Potential and flux reconstructions for optimal *a priori* and *a posteriori* error estimates

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informatics mathematics

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Outline

1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$
5. A posteriori estimates
   - Guaranteed upper bound
   - Polynomial-degree-robust local efficiency
   - Applications and numerical results
6. Tools
7. Conclusions and outlook

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\rightarrow$ continuous pw pol. ▸ pot. rec.

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- discontinuous pw polynomial → continuous pw pol. ▸ pot. rec.
- *a posteriori* analysis of mixed and nonconforming FEs:

  est. ≈ error

- *a priori* analysis of conforming FEs:

  global-best–local-best equivalence in H^1 Veeser (2016)

  approximation continuous pw pols $\approx_p$ discontinuous pw pols

Equilibrated flux reconstruction

- pw vec.-valued pol. with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ▸ fl. rec.
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Definition (Construction of s_h EV (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\psi_{\boldsymbol{a}} \xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}} s_h^{\boldsymbol{a}}.$$

Equivalent form: conforming FEs

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathbb{P}_{p'}(\mathcal{T}) \cap H_0^1(\Omega)$
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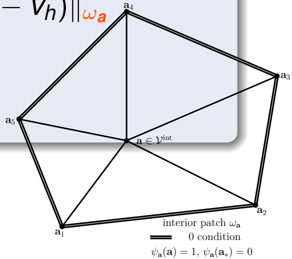

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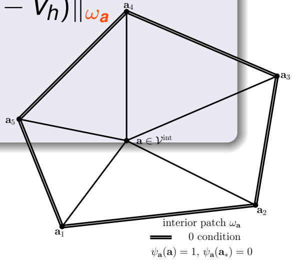

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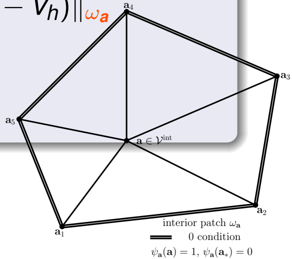

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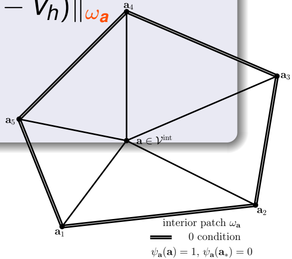

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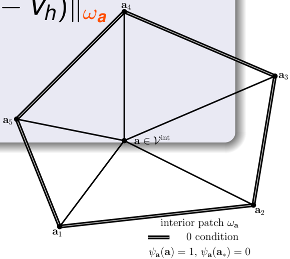

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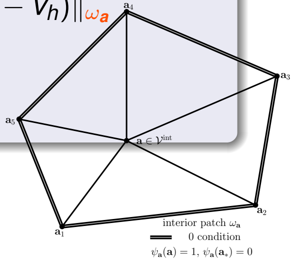

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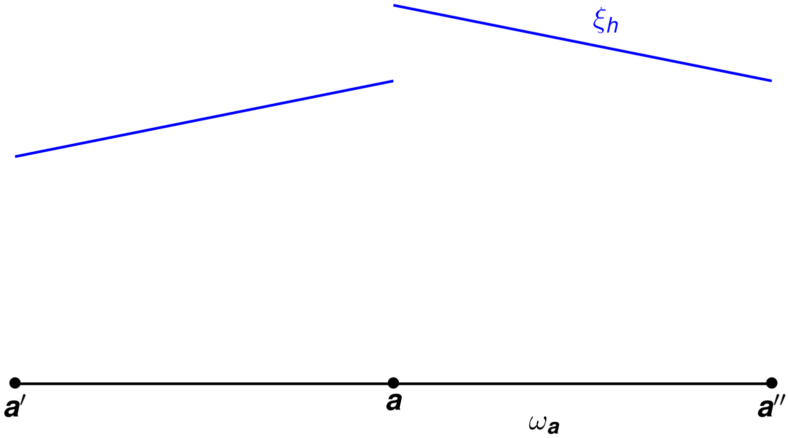
Potential reconstruction in 1D, $p = 1$, $p' = 2$



