Adaptive regularization, linearization, and discretization for the two-phase Stefan problem

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joint work with D. A. Di Pietro and S. Yousef

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Outline

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The Stefan problem

The Stefan problem

$$\begin{aligned} \partial_t u - \Delta\beta(u) &= f && \text{in } \Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0, T) \end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0, 1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$

Context

- ongoing Ph.D. thesis of Soleiman Yousef
- collaboration with IFP Energies Nouvelles

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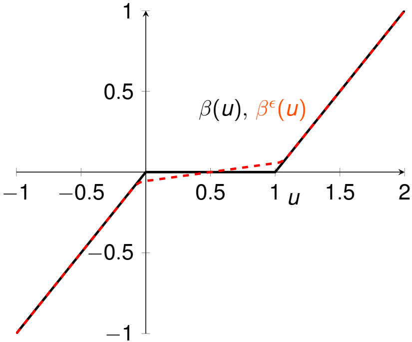
Numerical practice: regularization

Regularization of β with a parameter ϵ



Questions

Discretization

- ...

Question (Stopping and balancing criteria)

- *What is a good choice of the*
  - *regularization parameter ϵ?*
  - *time step?*
  - *space mesh?*
- *What is a good stopping criterion for the*
  - *nonlinear solver?*
  - *linear solver?*

Question (Error)

- *How big is the error $\|u|_{I_n} - u_{h\tau}^{n,\epsilon,k,i}\|$ on time step n, space mesh $\mathcal{K}^n$, regularization parameter ϵ, linearization step k, and algebraic solver step i? How big are the individual components? How is error distributed in time and space?*

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Previous results – a posteriori error estimates

Nonlinear steady problems

- Ladevèze (since 1990's), guaranteed upper bound
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), linearization errors
- Kim (2007), guaranteed estimates, loc. cons. methods

Linear unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness wrt the final time

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- Verfürth (1998), framework for energy norm control
- Ohlberger (2001), non energy-norm estimates

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1. Introduction
2. Error estimates for the dual norm of the residual
   - Residual and its dual norm
   - A posteriori error estimate
   - Error components identification and adaptivity
   - Efficiency
3. Energy error estimates
4. Numerical results
5. Conclusions and future directions

Weak formulation

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$$u \in Z \qquad \text{with } \beta(u) \in X$$

$$u(\cdot,0) = u_0 \qquad \text{in } \Omega$$

$$\langle \partial_t u, \varphi \rangle(s) + (\nabla \beta(u), \nabla \varphi)(s) = (f,\varphi)(s) \qquad \forall \varphi \in H_0^1(\Omega)$$

$$\text{a.e. } s \in (0,T)$$

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Residual and its dual norm

Residual $\mathcal{R}(u_{h\tau}) \in X'$ **for** $u_{h\tau} \in Z$ **such that** $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X} = \int_0^T \{ \langle \partial_t (u - u_{h\tau}), \varphi \rangle + (\nabla \beta(u) - \nabla \beta(u_{h\tau}), \nabla \varphi) \} (s) \, \mathrm{d}s, \quad \varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi \in X, \, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X}$$

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Time-localization of the dual norm of the residual

Time interval I_n

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L^2 in time . . .

$$\|\mathcal{R}(u_{h\tau})\|_{X'}^2 = \sum_{1 \le n \le N} \|\mathcal{R}(u_{h\tau})\|_{X_n'}^2$$

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Assumptions

Assumption A (Approximate solution)

The function $u_{h\tau}$ *is such that*

$$u_{h\tau} \in Z, \qquad \partial_t u_{h\tau} \in L^2(0,T;L^2(\Omega)), \qquad \beta(u_{h\tau}) \in X,$$
$$u_{h\tau}|_{I_n} \text{ is affine in time on } I_n \qquad \forall 1 \le n \le N.$$

Assumption B (Equilibrated flux reconstruction)

For all $1 \le n \le N$*, there exists a vector field* $\mathbf{t}_h^n \in \mathbf{H}(\mathrm{div};\Omega)$ *such that*

$$(\nabla\cdot\mathbf{t}_h^n, 1)_K = (f^n, 1)_K - (\partial_t u_{h\tau}^n, 1)_K \qquad \forall K \in \mathcal{K}^n.$$

We denote by $\mathbf{t}_{h\tau}$ *the space–time function such that* $\mathbf{t}_{h\tau}|_{I_n} := \mathbf{t}_h^n$.

