Multigrid for high-order finite elements: line search, *p*-robustness, a posteriori estimates, and adaptivity

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Inria Paris & Ecole des Ponts

Prague, April 28, 2022

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Outline

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Numerical methods for PDEs: example $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

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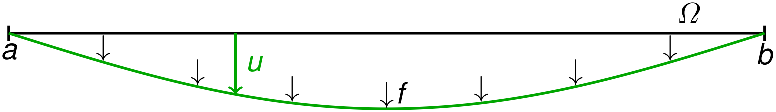

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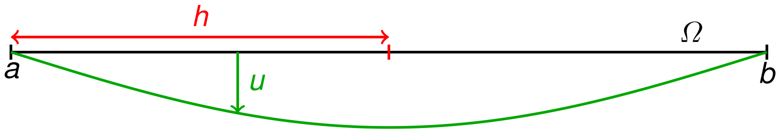
- computational mesh $\mathcal{T}_h$



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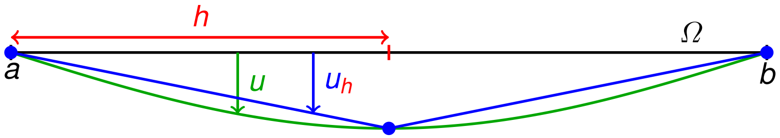

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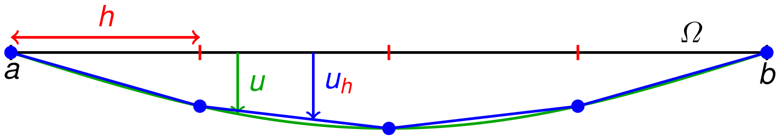
- **more computational resources** $\Rightarrow$ numerical approximation u_h **closer** to u



Numerical approximation u_h and its convergence to u

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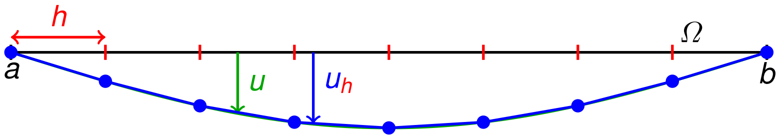

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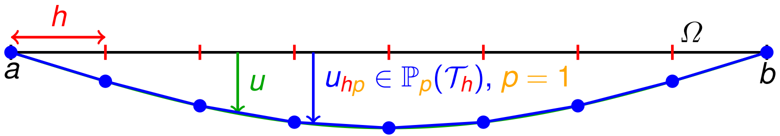

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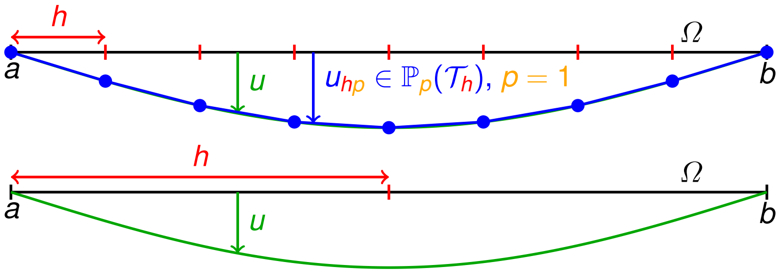

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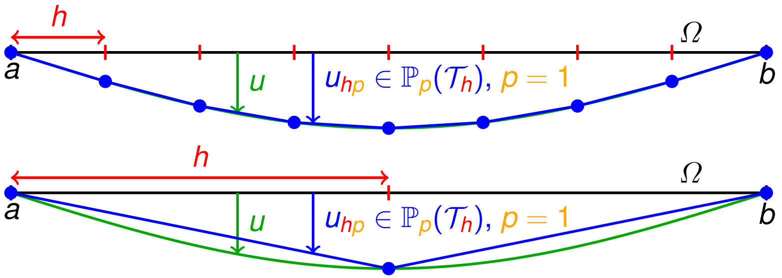

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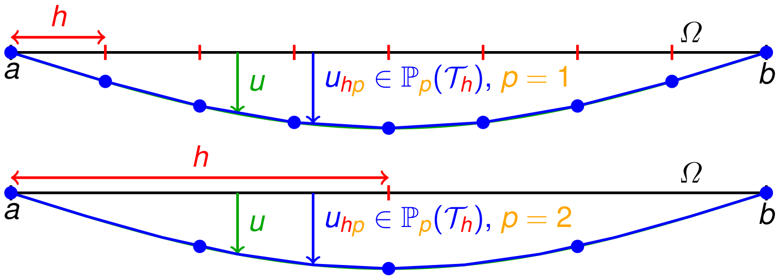

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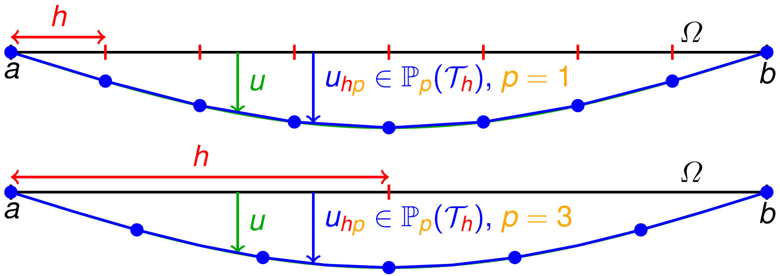

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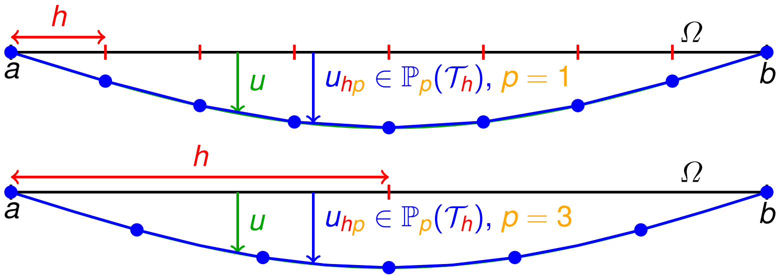

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$$\mathbb{A}_{hp}\mathrm{U}_{hp} = \mathrm{F}_{hp}$$

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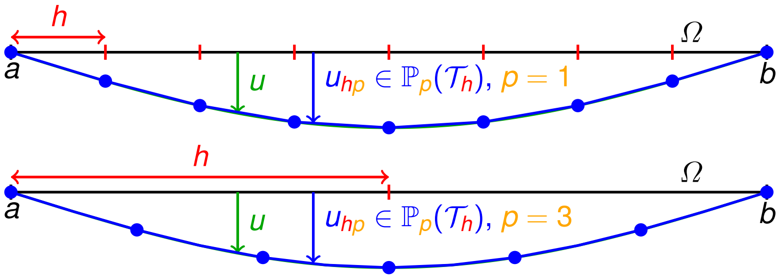

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$$\|\nabla(u_{hp} - u_{hp}^i)\| = \left\{\int_a^b |(u_{hp} - u_{hp}^i)'|^2\right\}^{\frac{1}{2}}$$

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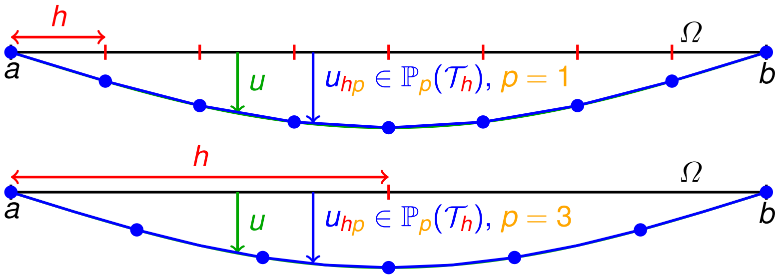

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Discretization error

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A priori estimate of the discretization error

Lowest-order finite elements $p = 1$

$$\|\nabla(u - u_h)\| \leq C h^1$$

A priori estimate of the discretization error, $u \in H^s(\Omega)$

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Any-order finite elements $p \ge 1$

$$\|\nabla(u - u_h)\| \le C(\#DoF)^{-\frac{p}{d}} \approx C h^p$$

A priori estimate of the discretization error, $u \in H^s(\Omega)$

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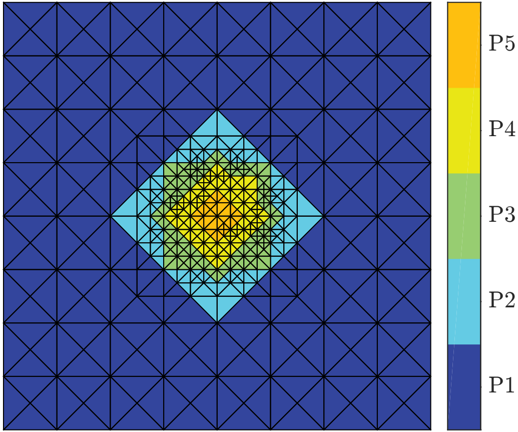
Rectified claim

High-order finite elements always pay-off (on graded meshes).

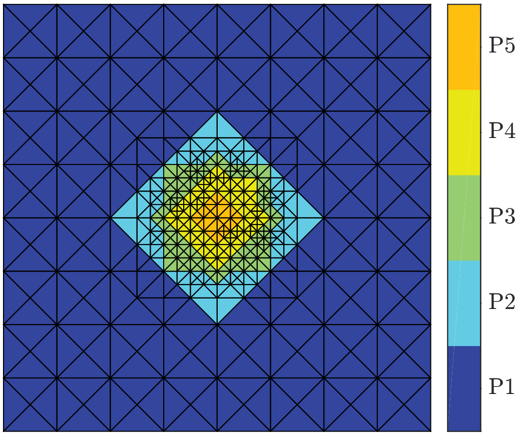
Most efficient error decrease: *hp* adaptivity (Babuška, Schwab, . . .)



