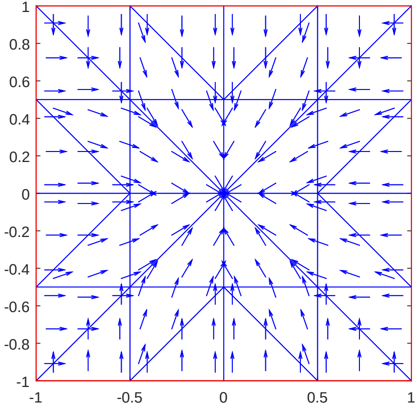
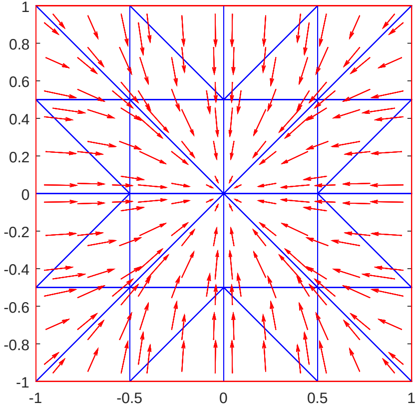


Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\iota}_\ell \neq f$



Equilibrated flux $\boldsymbol{\sigma}_\ell \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_\ell = f$

$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell), f \in \mathcal{P}_p(\mathcal{T}_\ell)} \to \boldsymbol{\sigma}_\ell := \sum_{\boldsymbol{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_\ell) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{\sigma}_\ell = f$$

Equilibrated flux reconstruction

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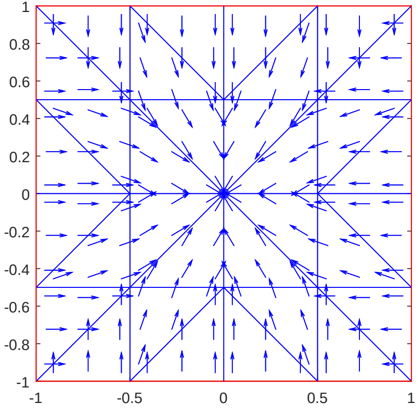
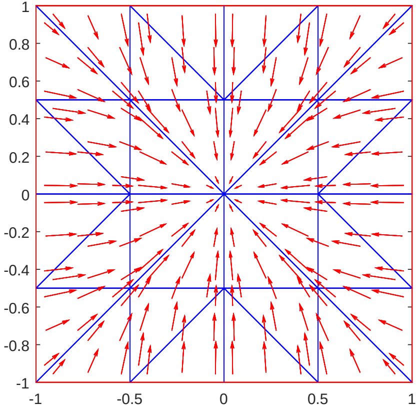


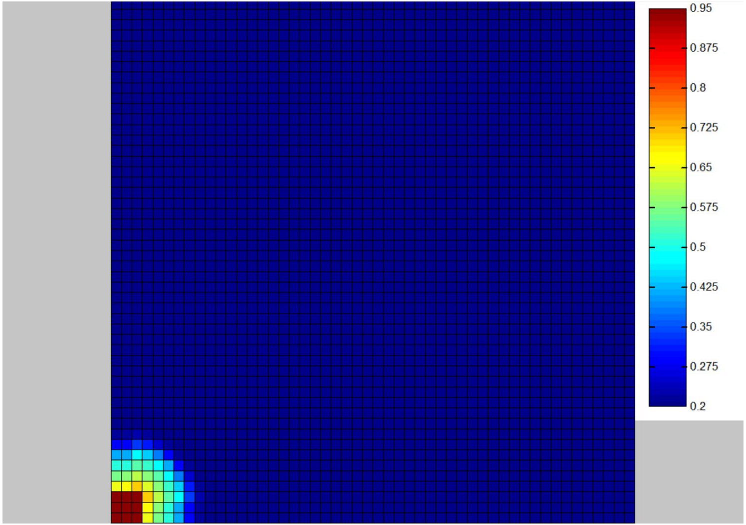
Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\iota}_\ell \neq f$



Equilibrated flux $\boldsymbol{\sigma}_\ell \in \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\sigma}_\ell = f$

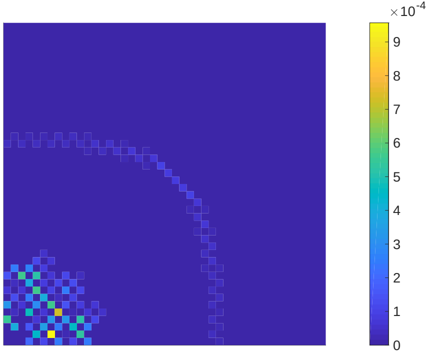
Outline

Two-phase flow, water saturation

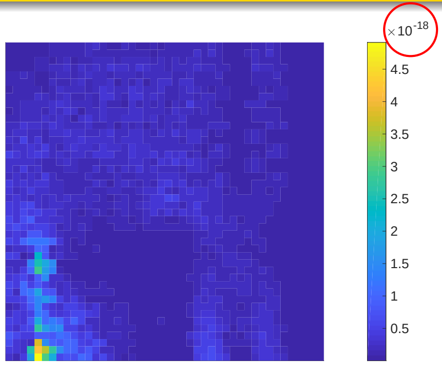


M. Vohralík, M. Wheeler, Computational Geosciences (2013)

Recovering mass balance: two-phase flow (inexact solver, water)



original mass balance misfit ($m^2 s^{-1}$)



corrected mass balance misfit ($m^2 s^{-1}$)

Setting

- fully implicit discretization of a two-phase oil–water flow
- cell-centered finite volumes on a square mesh
- time step 260, 1st Newton linearization, GMRes iteration 195

J. Papež, U. Rüde, M. Vohralík, B. Wohlmuth, Computer Methods in Applied Mechanics and Engineering (2020)

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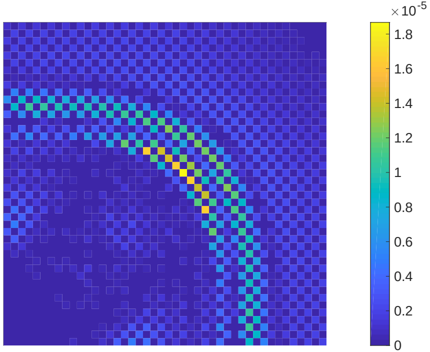

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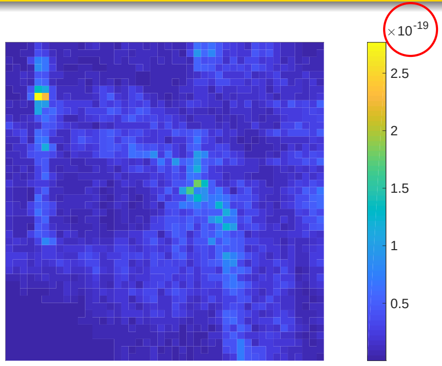

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