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A posteriori error estimate

Theorem (A posteriori error estimate)

Let Assumptions A and B hold. Then

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(\cdot,0)\|_{H^{-1}(\Omega)}$$

$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K\in\mathcal{K}^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n(t)\right)^2 \,\mathrm{d}t \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau}^n - \nabla\cdot\mathbf{t}_h^n\|_K,$$

$$\eta_{\mathrm{F},K}^n(t) := \|\nabla\beta(u_{h\tau}(t)) + \mathbf{t}_h^n\|_K,$$

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Sketch of the proof

- $\varphi \in X$ with $\|\varphi\|_X = 1$ given (f pw constant in time)
- adding and subtracting $(\mathbf{t}_{h\tau}, \nabla\varphi)$, Green theorem:

$$\langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X',X} = \mathfrak{T}_1 + \mathfrak{T}_2$$

$$:= \int_0^T \{(f - \partial_t u_{h\tau} - \nabla\cdot\mathbf{t}_{h\tau}, \varphi) - (\mathbf{t}_{h\tau} + \nabla\beta(u_{h\tau}), \nabla\varphi)\}(s)\mathrm{d}s$$

- $$\mathfrak{T}_1 = \sum_{n=1}^N \int_{I_n} \sum_{K\in\mathcal{K}^n} (f^n - \partial_t u_{h\tau}^n - \nabla\cdot\mathbf{t}_h^n, \varphi - \Pi_0^n\varphi)_K(s)\mathrm{d}s$$
$$\leq \sum_{n=1}^N \int_{I_n} \sum_{K\in\mathcal{K}^n} \underbrace{C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau}^n - \nabla\cdot\mathbf{t}_h^n\|_K}_{\eta_{\mathrm{R},K}^n} \|\nabla\varphi\|_K(s)\mathrm{d}s$$

- $$\mathfrak{T}_2 \leq \sum_{n=1}^N \int_{I_n} \sum_{K\in\mathcal{K}^n} \underbrace{\|\mathbf{t}_h^n + \nabla\beta(u_{h\tau})\|_K}_{\eta_{\mathrm{F},K}^n} \|\nabla\varphi\|_K(s)\mathrm{d}s$$

Outline

Distinguishing different error components

Theorem (An estimate distinguishing the error components)

For time n, linearization k, and regularization ϵ, there holds

$$\|\mathcal{R}(u_{h\tau}^{n,\epsilon,k})\|_{X'_n} \leq \eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k} + \eta_{\mathrm{lin}}^{n,\epsilon,k}.$$

- $\mathbf{l}_h^{n,\epsilon,k}$ a scheme linearized flux (not $\mathbf{H}(\mathrm{div},\Omega)$), $\mathbf{t}_h^{n,\epsilon,k}$ reconstructed $\mathbf{H}(\mathrm{div},\Omega)$ flux, Π^n interpolation op.

$$(\eta_{\mathrm{sp}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{K}^n} \left(\eta_{\mathrm{R},K}^{n,\epsilon,k} + \|\mathbf{l}_h^{n,\epsilon,k} + \mathbf{t}_h^{n,\epsilon,k}\|_K\right)^2,$$

$$(\eta_{\mathrm{tm}}^{n,\epsilon,k})^2 := \int_{I_n} \sum_{K\in\mathcal{K}^n} \|\nabla\Pi^n\beta(u_{h\tau}^{n,\epsilon,k})(t) - \nabla\Pi^n\beta(u_{h\tau}^{n,\epsilon,k})(t^n)\|_K^2 \,\mathrm{d}t,$$

$$(\eta_{\mathrm{reg}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{K}^n} \|\nabla\Pi^n\beta(u_{h\tau}^{n,\epsilon,k})(t^n) - \nabla\Pi^n\beta_\epsilon(u_{h\tau}^{n,\epsilon,k})(t^n))\|_K^2,$$

$$(\eta_{\mathrm{lin}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{K}^n} \|\nabla\Pi^n\beta_\epsilon(u_{h\tau}^{n,\epsilon,k})(t^n) - \mathbf{l}_h^{n,\epsilon,k}\|_K^2$$

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Adaptive algorithm

Goal

$$\frac{\sum_{n=1}^{N} \|\mathcal{R}(u_{h\tau}^{n,\epsilon,k})\|_{X_n'}^2}{\sum_{n=1}^{N} \|\mathbf{l}_h^{n,\epsilon,k}\|_{L^2(I_n;L^2(\Omega))}^2} \le \zeta^2$$

Computer resources limitations

$$\min_{K\in\mathcal{K}^n} h_K \ge \underline{h}, \qquad \tau^n \ge \underline{\tau}$$

Algorithm (Adaptive algorithm, initialization)

Choose an initial mesh $\mathcal{K}^0$, regularization parameter ϵ_0, and a tolerance $\zeta_{\mathrm{IC}} > 0$
repeat {Initial mesh and regularization parameter adaptation}
  Compute η_{IC}, adapt $\mathcal{K}^0$, and adjust ϵ_0.
until $\eta_{\mathrm{IC}} \le \zeta_{\mathrm{IC}} \|\nabla(\beta_{\epsilon_0}(u_h^0))\|$
Choose an initial time step τ^0, $\epsilon \leftarrow \epsilon_0$, $t^0 \leftarrow 0$, $n \leftarrow 0$