Mesh $\mathcal{T}_\ell$ and pol. degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Most efficient error decrease: *hp* adaptivity (**smooth** solution)



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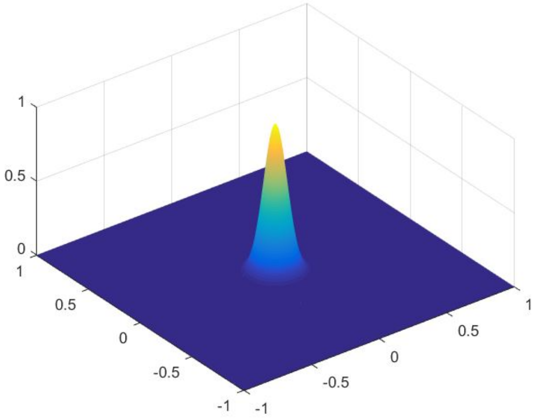
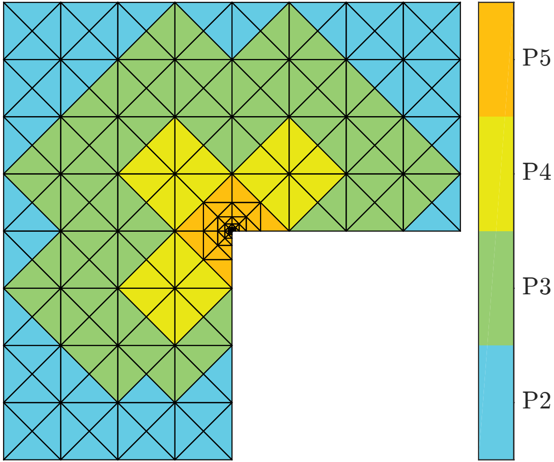


Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Most efficient error decrease: *hp* adaptivity (**singular** solution)



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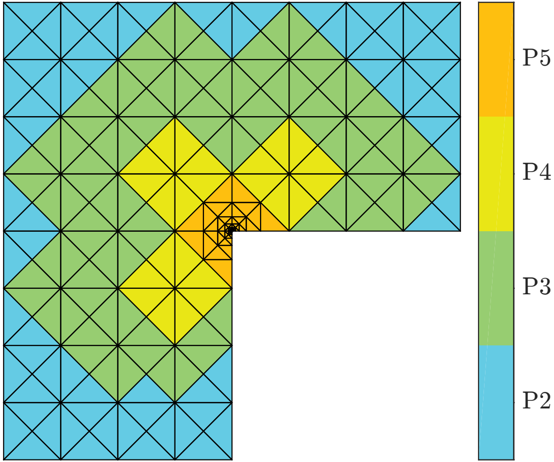

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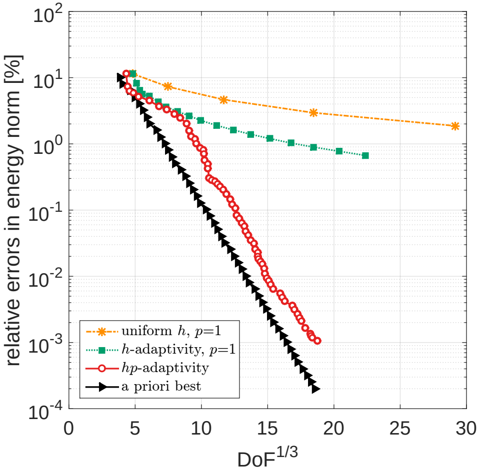

Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

From PDEs to numerical linear algebra

Algebraic problem

Find $\mathrm{U}_J \in \mathbb{R}^{|V_J^p|}$ such that

$$\mathbb{A}_J \mathrm{U}_J = \mathrm{F}_J$$

From PDEs to numerical linear algebra

Problem

Let $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, $\mathbf{K} \in [L^\infty(\Omega)]^{d\times d}$, and $f \in L^2(\Omega)$. Find $u : \Omega \to \mathbb{R}$ such that $-\nabla\cdot(\mathbf{K}\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$.

Weak solution

Find $u \in H_0^1(\Omega)$ such that

$$(\mathbf{K}\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega).$$

Finite elements

Find $u_J \in V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$ such that

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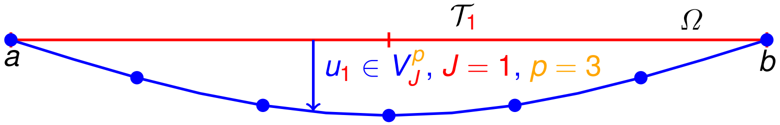

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Let $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, $\mathbf{K} \in [L^\infty(\Omega)]^{d\times d}$, and $f \in L^2(\Omega)$. Find $u : \Omega \to \mathbb{R}$ such that $-\nabla\cdot(\mathbf{K}\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$.

Weak solution

Find $u \in H_0^1(\Omega)$ such that

$$(\mathbf{K}\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega).$$

Finite elements

Find $u_J \in V_J^p := \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$ such that

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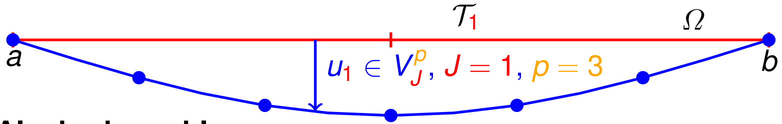

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Find $\mathrm{U}_J \in \mathbb{R}^{|V_J^p|}$ such that

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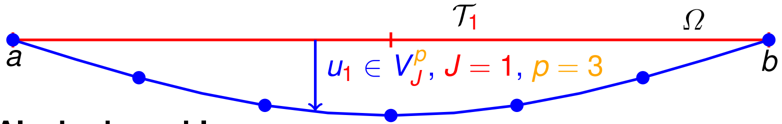

Finite elements

independent of the basis of V_J^p

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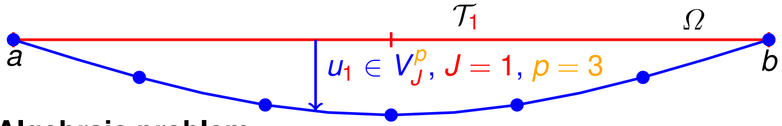

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Algebraic system

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Solvers for high-order finite elements

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Find $\mathrm{U}_J \in \mathbb{R}^{|V_J^p|}$ such that

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- $\mathbb{A}_J$ less and less sparse for big p
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Black-box iterative solvers

do not work well for high p & on highly graded meshes $\mathcal{T}_J$

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p-robust solver/preconditoner

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Outline

A hierarchy of meshes

Example: Two different mesh hierarchies with $J = 3$ refinements.

Assumption: The meshes $\{\mathcal{T}_j\}_{0\le j\le J}$ can be *quasi-uniform* or *graded*, satisfying:

- quasi-uniform $\mathcal{T}_0$,
- shape-regularity,
- maximum strength of refinement.

For given p and J, choose *increasing* polynomial degrees p_j, $j \in \{1, \dots, J\}$,

and define the spaces

$$V_j^{p_j} := \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$

Economic choice:

$p_0 = p_1 = \dots p_{J-1} = 1$,
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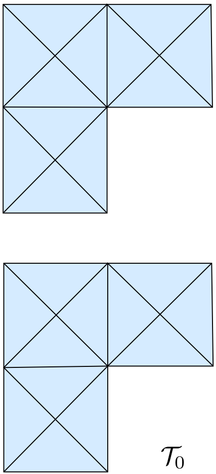

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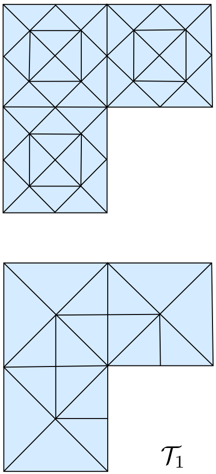

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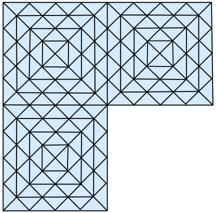

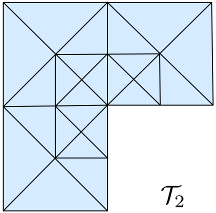

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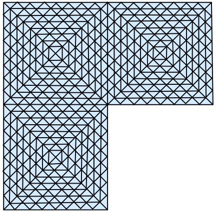

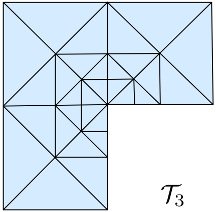

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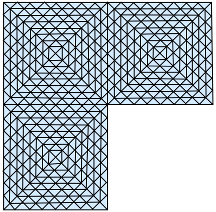

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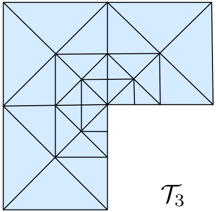

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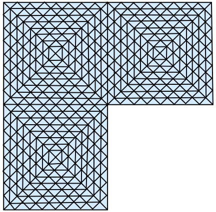

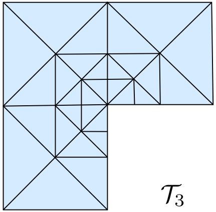

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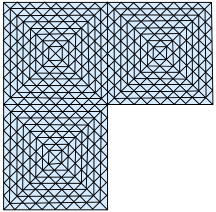

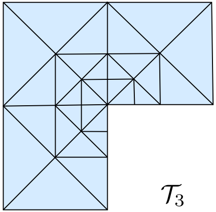

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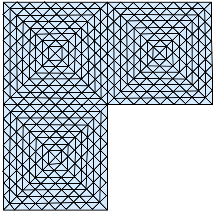

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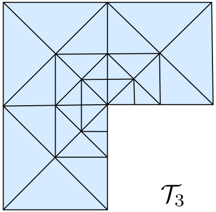

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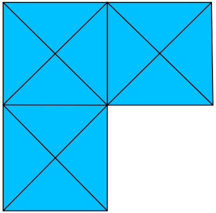

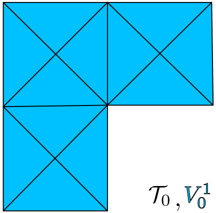

$\mathcal{T}_0$, V_0^1

A hierarchy of meshes and spaces

Example: Two different mesh hierarchies with $J = 3$ refinements.

Assumption: The meshes $\{\mathcal{T}_j\}_{0\le j\le J}$ can be *quasi-uniform* or *graded*, satisfying:

- quasi-uniform $\mathcal{T}_0$,
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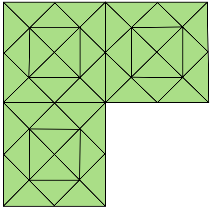
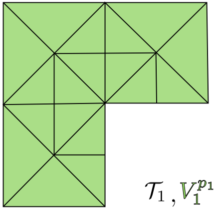
- maximum strength of refinement.



For given p and J, choose *increasing* polynomial degrees p_j, $j \in \{1, \ldots, J\}$,

$$1 = p_0 \le p_1 \le p_2 \le \ldots \le p_J = p,$$



$\mathcal{T}_1$, $V_1^{p_1}$

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Economic choice

$p_0 = p_1 = \ldots p_{J-1} = 1,$
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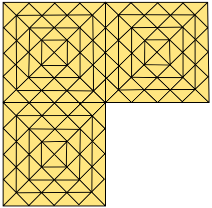

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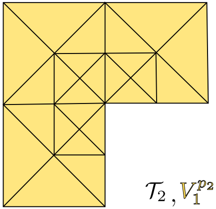

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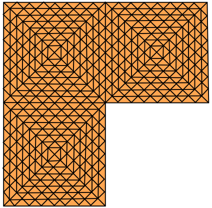

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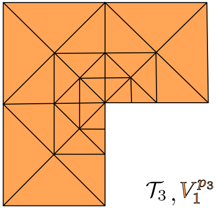

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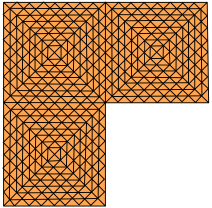

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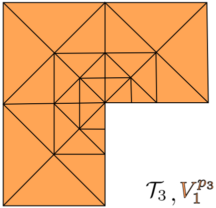

$\mathcal{T}_3, V_1^{p_3}$

Vertex patches: local subspaces/Jacobi blocks

Given a vertex of the mesh $\mathcal{T}_j$, $\mathbf{a} \in \mathcal{V}_j$, denote

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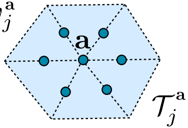

$$V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$

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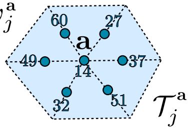