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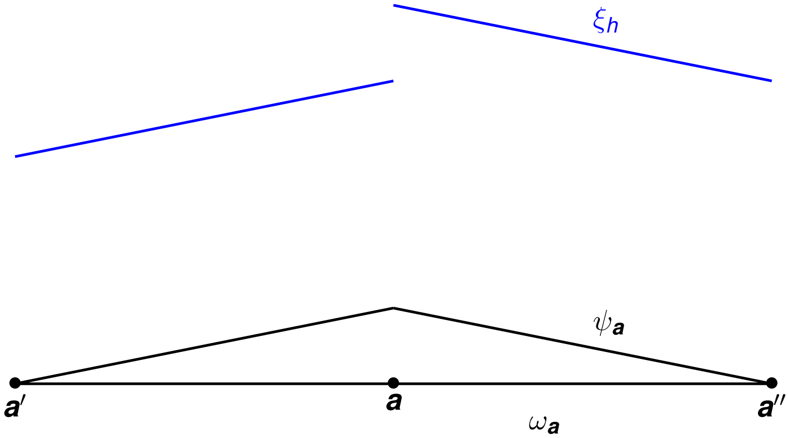

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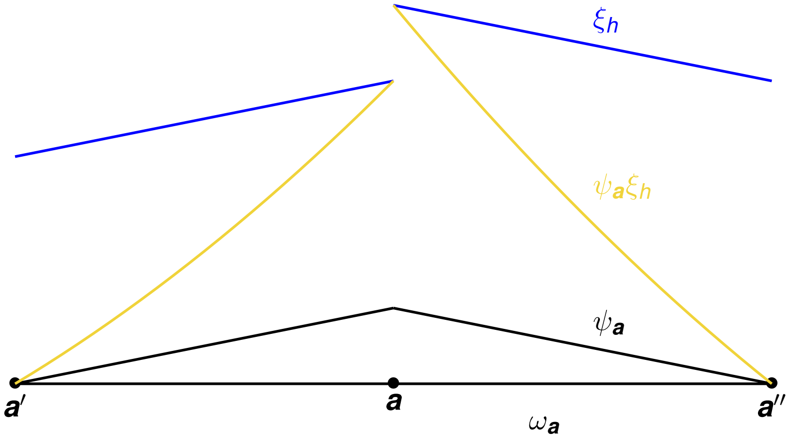

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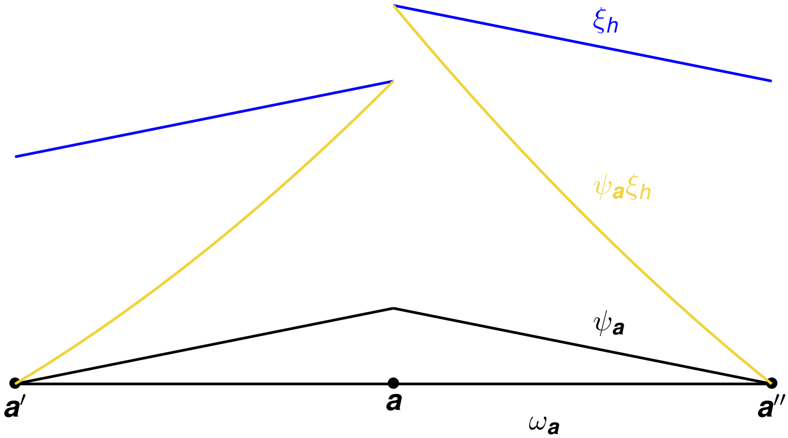

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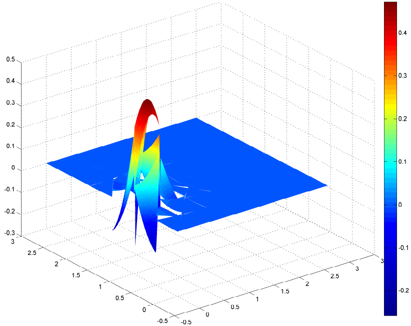
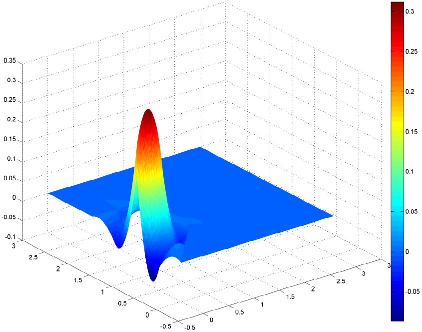


Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathbb{P}_p(\mathcal{T}) \to s_h \in \underbrace{\mathbb{P}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Stability of the potential reconstruction

Theorem (Local stability EV (2015, 2016), using ▸ Tools**)**

There holds

$$\min_{v_h \in \mathbb{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H_0^1(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

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Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

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Outline

1 Introduction
2 Potential reconstruction
3 Flux reconstruction
4 A priori estimates
- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
- Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

5 A posteriori estimates
- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools
7 Conclusions and outlook

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There holds

$$(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}}.$$

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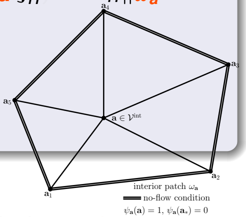

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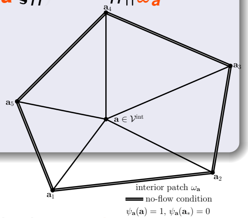

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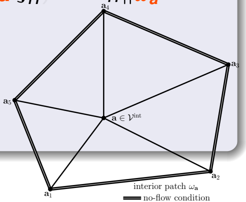

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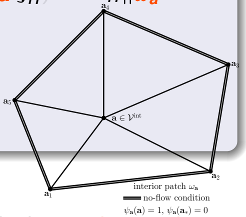

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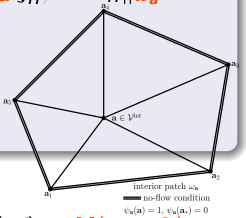

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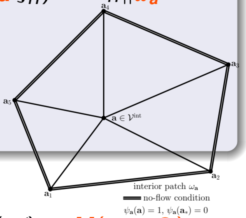

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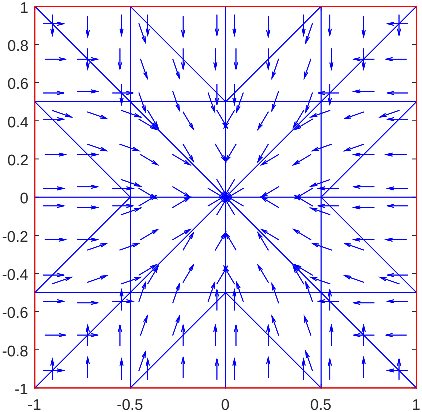
Equilibrated flux reconstruction

Equivalent form: mixed FEs

Find $(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \gamma_h^{\boldsymbol{a}}) \in \boldsymbol{V}_h^{\boldsymbol{a}} \times \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}})$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} - (\gamma_h^{\boldsymbol{a}}, \nabla{\cdot}\boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h), \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}}, \\
(\nabla{\cdot}\boldsymbol{\sigma}_h^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} &= (f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h{\cdot}\nabla\psi_{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} && \forall q_h \in \mathbb{P}_{p'}(\mathcal{T}_{\boldsymbol{a}})
\end{aligned}
$$

Equilibrated flux reconstruction



Flux ξ_h

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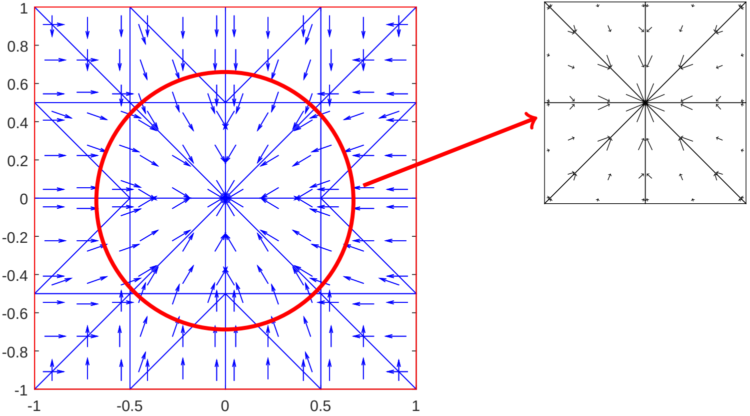

Flux ξ_h

$\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h$

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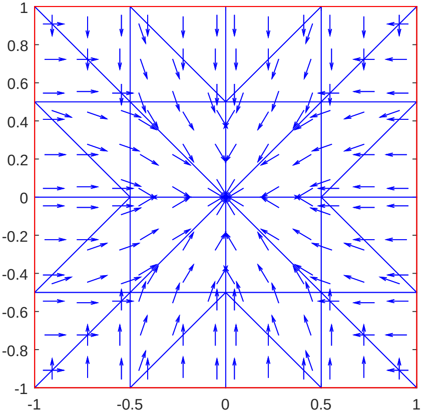