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Adaptive algorithm

Algorithm (Adaptive algorithm)

while $t^n \leq T$ **do** {Time loop}

$n \leftarrow n+1$, $\mathcal{K}^n \leftarrow \mathcal{K}^{n-1}$, $\tau^n \leftarrow \tau^{n-1}$, $u_h^{n,\epsilon,0} \leftarrow u_h^{n-1}$

repeat {Space refinement}

repeat {Space and time error balancing}

repeat {Regularization}

$k \leftarrow 0$

repeat {Nonlinear solver}

$k \leftarrow k+1$; $u_h^{n,\epsilon,k} = \Psi(u_h^{n,\epsilon,k-1}, \tau^n, \mathcal{K}^n)$; compute $\eta_{\rm sp}^{n,\epsilon,k}$, $\eta_{\rm tm}^{n,\epsilon,k}$, $\eta_{\rm reg}^{n,\epsilon,k}$, $\eta_{\rm lin}^{n,\epsilon,k}$

until $\eta_{\rm lin}^{n,\epsilon,k} \leq \Gamma_{\rm lin}\left(\eta_{\rm sp}^{n,\epsilon,k} + \eta_{\rm tm}^{n,\epsilon,k} + \eta_{\rm reg}^{n,\epsilon,k}\right)$

$k_n \leftarrow k$

if $\eta_{\rm reg}^{n,\epsilon,k_n} \leq \Gamma_{\rm reg}\left(\eta_{\rm sp}^{n,\epsilon,k_n} + \eta_{\rm tm}^{n,\epsilon,k_n}\right)$ does not hold **then**

$\epsilon \leftarrow \epsilon/2$

end if

until $\eta_{\rm reg}^{n,\epsilon,k_n} \leq \Gamma_{\rm reg}\left(\eta_{\rm sp}^{n,\epsilon,k_n} + \eta_{\rm tm}^{n,\epsilon,k_n}\right)$

$\epsilon_n \leftarrow \epsilon$

if $\eta_{\rm tm}^{n,\epsilon_n,k_n} < \gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n}$ **then**

$\tau^n \leftarrow 2\tau^n$

else if $\eta_{\rm tm}^{n,\epsilon_n,k_n} > \Gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n}$ and $\tau^n \geq 2\underline{\tau}$ **then**

$\tau^n \leftarrow \tau^n/2$

end if

until $\gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n} \leq \eta_{\rm tm}^{n,\epsilon_n,k_n} \leq \Gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n}$ or $\tau^n = \underline{\tau}$

Refine the cells $K \in \mathcal{K}^n$ such that $\eta_{{\rm sp},K}^{n,\epsilon_n,k_n} \geq c_{\rm ref} \max_{L\in\mathcal{K}^n} \eta_{{\rm sp},L}^{n,\epsilon_n,k_n}$

until $\eta_{\rm sp}^{n,\epsilon_n,k_n} + \eta_{\rm tm}^{n,\epsilon_n,k_n} + \eta_{\rm reg}^{n,\epsilon_n,k_n} + \eta_{\rm lin}^{n,\epsilon_n,k_n} \leq \zeta \|\mathbf{l}_h^{n,\epsilon_n,k_n}\|_{L^2(I_n;L^2(\Omega))}$ or $h_K = \underline{h}$ for all K

Derefine cells $\eta_{{\rm sp},K}^{n,\epsilon_n,k_n} \leq c_{\rm deref} \max_{L\in\mathcal{K}^n} \eta_{{\rm sp},L}^{n,\epsilon_n,k_n}$; $u_h^n \leftarrow u_h^{n,\epsilon_n,k_n}$; $t^n \leftarrow t^{n-1} + \tau^n$; $\epsilon \leftarrow 2\epsilon$

end while

Outline

Efficiency assumptions

Assumption C (Technicalities)

All the meshes are shape-regular and all the approximations are piecewise polynomial.

Residual estimators

$$\left(\eta_{\mathrm{res},1}^{n,\epsilon_n,k_n}\right)^2 := \tau^n \sum_{K\in\mathcal{K}^{n-1,n}} h_K^2 \|f^n - \partial_t u_{h\tau}^{n,\epsilon_n,k_n} + \nabla\cdot \mathbf{l}_h^{n,\epsilon_n,k_n}\|_K^2,$$

$$\left(\eta_{\mathrm{res},2}^{n,\epsilon_n,k_n}\right)^2 := \tau^n \sum_{F\in\mathcal{F}^{\mathrm{i},n-1,n}} h_F \|[\![\mathbf{l}_h^{n,\epsilon_n,k_n}]\!]\cdot\mathbf{n}_F\|_F^2$$