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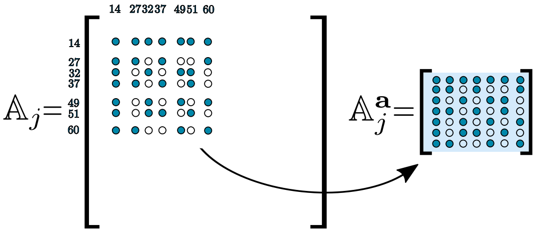

V -cycle multigrid

- **V-cycle** geometric multigrid

V(0,1)-cycle multigrid

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- **zero** pre- and a **single** post-smoothing step

V(0,1)-cycle multigrid

$u_J^i \in V_J^p \qquad u_J^{i+1} = u_J^i + \sum_{j=0}^{J} \lambda_j^i \rho_j^i \in V_J^p \qquad \eta_{\mathrm{alg}}^i = \left(\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\| \right)^2 \right)^{\frac{1}{2}}$

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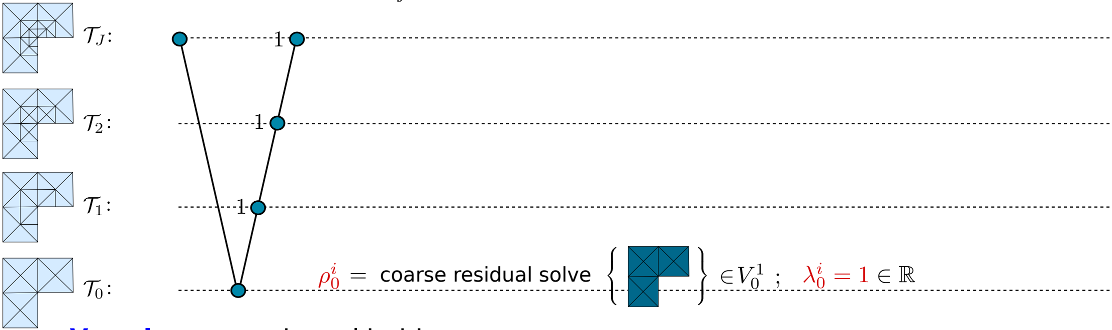

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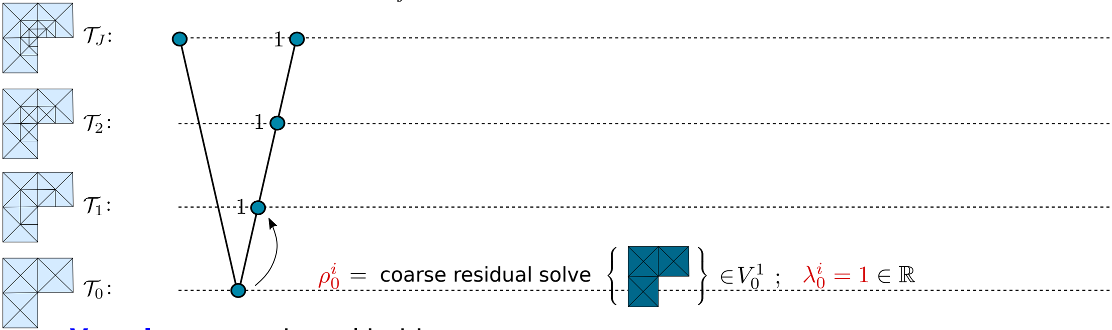

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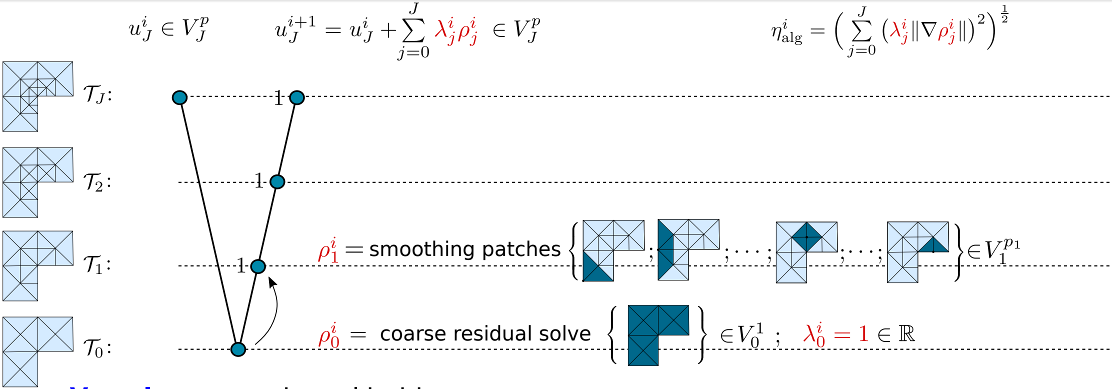
V(0,1)-cycle multigrid with block-Jacobi smoothing and line search



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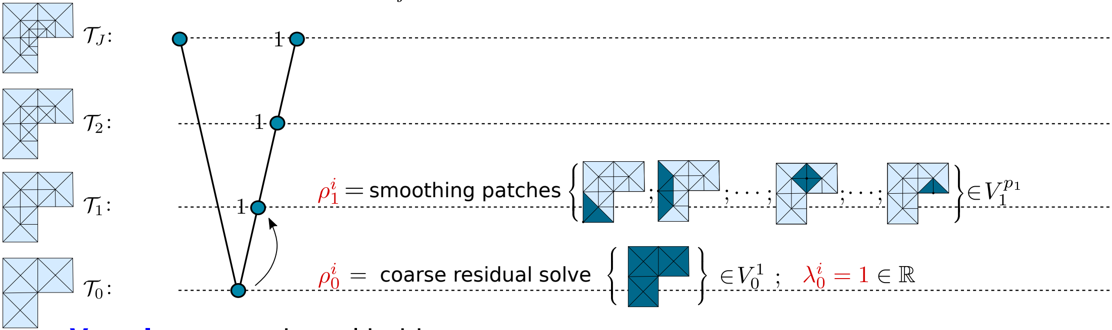

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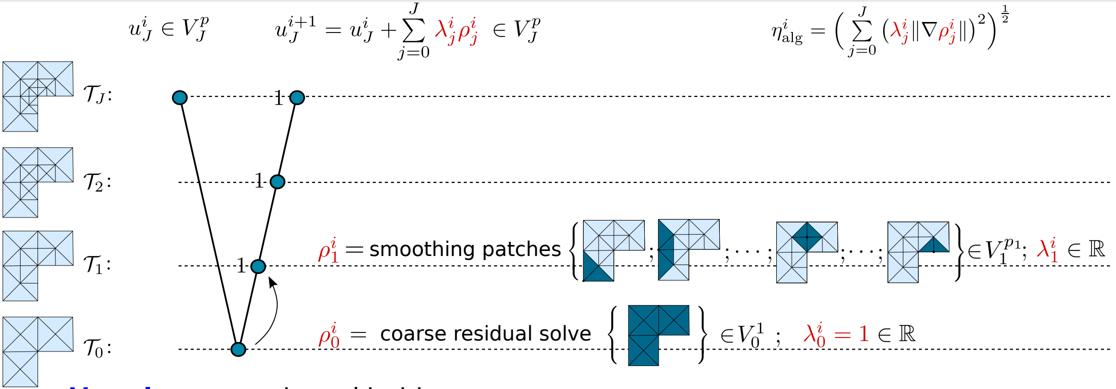

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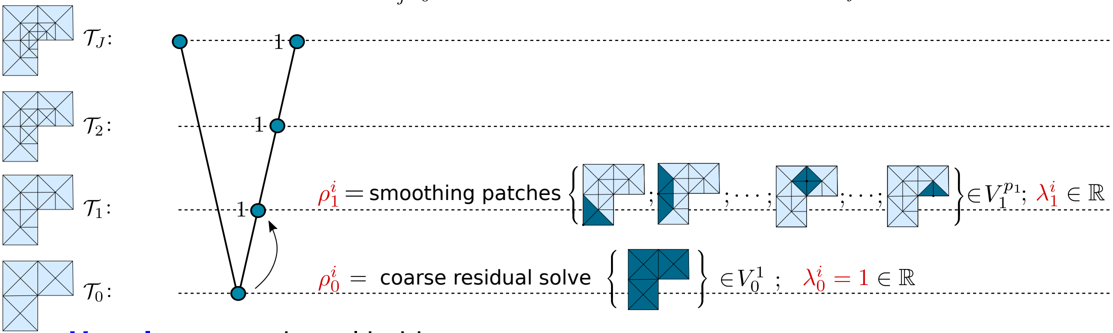

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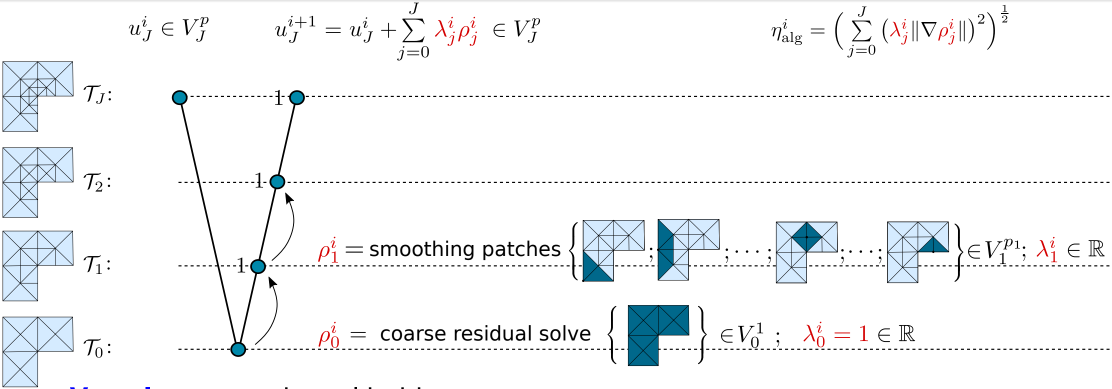

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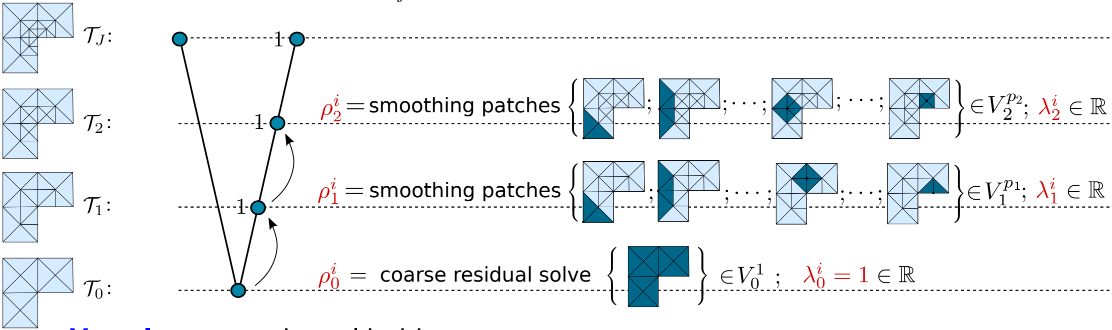