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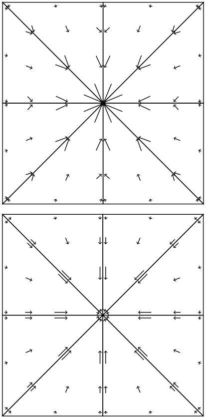

$\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h$

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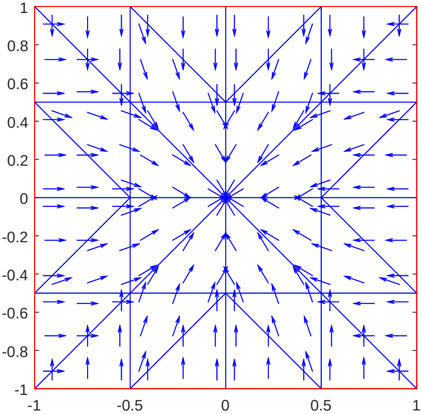
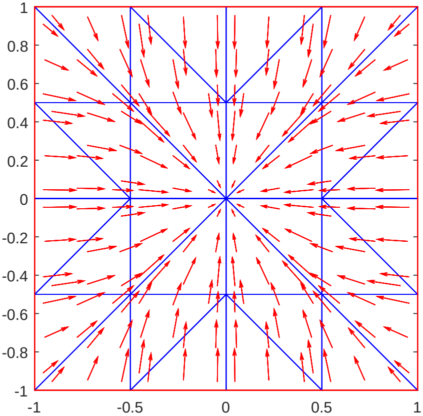


Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T}),\, f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\xi}_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}} \quad \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\boldsymbol{RTN}_{p'}(\mathcal{T})}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega),\ \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), EV (2016; 3D), using Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

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Corollary (Global stability; $p' = p+1$)

$\boldsymbol{\sigma}_h$ is closer to ξ_h than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ such that $\nabla\cdot\boldsymbol{\sigma} = f$:

$$\|\xi_h - \boldsymbol{\sigma}_h\| \lesssim \|\xi_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}$, and polynomial degree p
- proof taking $\xi_h|_K := \arg\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}$, applying with $p' = p$, and using its

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Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

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Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Primal weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Conforming finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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Corollary (Localized *a priori* error estimate)

$$\underbrace{\|\nabla(u-u_h)\|^2}_{\min_{v_h \in V_h} \|\nabla(u-v_h)\|^2} \lesssim_p \sum_{K\in\mathcal{T}} \min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u-v_h)\|_K^2$$

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From ..., *there holds*

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no interface constraints

regularity only in K counts

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From $H_0^1(\Omega)$ global – local*, there holds*

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local-best approximation of u on each K
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hp interpolantion/stable local commuting projectors

hp interpolantion estimates

- Demkowicz and Buffa (2005): log(p) factors
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})$
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})$ regularity but not commuting and only optimal in h
- Melenk and Rojik (2019): optimal *hp* approximation estimates (no log(p) factors) but higher regularity requested

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Falk and Winther (2014): local but not L^2-stable
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Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

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$$\approx_p \sum_{K\in\mathcal{T}} \left[\underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}\right].$$

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global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K\in\mathcal{T}} \left[\underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2} \right].$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

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global-best on Ω
normal trace-continuity constraint
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$$\approx_p \sum_{K\in\mathcal{T}} \Bigg[\underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2}_{\text{local-best on each } K} \Bigg].$$

local-best on each K
no normal trace-continuity constraint
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- proof using ▸ flux reconstruction with $p' = p$ & ▸ $\boldsymbol{H}(\mathrm{div})$ stability

Optimal *hp* approximation estimate

Theorem (Localized *hp* approximation, Ern, Gudi, Smears, & V. (2019))

For any $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on all* $K \in \mathcal{T}$,

$$\boldsymbol{v}|_K \in \boldsymbol{H}^s(K), \; s > 0,$$

there holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \left[\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K\in\mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{v}\|^2_{\boldsymbol{H}^s(K)} + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v}\|_K^2 & \text{if } s < 1, \\ \sum_{K\in\mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{v}\|^2_{\boldsymbol{H}^s(K)} & \text{if } s \geq 1. \end{cases}$$