Assumption D (Approximation property)

For all $1 \le n \le N$, there holds

$$\tau^n \sum_{K\in\mathcal{K}^{n-1,n}} \|\mathbf{l}_h^{n,\epsilon_n,k_n} + \mathbf{t}_h^{n,\epsilon_n,k_n}\|_K^2 \le C\left(\left(\eta_{\mathrm{res},1}^{n,\epsilon_n,k_n}\right)^2 + \left(\eta_{\mathrm{res},2}^{n,\epsilon_n,k_n}\right)^2\right).$$

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Assumption D (Approximation property)

For all $1 \le n \le N$*, there holds*

$$\tau^n \sum_{K\in\mathcal{K}^{n-1,n}} \|\mathbf{l}_h^{n,\epsilon_n,k_n} + \mathbf{t}_h^{n,\epsilon_n,k_n}\|_K^2 \le C\left(\left(\eta_{\mathrm{res},1}^{n,\epsilon_n,k_n}\right)^2 + \left(\eta_{\mathrm{res},2}^{n,\epsilon_n,k_n}\right)^2\right).$$

Efficiency assumptions

Theorem (Efficiency)

Let, for all $1 \leq n \leq N$, the stopping and balancing criteria be satisfied with the parameters $\Gamma_{\rm lin}$, $\Gamma_{\rm reg}$, and $\Gamma_{\rm tm}$ small enough. Let Assumptions C and D hold. Then

$$\eta_{\rm sp}^{n,\epsilon_n,k_n} + \eta_{\rm tm}^{n,\epsilon_n,k_n} + \eta_{\rm reg}^{n,\epsilon_n,k_n} + \eta_{\rm lin}^{n,\epsilon_n,k_n} \lesssim \|\mathcal{R}(u_{h\tau}^{n,\epsilon_n,k_n})\|_{X_n'}.$$

Sketch of the proof, part 1

- stopping and balancing criteria:

$$\eta_{\mathrm{sp}}^{n,\epsilon_n,k_n} + \eta_{\mathrm{tm}}^{n,\epsilon_n,k_n} + \eta_{\mathrm{reg}}^{n,\epsilon_n,k_n} + \eta_{\mathrm{lin}}^{n,\epsilon_n,k_n} \lesssim \eta_{\mathrm{sp}}^{n,\epsilon_n,k_n}$$

- inverse inequality, $K \in \mathcal{K}^n$:

$$\eta_{\mathrm{R},K}^{n,\epsilon_n,k_n} \lesssim \left\{ \sum_{K' \in \mathcal{K}^{n-1,n},\, K' \subset K} h_{K'}^2 \| f^n - \partial_t u_{h\tau}^{n,\epsilon_n,k_n} + \nabla\cdot \mathbf{l}_h^{n,\epsilon_n,k_n} \|_{K'}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K' \in \mathcal{K}^{n-1,n},\, K' \subset K} \| \mathbf{l}_h^{n,\epsilon_n,k_n} + \mathbf{t}_h^{n,\epsilon_n,k_n} \|_{K'}^2 \right\}^{\frac{1}{2}}$$

- approximation property:

$$\eta_{\mathrm{sp}}^{n,\epsilon_n,k_n} \lesssim \eta_{\mathrm{res},1}^{n,\epsilon_n,k_n} + \eta_{\mathrm{res},2}^{n,\epsilon_n,k_n}$$

Sketch of the proof, part 2

- denote

$$\left(\eta_{\mathrm{LRQT}}^{n,\epsilon_n,k_n}\right)^2 := \int_{I_n} \sum_{K\in\mathcal{K}^{n-1,n}} \|\nabla\beta(u_{h\tau}^{n,\epsilon_n,k_n}(t)) - \mathbf{l}_h^{n,\epsilon_n,k_n}\|_K^2 \mathrm{d}t$$

- element bubble function technique:

$$\eta_{\mathrm{res},1}^{n,\epsilon_n,k_n} \lesssim \|\mathcal{R}(u_{h\tau}^{n,\epsilon_n,k_n})\|_{X_n'} + \eta_{\mathrm{LRQT}}^{n,\epsilon_n,k_n}$$

- face bubble function technique:

$$\eta_{\mathrm{res},2}^{n,\epsilon_n,k_n} \lesssim \|\mathcal{R}(u_{h\tau}^{n,\epsilon_n,k_n})\|_{X_n'} + \eta_{\mathrm{LRQT}}^{n,\epsilon_n,k_n}$$

- thus:

$$\eta_{\mathrm{sp}}^{n,\epsilon_n,k_n} \lesssim \|\mathcal{R}(u_{h\tau}^{n,\epsilon_n,k_n})\|_{X_n'} + \eta_{\mathrm{LRQT}}^{n,\epsilon_n,k_n}$$