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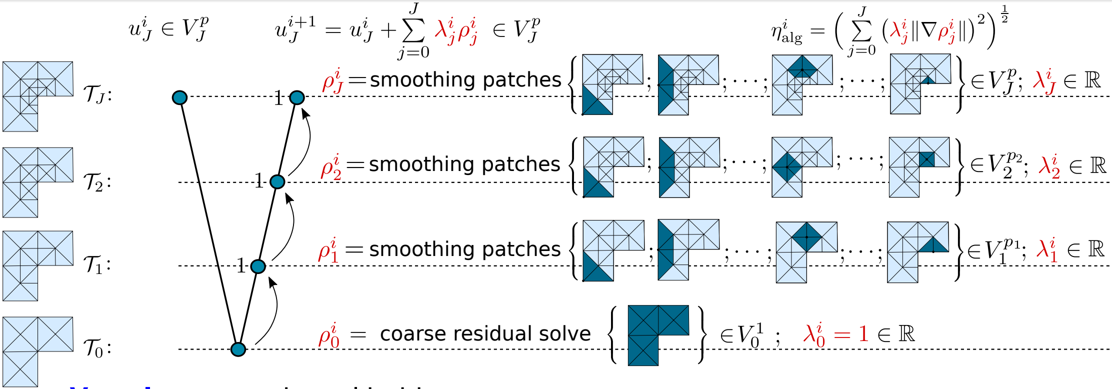

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Functional writing

Let $u_J^i \in V_J^p$ be *arbitrary*. We construct $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ as follows:

Coarse solve: Define $\rho_0^i \in V_0^1$ by: $\underbrace{(\mathbf{K}\nabla\rho_0^i, \nabla v_0)}_{\text{global lifting}} = \underbrace{(f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0)}_{\text{global algebraic residual}} \quad \forall v_0 \in V_0^1.$

Patchwise smoothing (local solves): For $j = 1 : J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by:

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Level-wise step-sizes by line search: $\lambda_j^i := \dfrac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i), \nabla\rho_j^i)}{\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|^2}.$

Functional writing

Let $u_J^i \in V_J^p$ be *arbitrary*. We construct $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ as follows:

Coarse solve: Define $\rho_0^i \in V_0^1$ by:
$$\underbrace{(\mathbf{K}\nabla \rho_0^i, \nabla v_0)}_{\text{global lifting}} = \underbrace{(f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0)}_{\text{global algebraic residual}} \quad \forall v_0 \in V_0^1.$$

Patchwise smoothing (local solves): For $j = 1 : J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by:
$$\underbrace{(\mathbf{K}\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}}_{\text{local lifting}} = \underbrace{(f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \left(\mathbf{K}\nabla\left(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i\right), \nabla v_{j,\mathbf{a}}\right)_{\omega_j^{\mathbf{a}}}}_{\text{local algebraic residual}} \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

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The power of line search: theory

- current approximation $u^i_{J,j-1} := u^i_J + \sum_{k=0}^{j-1} \lambda^i_k \rho^i_k$
- j-level update (correction direction) by Schwarz/block-Jacobi smoothing: ρ^i_j

Lemma (Line search)

The choice

$$\lambda^i_j := \frac{(f, \rho^i_j) - (\mathbf{K}\nabla u^i_{J,j-1}, \nabla \rho^i_j)}{\|\mathbf{K}^{\frac{1}{2}} \nabla \rho^i_j\|^2}$$

minimizes the error $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\|^2$ *over all possible* $\lambda \in \mathbb{R}$

Proof. (Minimization of a quadratic function $\mathbb{R} \to \mathbb{R}$).

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\|^2 = \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_{J,j-1})\|^2 - 2\lambda \underbrace{(\mathbf{K}\nabla(u_J - u^i_{J,j-1}), \nabla\rho^i_j)}_{(f,\rho^i_j) - (\mathbf{K}\nabla u^i_{J,j-1}, \nabla\rho^i_j)} + \lambda^2 \|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_j\|^2$$

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Lemma (Line search: Pythagorean formula for the algebraic error)

The choice

$$\lambda^i_j := \frac{(f, \rho^i_j) - (\mathbf{K}\nabla u^i_{J,j-1}, \nabla \rho^i_j)}{\left\|\mathbf{K}^{\frac12}\nabla \rho^i_j\right\|^2}$$

minimizes the error $\left\|\mathbf{K}^{\frac12}\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\right\|^2$ *over all possible* $\lambda \in \mathbb{R}$ *and gives*

$$\underbrace{\left\|\mathbf{K}^{\frac12}\nabla(u_J - (u^i_{J,j-1} + \lambda^i_j\rho^i_j))\right\|^2}_{\text{new error}} = \underbrace{\left\|\mathbf{K}^{\frac12}\nabla(u_J - u^i_{J,j-1})\right\|^2}_{\text{old error}} - \underbrace{(\lambda^i_j)^2\left\|\mathbf{K}^{\frac12}\nabla \rho^i_j\right\|^2}_{\text{computable decrease}}.$$

Proof. (Minimization of a quadratic function $\mathbb{R} \to \mathbb{R}$).

$$\left\|\mathbf{K}^{\frac12}\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\right\|^2 = \left\|\mathbf{K}^{\frac12}\nabla(u_J - u^i_{J,j-1})\right\|^2 - 2\lambda\underbrace{(\mathbf{K}\nabla(u_J - u^i_{J,j-1}), \nabla\rho^i_j)}_{(f,\rho^i_j) - (\mathbf{K}\nabla u^i_{J,j-1}, \nabla \rho^i_j)} + \lambda^2\left\|\mathbf{K}^{\frac12}\nabla \rho^i_j\right\|^2$$

The power of line search: numerics (global step-size on level J only)

		Sine		Peak		L-shape	
J	p	AS	MG(0,1)-J	AS	MG(0,1)-J	AS	MG(0,1)-J
3	1	21	-	19	68	17	44
4	1	23	-	20	-	18	-
5	1	22	-	20	-	17	-

- for $p = 1$: **AS** and **MG(0,1)-J** **only differ** by the global optimal step-size.

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The power of line search: numerics (global step-size on level J only)

		Sine		Peak		L-shape	
J	p	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J
3	1	21	-	19	68	17	44
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	10	-
4	1	23	-	20	-	18	-
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	9	-
5	1	22	-	20	-	17	-
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	6	13	-	14	-	9	-
	9	13	-	13	-	8	-

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Outline

MG solver and a posteriori estimator of the algebraic error

Definition (MG solver)

Initialize $u_J^0 = 0$ *and let* $i = 0$*. Perform the following steps:*

1. *Construct* $\{\rho_j^i\}_{j=0}^J$ *and* $\{\lambda_j^i\}_{j=0}^J$ *as detailed above.*
2. *Update the current approximation* $u_J^{i+1} := u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i$*.*
3. *If* $\underbrace{\eta_{\mathrm{alg}}^i \text{ is small enough}}_{\text{stopping criterion}}$*, then stop the solver; otherwise increase* $i := i + 1$*.*

Definition (A posteriori estimator of the algebraic error)

Let $u_J^i \in V_J^p$ *be arbitrary. Define the a posteriori estimator of the algebraic error*

$$\eta_{\mathrm{alg}}^i := \left\{ \sum_{j=0}^J \left(\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|\right)^2 \right\}^{\frac{1}{2}}.$$

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Pythagorean error formula and bound on the algebraic error

Proposition (Pythagorean error representation)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be the next iterate. Then

$$\underbrace{\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2}_{\text{new error}} = \underbrace{\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J}\left(\lambda_j^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|\right)^2}_{\left(\eta_{\mathrm{alg}}^i\right)^2}.$$

Corollary (Guaranteed lower bound on the algebraic error)

There holds:

$$\eta_{\mathrm{alg}}^i \leq \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|.$$

- similar situation to the conjugate gradients method, see Meurant (1997) and Strakoš and Tichý (2002)
- here one additional iteration $i \to i+1$ is sufficient for reliable η_{alg}^i

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Main results

Theorem (p-robust error contraction of the multilevel solver)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be given by one MG step. Then

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\| \leq \alpha\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, d, \mathbf{K}, J) < 1.$$

Theorem (p-robust reliable and efficient bound on the algebraic error)

Let η_{alg}^i be the algebraic error estimator. Then, on top of $\eta_{\mathrm{alg}}^i \leq \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|$,

$$\eta_{\mathrm{alg}}^i \geq \beta\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad \beta = \sqrt{1-\alpha^2}.$$

- α is **independent** of the **polynomial degree** p
- the dependence on J is at most *linear* under minimal H^1-regularity
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Additional results

Corollary (Equivalence of the two main results)

The solver **contraction** *is* **equivalent** *to the* **efficiency** *of the estimator* η_{alg}^{i}.

Proof.

By the Pythagorean formula, there holds:

$$\begin{aligned} &(\eta_{\mathrm{alg}}^{i})^2 \geq \beta^2 \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \quad \text{(estimator efficiency)} \\ \Leftrightarrow\ & \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 - \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\|^2 \geq \beta^2 \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \\ \Leftrightarrow\ & \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\|^2 \leq (1-\beta^2) \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \quad \text{(solver contraction).} \end{aligned}$$

Corollary (Equivalence of error–global estimator–local estimators)

There holds

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \approx (\eta_{\mathrm{alg}}^{i})^2 = \sum_{j=0}^{J} (\lambda_j^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|)^2 = \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2.$$

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\Leftrightarrow \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 &\leq (1-\beta^2)\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 \quad \text{(solver contraction).}
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\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \sum_{j=0}^{J}\left(\lambda_j^i \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|\right)^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^{J}\lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j}\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|_{\omega_j^{\mathbf{a}}}^2.
$$

Additional results

Corollary (Equivalence of the two main results)

The solver **contraction** *is* **equivalent** *to the* **efficiency** *of the estimator* $\eta^i_{\rm alg}$.

Proof.

By the Pythagorean formula, there holds:

$$\begin{aligned} &\left(\eta^i_{\rm alg}\right)^2 \geq \beta^2 \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 \quad \text{(estimator efficiency)} \\ \Leftrightarrow\ &\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 - \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^{i+1}_J)\right\|^2 \geq \beta^2 \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 \\ \Leftrightarrow\ &\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^{i+1}_J)\right\|^2 \leq (1-\beta^2)\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 \quad \text{(solver contraction).} \end{aligned}$$

Corollary (Equivalence of error–global estimator–local estimators)

There holds

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 \approx \left(\eta^i_{\rm alg}\right)^2 = \sum_{j=0}^{J}\left(\lambda^i_j \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_j\right\|\right)^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_0\right\|^2 + \sum_{j=1}^{J}\lambda^i_j \sum_{\mathbf{a}\in\mathcal{V}_j}\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_{j,\mathbf{a}}\right\|^2_{\omega^{\mathbf{a}}_j}.$$

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Alternatives

- **patches:** larger subdomains for $V_j^{\mathbf{a}}$
- **smoothing:** modifying $\rho_{j,\mathbf{a}}^i$
  - *damped additive Schwarz* (dAS)
  - *weighted restricted additive Schwarz* (wRAS)

$$\rho_0^i + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i \quad \text{(AS)}, \qquad \rho_0^i + w \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i \quad \text{(dAS)}, \qquad \rho_0^i + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \mathcal{I}_j^{p_j}(\psi_j^{\mathbf{a}} \rho_{j,\mathbf{a}}^i) \quad \text{(wRAS)}$$

  - hat function $\psi_j^{\mathbf{a}}$ for vertex $\mathbf{a} \in \mathcal{V}_j$
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Some of these variants are *parallelizable* also level-wise.

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Outline

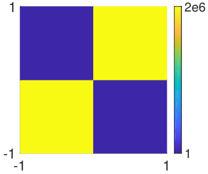
5 test cases

Sine: $u(x,y) = \sin(2\pi x)\sin(2\pi y),\ \Omega := (-1,1)^2$

Peak: $u(x,y) = x(x-1)y(y-1)\mathrm{e}^{-100((x-0.5)^2-(y-0.117)^2)};\ \Omega := (0,1)^2$

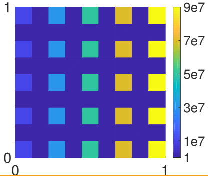
L-shape: $u(r,\theta) = r^{2/3}\sin(2\theta/3);\quad \Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$

Checkerboard: $u(r,\varphi) = r^\gamma \mu(\varphi);\quad \Omega := (-1,1)^2$



with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^6)$ or no jump

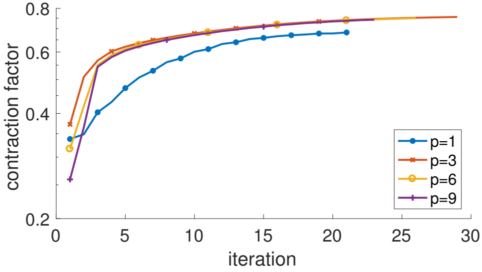
Skyscraper: unknown analytic solution; $\Omega := (0,1)^2$



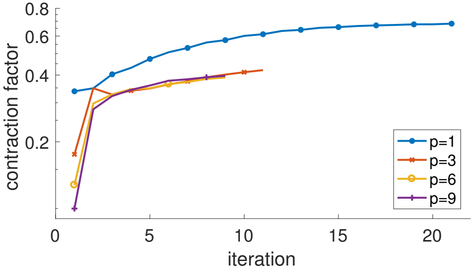
with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^7)$ or $\mathcal{J}(\mathbf{K}) = O(1)$

Confirmation of p-robustness: contraction factors

L-shape problem, $J = 3$, $p_j = 1$ (left) and $p_j = p$ (right), $j \in \{1, \dots, J-1\}$



$1 \to 1, p$



$1, p \to p$

Confirmation of p-robustness: iteration numbers

Stopping criterion: $$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \leq 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Numerical **K**- and J-robustness observed even in low-regularity cases.

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			$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
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			$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
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	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
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			$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
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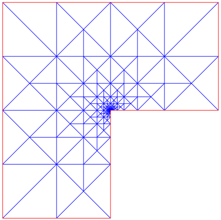
Tests for graded meshes and H^1-regular solutions

L-shape, $\mathbf{K} = I$, 1, $p \to p$

J	p	i_{s}
5	1	16
	3	7
	6	6
	9	5

J	p	i_{s}
10	1	15
	3	6
	6	5
	9	5

J	p	i_{s}
15	1	17
	3	11
	6	5
	9	4



These H^1-regular test cases indicate the possibility of *linear J-dependence*, in accordance with the theoretical results.

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J	p	i_s	J	p	i_s	J	p	i_s
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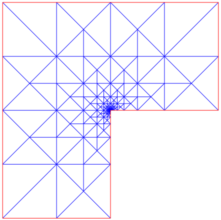
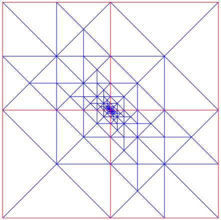
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



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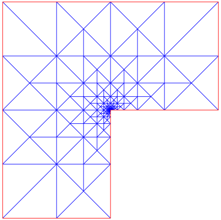
Tests for graded meshes and H^1-regular solutions

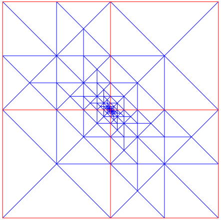
L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



These H^1-regular test cases indicate the possibility of *linear J-dependence*, in accordance with the theoretical results.

Three space dimensions

Test cases: uniform mesh refinement, $p_j = 1$, $j \in \{1, \ldots, J-1\}$, and $J = 4$.

Cube: $\Omega := (0,1)^3$,
$u(x,y,z) = x(x-1)y(y-1)z(z-1)$,
$\mathbf{K} = I$.

Nested cubes: $\Omega := (-1,1)^3$,
unknown analytic solution,
$\mathbf{K} = I$ and $10^5 * I$ in $(-0.5, 0.5)^3$.

Checkers cubes: $\Omega := (0,1)^3$,
unknown analytic solution,
$\mathbf{K} = I$ and $10^6 * I$ in $(0,0.5)^3 \cup (0.5,1)^3$

Three space dimensions

Test cases: uniform mesh refinement, $p_j = 1$, $j \in \{1, \dots, J-1\}$, and $J = 4$.

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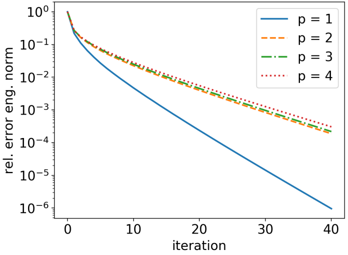

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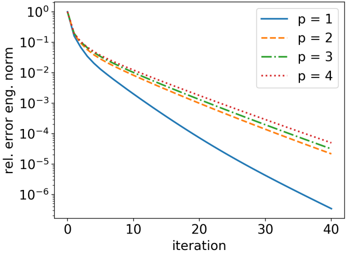

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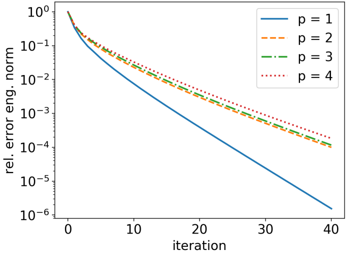

Three space dimensions

Test cases: uniform mesh refinement, $p_j = 1$, $j \in \{1, \dots, J-1\}$, and $J = 4$.

Cube: $\Omega := (0,1)^3$,
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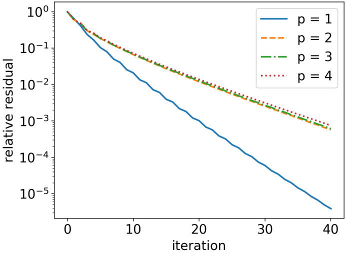

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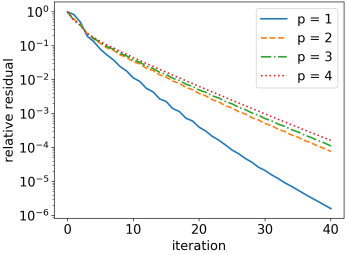

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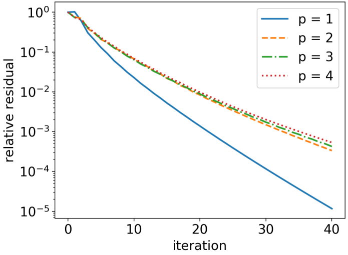
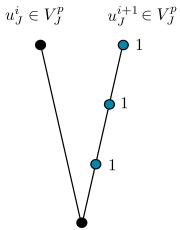


Outline

Adaptive number of smoothing steps



Variable number of smoothing steps/multigrid cycles:

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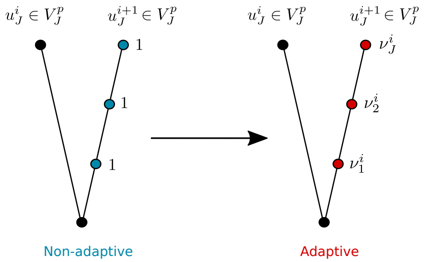

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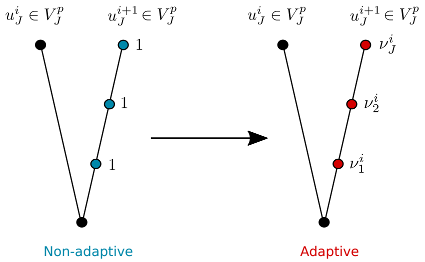

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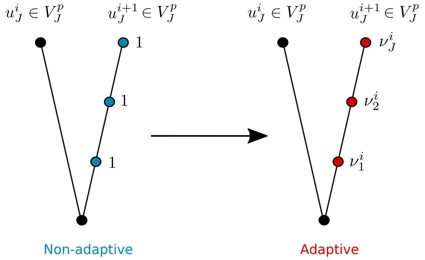

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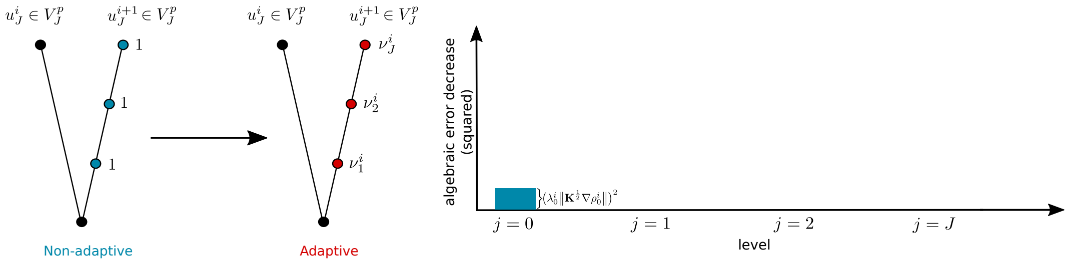

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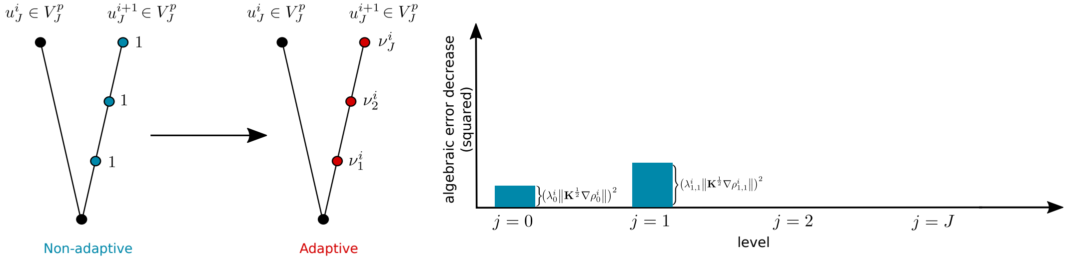

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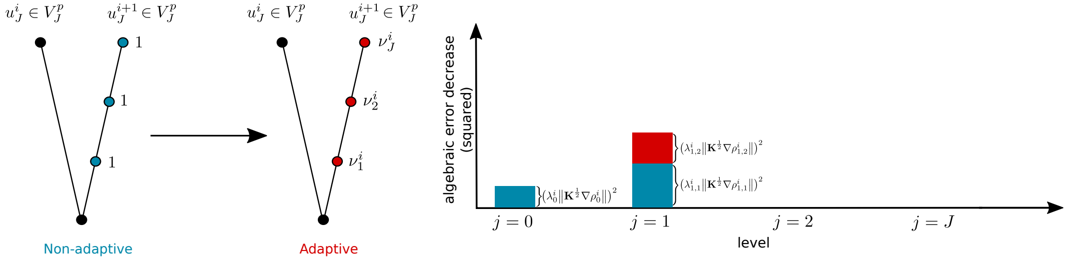