- triangle inequality and stopping and balancing criteria:

$$\eta_{\mathrm{LRQT}}^{n,\epsilon_n,k_n} \le \eta_{\mathrm{lin}}^{n,\epsilon_n,k_n} + \eta_{\mathrm{reg}}^{n,\epsilon_n,k_n} + \eta_{\mathrm{tm}}^{n,\epsilon_n,k_n} \le C(\Gamma_{\mathrm{lin}}, \Gamma_{\mathrm{reg}}, \Gamma_{\mathrm{tm}})\eta_{\mathrm{sp}}^{n,\epsilon_n,k_n}$$

- parameters $\Gamma_{\mathrm{lin}}, \Gamma_{\mathrm{reg}}, \Gamma_{\mathrm{tm}}$ small enough:

$$\eta_{\mathrm{sp}}^{n,\epsilon_n,k_n} \lesssim \|\mathcal{R}(u_{h\tau}^{n,\epsilon_n,k_n})\|_{X_n'}$$

Outline

Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

$$\frac{L_\beta}{2}\|u-u_{h\tau}\|^2_{X'}+\frac{L_\beta}{2}\|(u-u_{h\tau})(\cdot,T)\|^2_{H^{-1}(\Omega)}+\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_T}$$

$$\leq\frac{L_\beta}{2}(2e^T-1)\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'}+\|(u-u_{h\tau})(\cdot,0)\|^2_{H^{-1}(\Omega)}\right)$$

Theorem (Temperature and enthalpy errors, tight Gronwall)

Let $u_{h\tau}\in Z$ such that $\beta(u_{h\tau})\in X$ be arbitrary. There holds

$$\frac{L_\beta}{2}\|u-u_{h\tau}\|^2_{X'}+\frac{L_\beta}{2}\|(u-u_{h\tau})(\cdot,T)\|^2_{H^{-1}(\Omega)}+\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_T}$$

$$+2\int_0^T\left(\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_t}+\int_0^t\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_s}e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t$$

$$\leq\frac{L_\beta}{2}\Bigg\{(2e^T-1)\|(u-u_{h\tau})(\cdot,0)\|^2_{H^{-1}(\Omega)}+\|\mathcal{R}(u_{h\tau})\|^2_{X'}$$

$$+2\int_0^T\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'_t}+\int_0^t\|\mathcal{R}(u_{h\tau})\|^2_{X'_s}e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t\Bigg\}.$$

Relation residual–energy norm

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$$\leq\frac{L_\beta}{2}\Big\{(2e^T-1)\|(u-u_{h\tau})(\cdot,0)\|^2_{H^{-1}(\Omega)}+\|\mathcal{R}(u_{h\tau})\|^2_{X'}$$
$$+2\int_0^T\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'_t}+\int_0^t\|\mathcal{R}(u_{h\tau})\|^2_{X'_s}e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t\Big\}.$$

Sketch of the proof, part 1

Lemma (Duality estimate)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$. Then, for a.e. $t \in (0, T)$,

$$\frac{2}{L_\beta}\|\beta(u) - \beta(u_{h\tau})\|^2_{Q_t} + \|(u - u_{h\tau})(\cdot, t)\|^2_{H^{-1}(\Omega)}$$
$$\leq \|(u - u_{h\tau})(\cdot, 0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X_t'} + \|u - u_{h\tau}\|^2_{X_t'}.$$

- $W(\cdot, t) \in H^1_0(\Omega)$ the solution to

$$(\nabla W(\cdot, t), \nabla \psi) = ((u - u_{h\tau})(\cdot, t), \psi) \qquad \forall \psi \in H^1_0(\Omega)$$

- duality:

$$\|\nabla W(\cdot, t)\| = \|(u - u_{h\tau})(\cdot, t)\|_{H^{-1}(\Omega)}$$

- there holds

$$\langle \mathcal{R}(u_{h\tau}), W \rangle_{X_t', X_t} \leq \frac{1}{2}\|\mathcal{R}(u_{h\tau})\|^2_{X_t'} + \frac{1}{2}\|u - u_{h\tau}\|^2_{X_t'}$$

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- duality:

$$\|\nabla W(\cdot, t)\| = \|(u - u_{h\tau})(\cdot, t)\|_{H^{-1}(\Omega)}$$

- there holds

$$\langle \mathcal{R}(u_{h\tau}), W\rangle_{X'_t, X_t} \leq \frac{1}{2}\|\mathcal{R}(u_{h\tau})\|^2_{X'_t} + \frac{1}{2}\|u - u_{h\tau}\|^2_{X'_t}$$