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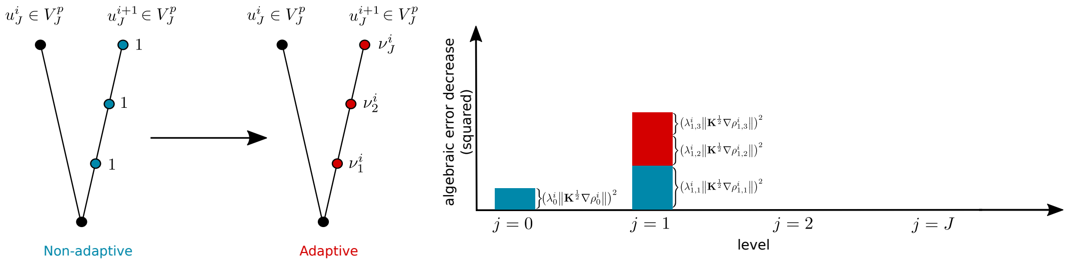

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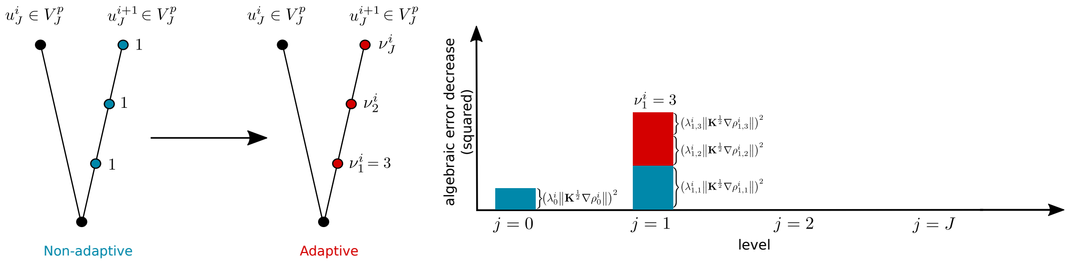

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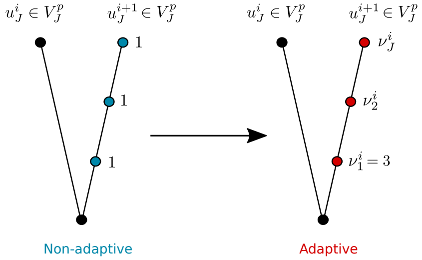

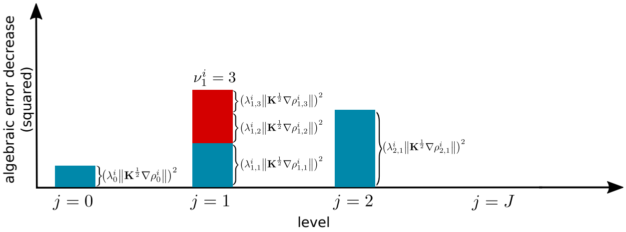

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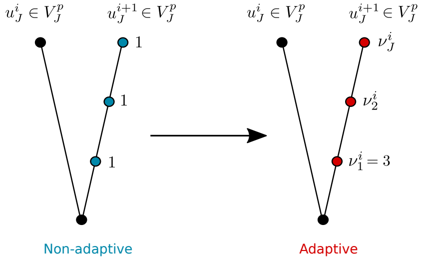

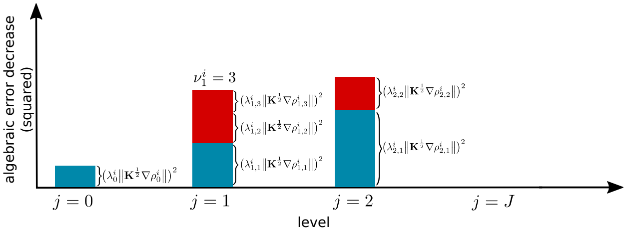

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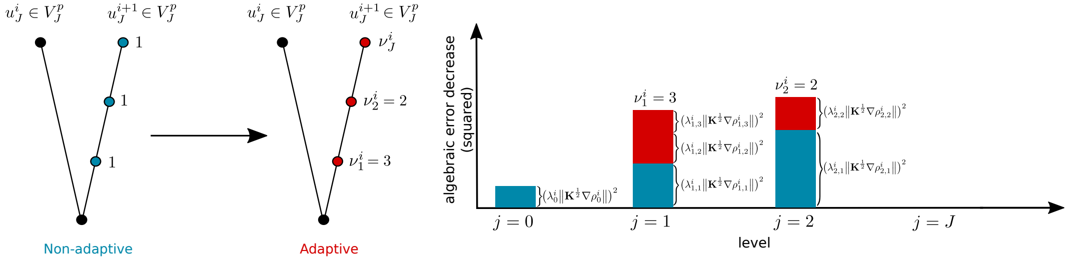

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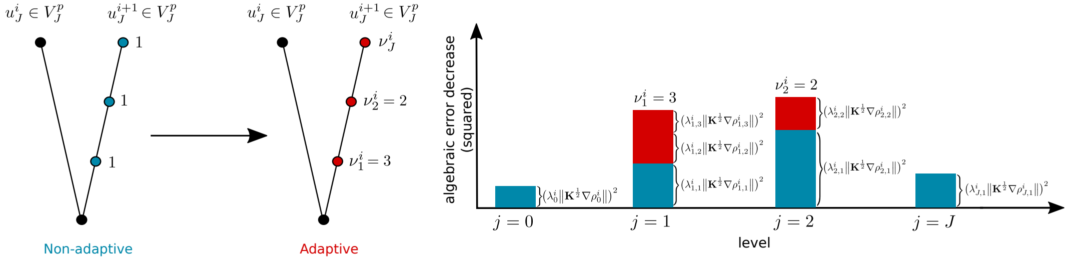

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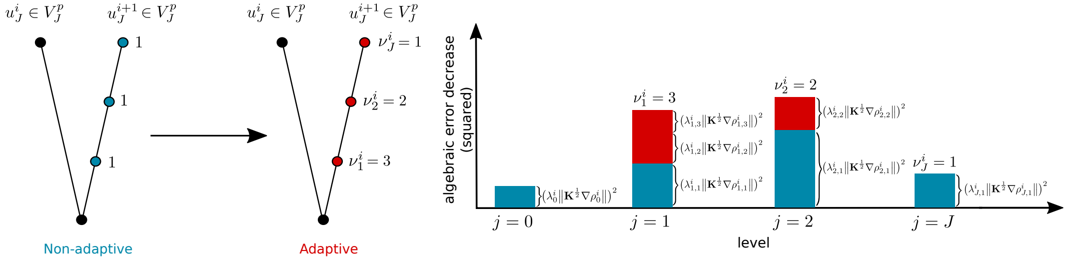

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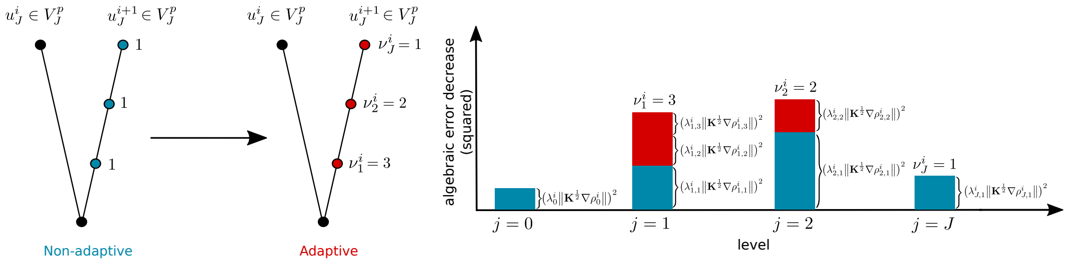

$$(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\nu}^i\|)^2 \geq \theta^2 \Big(\sum_{k=0}^{j-1} \sum_{\ell=1}^{\nu_k^i} (\lambda_{k,\ell}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{k,\ell}^i\|)^2 + \sum_{\ell=1}^{\nu-1} (\lambda_{j,\ell}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\ell}^i\|)^2 \Big)$$

Variable number of smoothing steps/multigrid cycles:

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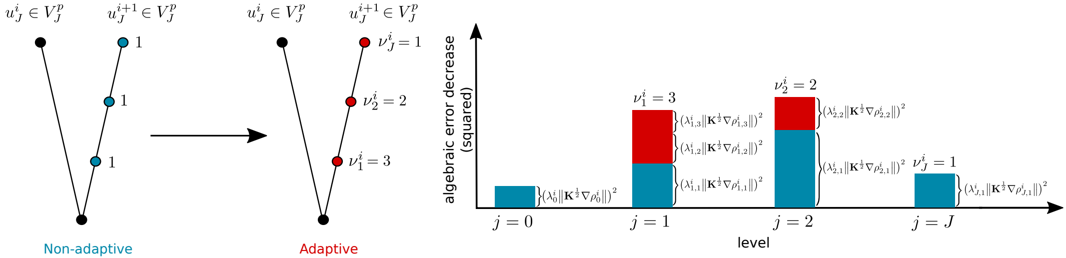

$$\underbrace{\left(\lambda_{j,\nu}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2\left(\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda_{k,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2 + \sum_{\ell=1}^{\nu-1}\left(\lambda_{j,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2\right)$$

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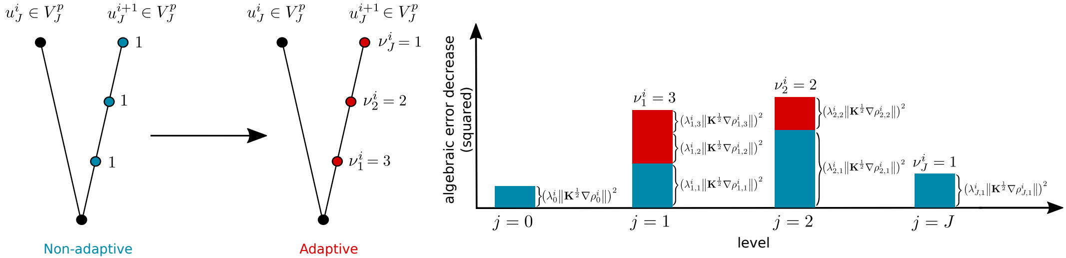

$$\underbrace{\left(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2 \left(\underbrace{\sum_{k=0}^{j-1} \sum_{\ell=1}^{\nu_k^i} \left(\lambda_{k,\ell}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{k,\ell}^i\|\right)^2}_{\text{all smoothings on previous levels}} + \sum_{\ell=1}^{\nu-1} \left(\lambda_{j,\ell}^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_{j,\ell}^i\|\right)^2 \right)$$

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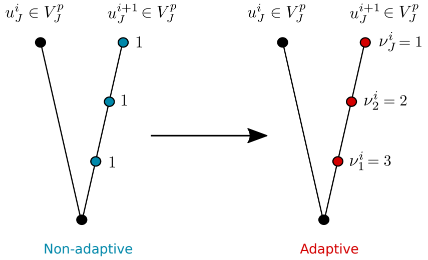

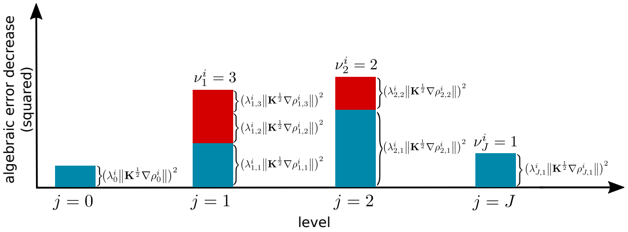

$$\underbrace{\left(\lambda^i_{j,\nu}\left\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\nu}\right\|\right)^2}_{\text{current smoothing}} \ge {\color{red}\theta}^2\left(\underbrace{\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu^i_k}\left(\lambda^i_{k,\ell}\left\|\mathbf{K}^{\frac12}\nabla\rho^i_{k,\ell}\right\|\right)^2}_{\text{all smoothings on previous levels}} + \underbrace{\sum_{\ell=1}^{\nu-1}\left(\lambda^i_{j,\ell}\left\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\ell}\right\|\right)^2}_{\text{previous smoothings on current level}}\right)$$

Variable number of smoothing steps/multigrid cycles:

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Adaptive vs. fixed number of smoothing steps

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1

Adaptive vs. fixed number of smoothing steps

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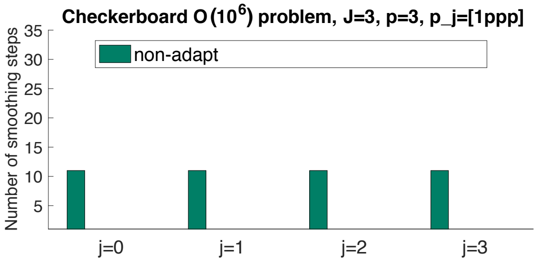
	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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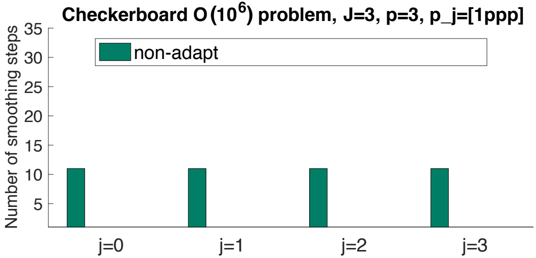
	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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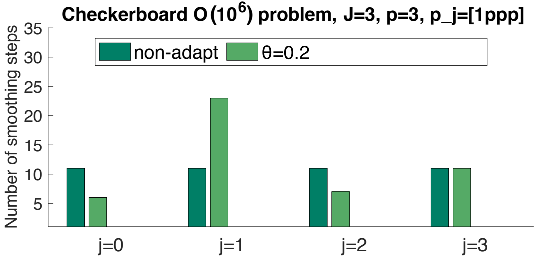
	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1

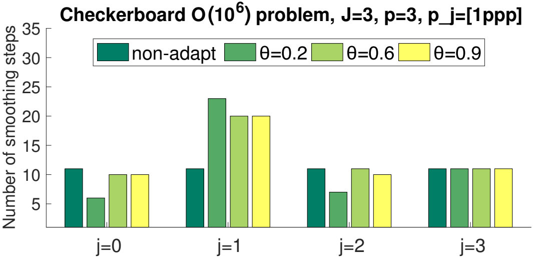


Adaptive vs. fixed number of smoothing steps

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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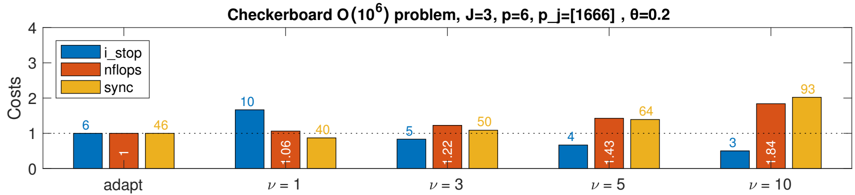
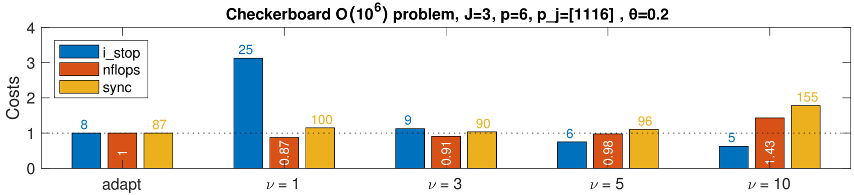
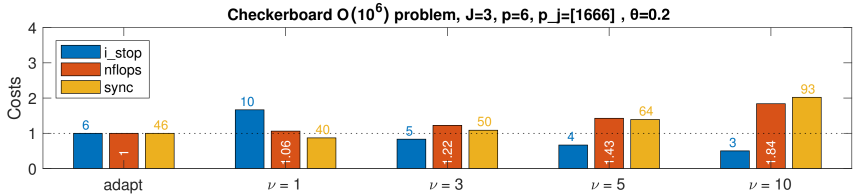


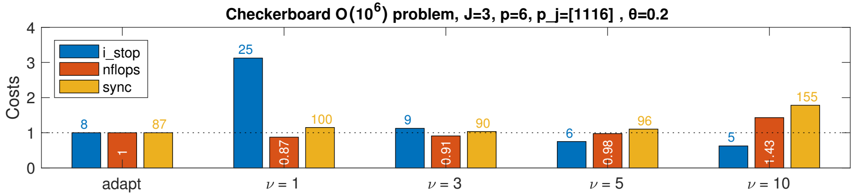


$$\textbf{nflops} := \frac{|\mathcal{V}_0|^3}{3} + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \frac{\mathrm{ndof}(V_j^{\mathbf{a}})^3}{3} + \sum_{i=1}^{i_s} \left[2|\mathcal{V}_0|^2 + \sum_{j=1}^{J} \nu_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} 2\mathrm{ndof}(V_j^{\mathbf{a}})^2 \right] + \sum_{i=1}^{i_s} \sum_{j=1}^{J} \left[2\,\mathrm{nnz}(\mathcal{I}_{j-1}^{j}) + 2\,\mathrm{nnz}(\mathcal{I}_{j}^{j-1}) + 2\nu_j^i\,\mathrm{nnz}(\mathbb{A}_j) + 3\nu_j^i(2\,\mathrm{size}(\mathbb{A}_j)) \right];$$

$$\textbf{sync} := i_s + \sum_{i=1}^{i_s} \sum_{j=1}^{J} \nu_j^i.$$

Adaptive vs. fixed number of smoothing steps





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Comparison with other multilevel solvers

We compare our methods with [1,2,3] in terms of the number of iterations (and CPU times[4]).

not p-robust

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[1] Antonietti et al. *J. Sci. Comput.* 2017.

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[3] Schöberl. "C++11 Implementation of Finite Elements in NGSolve". *Tech. report.* 2014.

[4] The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée "NEF" computation cluster, however, in a sequential Matlab script.

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		∼MG(0,1)-bJ $1, p \to p$		∼MG(0,adapt)-bJ (wRAS) $1 \nearrow p$		PCG(MG(3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m

MG(1,1)-PCG(iChol): not p-robust; MG(3,3)-GS: not p-robust

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Adaptive local smoothing

Recall: $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \approx \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}.$

Local smoothing in adaptively-refined meshes

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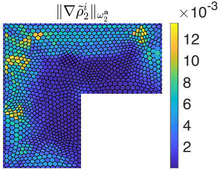

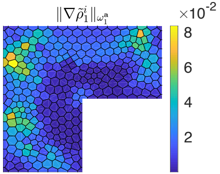

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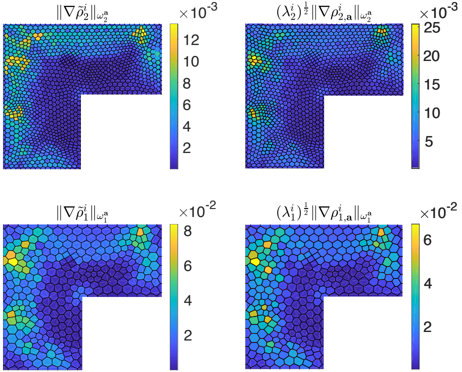

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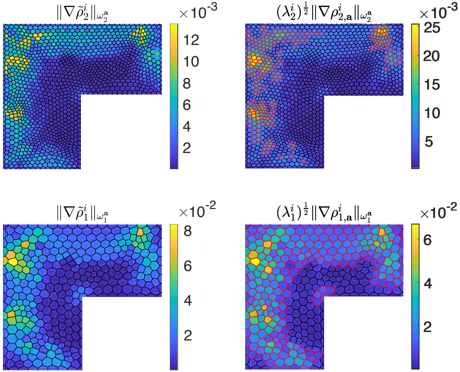

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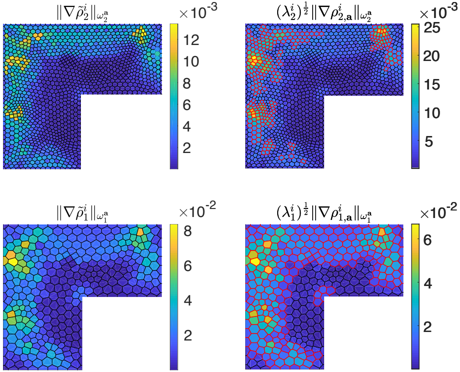

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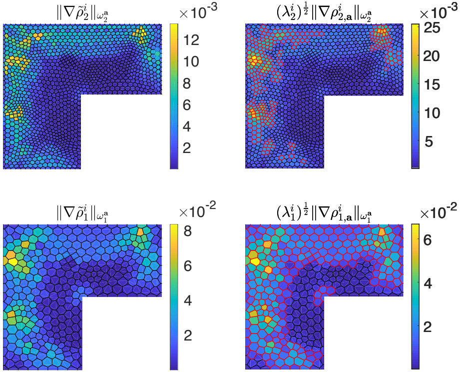

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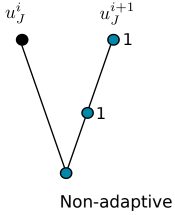

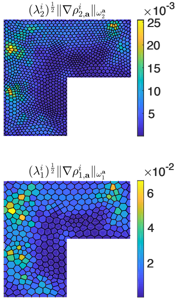

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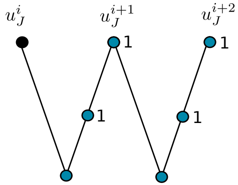
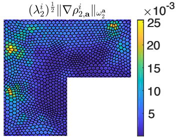
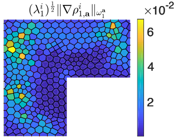


Non-adaptive





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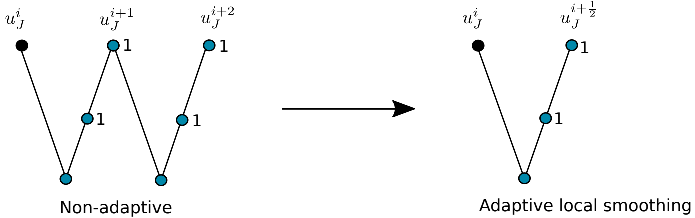

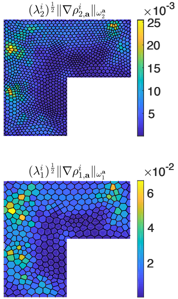

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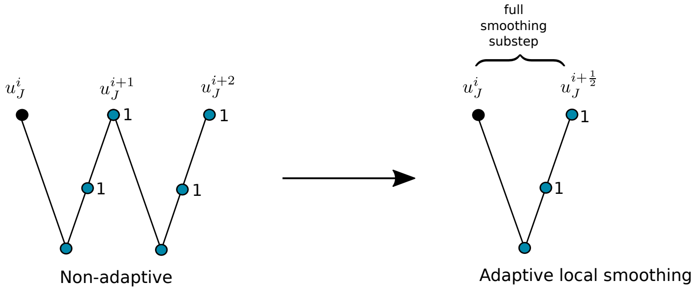

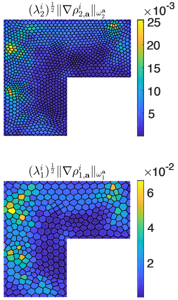

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[11] Dörfler. "A convergent adaptive algorithm for Poisson's equation". *SIAM J. Numer. Anal.* 1996.