Sketch of the proof, part 2

- definition of the residual:

$$\langle \mathcal{R}(u_{h\tau}), W\rangle_{X_t', X_t} = \mathfrak{R}_1 + \mathfrak{R}_2$$

$$:= \int_0^t \langle \partial_t (u - u_{h\tau}), W\rangle(s)\mathrm{d}s + \int_0^t (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla W)(s)\mathrm{d}s$$

- definition of W gives for $\mathfrak{R}_1$:

$$\int_0^t (\partial_t \nabla W, \nabla W)(s)\mathrm{d}s = \frac{1}{2}\left(\|\nabla W(\cdot, t)\|^2_{L^2(\Omega)} - \|\nabla W(\cdot, 0)\|^2_{L^2(\Omega)}\right)$$

$$= \frac{1}{2}\left(\|(u - u_{h\tau})(\cdot, t)\|^2_{H^{-1}(\Omega)} - \|u_0 - u_{h\tau}(\cdot, 0)\|^2_{H^{-1}(\Omega)}\right)$$

- definition of W and Lipschitz continuity of β gives for $\mathfrak{R}_2$:

$$\int_0^t (u - u_{h\tau}, \beta(u) - \beta(u_{h\tau}))(s)\mathrm{d}s$$

$$\geq \frac{1}{L_\beta}\int_0^t (\beta(u) - \beta(u_{h\tau}), \beta(u) - \beta(u_{h\tau}))(s)\mathrm{d}s = \frac{1}{L_\beta}\|\beta(u) - \beta(u_{h\tau})\|^2_{Q_t}$$

As tight as possible employment of Gronwall lemma

Gronwall lemma

$$\xi(t) \le \alpha(t) + \int_0^t \xi(s)\mathrm{d}s \Longrightarrow \xi(t) \le \alpha(t) + \int_0^t \alpha(s) e^{t-s} \mathrm{d}s$$

Lemma (As tight as possible employment of Gronwall lemma)

Additionally, there holds

$$\begin{aligned} & \|u - u_{h\tau}\|_{X'}^2 \\ \le & (e^T - 1)\|(u - u_{h\tau})(\cdot, 0)\|_{H^{-1}(\Omega)}^2 \\ & + \int_0^T \left(\|\mathcal{R}(u_{h\tau})\|_{X_t'}^2 + \int_0^t \|\mathcal{R}(u_{h\tau})\|_{X_s'}^2 e^{t-s}\,\mathrm{d}s \right) \mathrm{d}t \\ & - \frac{2}{L_\beta} \int_0^T \left(\|\beta(u) - \beta(u_{h\tau})\|_{Q_t}^2 + \int_0^t \|\beta(u) - \beta(u_{h\tau})\|_{Q_s}^2 e^{t-s}\,\mathrm{d}s \right) \mathrm{d}t. \end{aligned}$$

As tight as possible employment of Gronwall lemma

Gronwall lemma

$$\xi(t) \leq \alpha(t) + \int_0^t \xi(s)\mathrm{d}s \Longrightarrow \xi(t) \leq \alpha(t) + \int_0^t \alpha(s)e^{t-s}\mathrm{d}s$$

Lemma (As tight as possible employment of Gronwall lemma)

Additionally, there holds

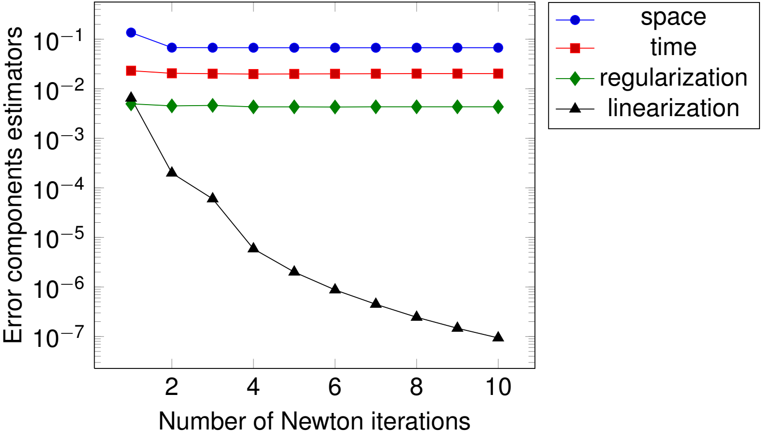
$$\begin{aligned}
&\|u - u_{h\tau}\|_{X'}^2 \\
\leq &(e^T - 1)\|(u - u_{h\tau})(\cdot, 0)\|_{H^{-1}(\Omega)}^2 \\
&+ \int_0^T \left(\|\mathcal{R}(u_{h\tau})\|_{X_t'}^2 + \int_0^t \|\mathcal{R}(u_{h\tau})\|_{X_s'}^2 e^{t-s}\,\mathrm{d}s \right)\,\mathrm{d}t \\
&- \frac{2}{L_\beta}\int_0^T \left(\|\beta(u) - \beta(u_{h\tau})\|_{Q_t}^2 + \int_0^t \|\beta(u) - \beta(u_{h\tau})\|_{Q_s}^2 e^{t-s}\,\mathrm{d}s \right)\mathrm{d}t.
\end{aligned}$$

Outline

Linearization stopping criterion

Linearization stopping criterion

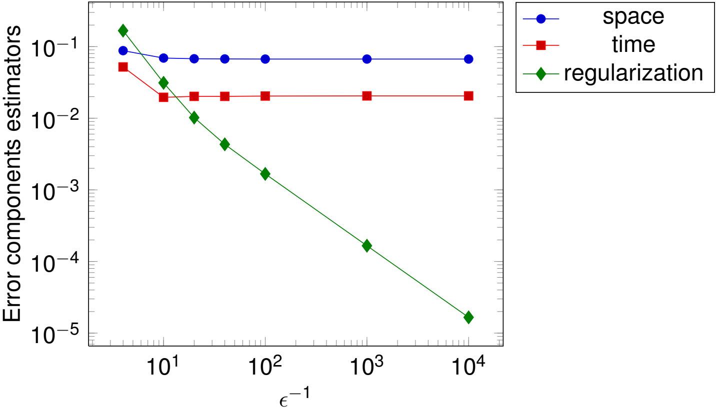
$$\eta_{\mathrm{lin}}^{n,\epsilon,k} \le \Gamma_{\mathrm{lin}} \left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k} \right)$$



Regularization stopping criterion

Regularization stopping criterion

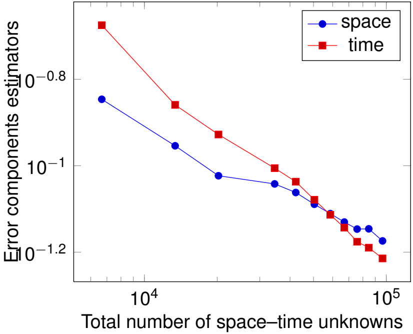
$$\eta_{\rm reg}^{n,\epsilon,k_n} \le \Gamma_{\rm reg}\left(\eta_{\rm sp}^{n,\epsilon,k_n} + \eta_{\rm tm}^{n,\epsilon,k_n}\right)$$

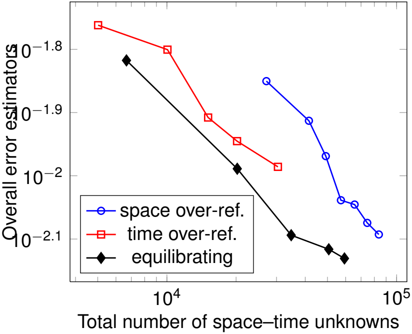


Equilibrating time and space errors

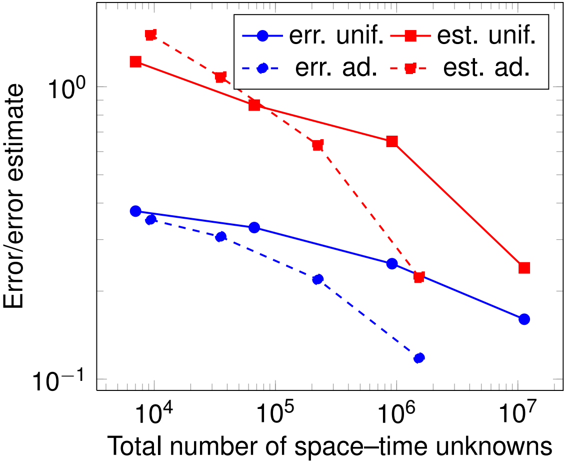
Equilibrating time and space errors

$$\gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n} \le \eta_{\rm tm}^{n,\epsilon_n,k_n} \le \Gamma_{\rm tm}\eta_{\rm sp}^{n,\epsilon_n,k_n}$$