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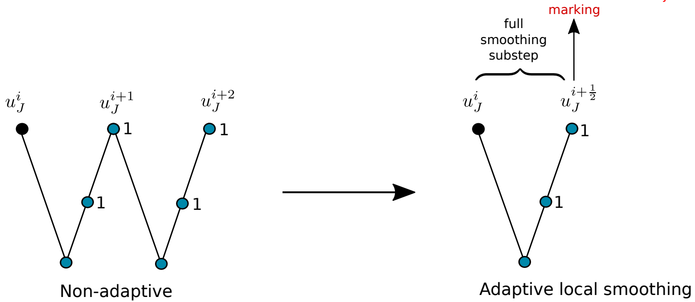
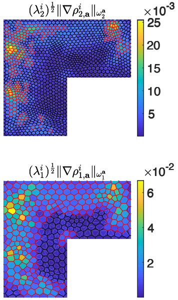
Recall: $\left\|\mathbf{K}^{\frac12}\nabla(u_J - u_J^i)\right\|^2 \approx \left\|\mathbf{K}^{\frac12}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \left\|\mathbf{K}^{\frac12}\nabla\rho_{j,\mathbf{a}}^i\right\|_{\omega_j^{\mathbf{a}}}^2.$





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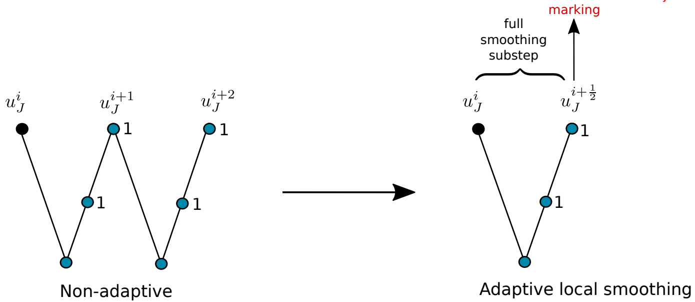

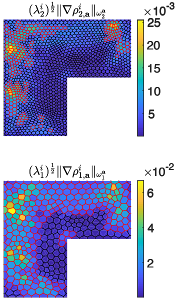

bulk-chasing criterion[11]: $\theta^2\left(\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^{J}\lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j}\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|^2_{\omega_j^{\mathbf{a}}}\right) \le \sum_{j\in\mathcal{M}}\lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j}\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|^2_{\omega_j^{\mathbf{a}}}$

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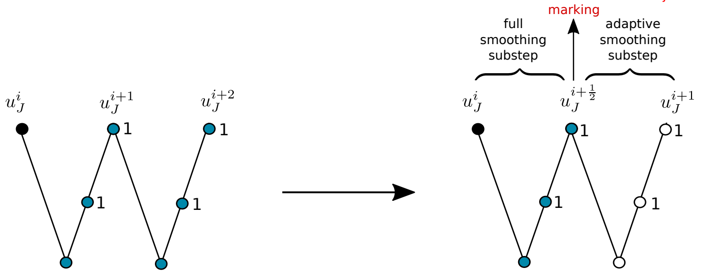
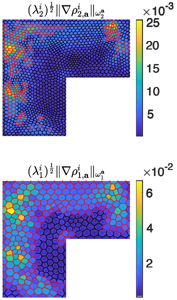
Adaptive local smoothing

Recall: $\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\right\|^2 \approx \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|^2_{\omega_j^{\mathbf{a}}}$.



Non-adaptive

Adaptive local smoothing



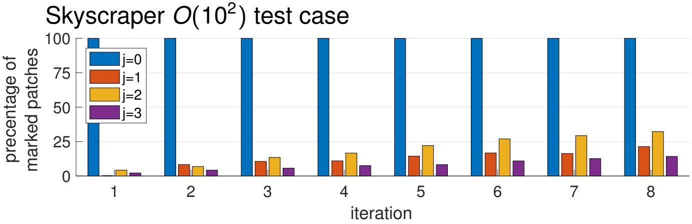
bulk-chasing criterion[11]: $\theta^2\left(\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\right\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|^2_{\omega_j^{\mathbf{a}}}\right) \leq \sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\right\|^2_{\omega_j^{\mathbf{a}}}$

Local smoothing in adaptively-refined meshes vs adaptive local smoothing on a given mesh hierarchy

- Bai and Brandt. “Local mesh refinement multilevel techniques.” *SIAM J. Sci. Statist. Comput.* 1987.
- Rüde. “Mathematical and computational techniques for multilevel adaptive methods.” SIAM 1993.
- Xu, Chen, and Nochetto. “Optimal multilevel methods for H(grad), H(curl), and H(div) systems on graded and unstructured grids”. Springer 2009.

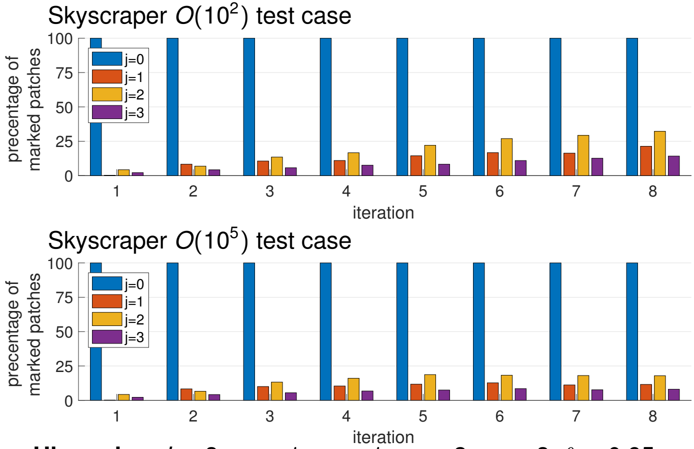
[11] Dörfler. “A convergent adaptive algorithm for Poisson’s equation”. *SIAM J. Numer. Anal.* 1996.

Does the adaptivity pay off?



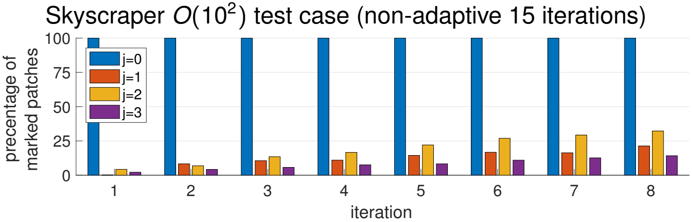
Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

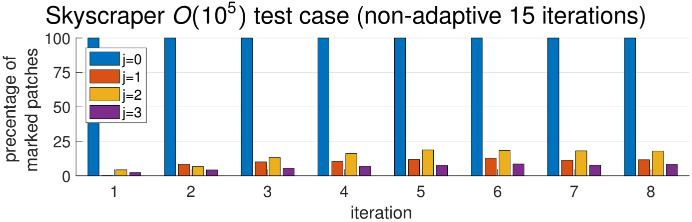
Does the adaptivity pay off?



Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

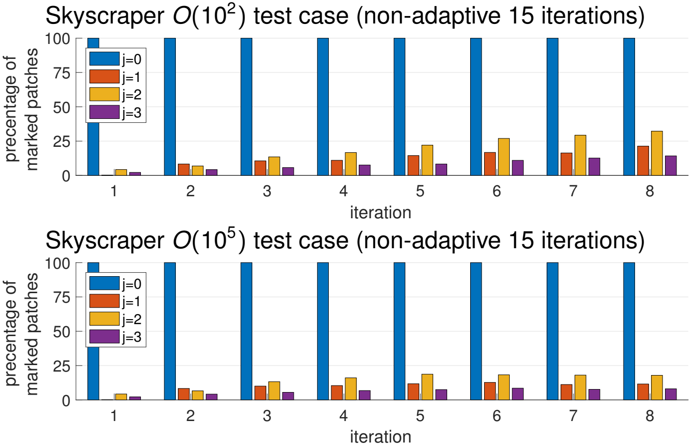
Does the adaptivity pay off?

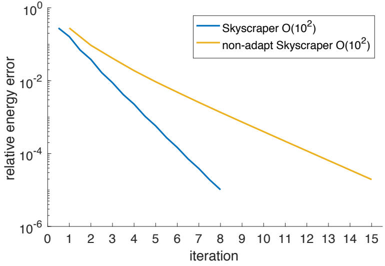




Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

Does the adaptivity pay off?





Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

Outline

Conclusions

Multigrid iterative solver that

- is genuinely steered by an a posteriori estimator that certifies the algebraic error in the energy norm
- features a Pythagorean formula for the decrease of the algebraic error in terms of level-wise and patch-wise computable error reductions
- contracts the algebraic error independently of the polynomial degree p
- is naturally non symmetric: first the roughest modes are captured by the coarse solve, and then smoothing, by additive Schwarz (block-Jacobi), is performed on each mesh level
- is naturally minimalist: only one post-smoothing step is sufficient
- is parameter-free (no damping or number of smoothing steps or other parameters need to be defined)
- calls for algebraic adaptivity: adaptive number of smoothing steps and adaptive choice of patches to perform smoothing

Future directions and references

Future directions

- more complex model problems
- use in applications
- proofs of optimality of numerical methods wrt computational cost

References

- A. Miraçi, J. Papež, M. Vohralík, A multilevel algebraic error estimator and the corresponding iterative solver with p-robust behavior, *SIAM J. Numer. Anal.* **58** (2020), 2856–2884.
- A. Miraçi, J. Papež, M. Vohralík, A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps, *SIAM J. Sci. Comput.* **43** (2021), S117–S145.
- A. Miraçi, J. Papež, M. Vohralík, Contractive local adaptive smoothing based on Dörfler's marking in a-posteriori-steered p-robust multigrid solvers, *Comput. Methods Appl. Math.* **21** (2021), 445–468.

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