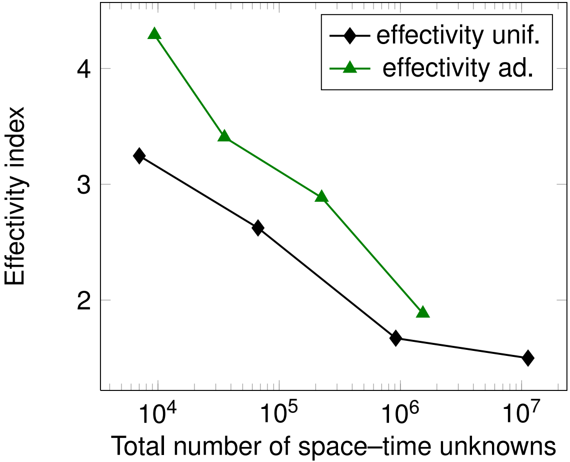




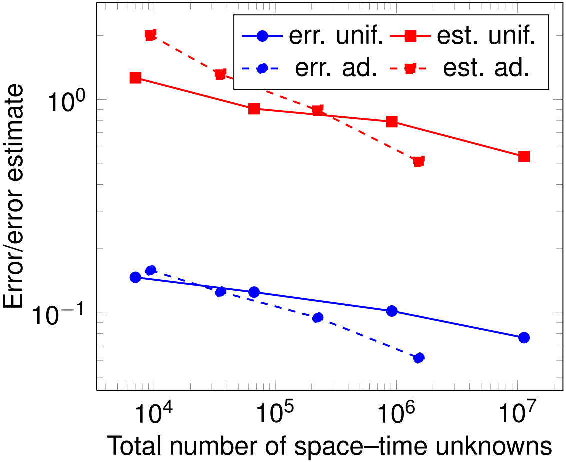
Error and estimate (dual norm of the residual)



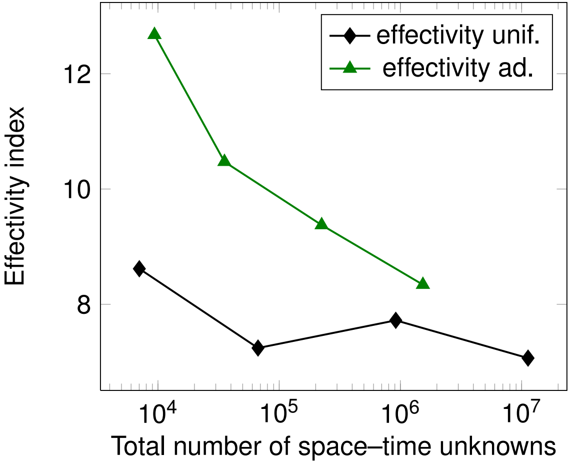
Effectivity indices (dual norm of the residual)



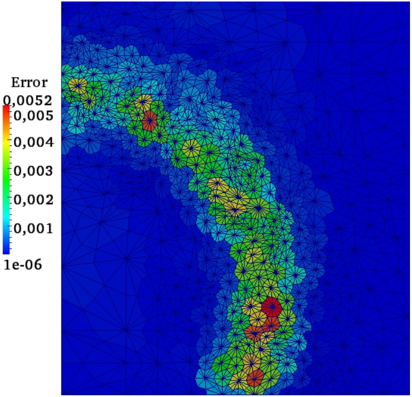
Error and estimate (energy norm)

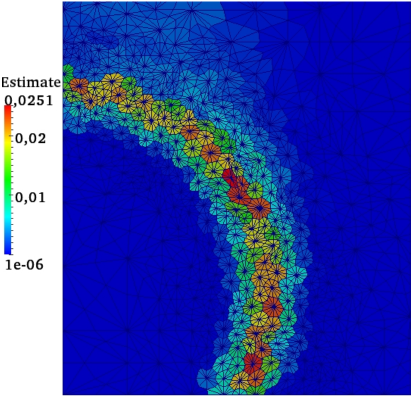


Effectivity indices (energy norm)



Actual and estimated error distribution





Computational efficiency

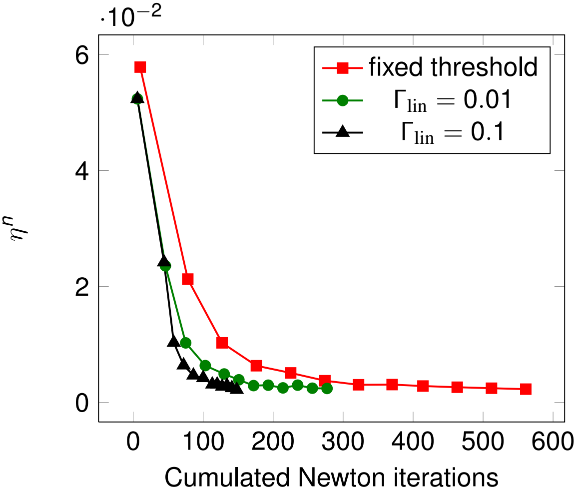


Figure: Number of cumulated Newton iterations vs. error estimate

Outline

Conclusions

Complete adaptivity

- only a **necessary number** of **linearization iterations**
- **optimal** choice of the **regularization parameter**
- **space-time** mesh **adaptivity**
- **"smart online decisions"**: linearization step / regularization / time step refinement / space mesh refinement
- important **computational savings**
- guaranteed upper bound via **a posteriori error estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

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Bibliography

- DI PIETRO D. A., VOHRALÍK M., YOUSEF S., Adaptive regularization, linearization, and discretization and a posteriori error control for the two-phase Stefan problem, HAL Preprint 00690862.

Thank you for your attention!