- $\lesssim$: only depends on d, shape-regularity of $\mathcal{T}$, and s
- $\boldsymbol{H}(\mathrm{div})$ stability of flux reconstruction with $p' = p$ & $p' = p+1$
- contours known (quasi-)interpolates
- **fully optimal** *hp* approximation estimate (minimal elementwise regularity, no logarithmic factor in p)

Optimal *hp* approximation estimate

Theorem (Localized *hp* approximation, Ern, Gudi, Smears, & V. (2019))

For any $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on all* $K \in \mathcal{T}$,

$$\boldsymbol{v}|_K \in \boldsymbol{H}^s(K),\ s > 0,$$

there holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \left[\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2 \right]$$

$$\lesssim_s \begin{cases} \sum_{K\in\mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(K)}^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v}\|_K^2 & \text{if } s < 1, \\ \sum_{K\in\mathcal{T}} \frac{h_K^{2\min(s,p+1)}}{(p+1)^{2s}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(K)}^2 & \text{if } s \geq 1. \end{cases}$$

- $\lesssim$: only depends on d, shape-regularity of $\mathcal{T}$, and s
- $\boldsymbol{H}(\mathrm{div})$ stability of flux reconstruction with $p' = p$ & $p' = p + 1$
- contours known (quasi-)interpolates
- **fully optimal** *hp* approximation estimate (minimal elementwise regularity, no logarithmic factor in p)

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}, \boldsymbol{v}) - (u, \nabla\cdot\boldsymbol{v}) &= 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega),\\
(\nabla\cdot\boldsymbol{\sigma}, q) &= (f, q) \quad \forall q \in L^2(\Omega)
\end{aligned}
$$

Mixed finite elements

Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) - (u_h, \nabla\cdot\boldsymbol{v}_h) &= 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,\\
(\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \quad \forall q_h \in \mathbb{P}_p(\mathcal{T})
\end{aligned}
$$

Theorem (Optimal *hp a priori* error estimate, Ern, Gudi, Smears, & V. (2019)**)**

From ..., there holds

$$
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h\\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \ldots
$$

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Dual mixed weak formulation

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{v}) - (u, \nabla\cdot\boldsymbol{v}) &= 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega), \\ (\nabla\cdot\boldsymbol{\sigma}, q) &= (f, q) \quad \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

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From , *there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \lesssim_s \frac{h^{\min(s,p+1)}}{(p+1)^s}.$$

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Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div},\Omega) \times L^2(\Omega)$ such that

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Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

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Theorem (Optimal *hp a priori* error estimate, Ern, Gudi, Smears, & V. (2019)**)**

From $\boldsymbol{H}(\mathrm{div},\Omega)$ *hp* approx.*, there holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h\\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \lesssim_s \frac{h^{\min(s,p+1)}}{(p+1)^s}.$$

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Mixed finite elements

Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

$$\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{v}_h) - (u_h, \nabla\cdot\boldsymbol{v}_h) &= 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ (\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \quad \forall q_h \in \mathbb{P}_p(\mathcal{T}) \end{aligned}$$

Theorem (Optimal *hp a priori* error estimate, Ern, Gudi, Smears, & V. (2019)**)**

From ‣ $\boldsymbol{H}(\mathrm{div}, \Omega)$ *hp* approx.*, there holds*

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Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{v} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) =$ ▸ flux reconstruction *of*

$$\boldsymbol{\xi}_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K),\, \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 \quad \text{for all } K \in \mathcal{T}$$

with $p' = p$ *is locally defined,*

$\nabla\cdot(P_p\boldsymbol{v}) = \Pi_p(\nabla\cdot\boldsymbol{v})$ *commuting,*

$P_p\boldsymbol{v} = \boldsymbol{v}$ *if* $\boldsymbol{v} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ *projector,*

Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019)**)**

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p \boldsymbol{v} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) =$ ▸ flux reconstruction *of*

$\xi_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K), \nabla\cdot \boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$ for all $K \in \mathcal{T}$ with $p' = p$ is locally defined,

$\nabla\cdot(P_p \boldsymbol{v}) = \Pi_p(\nabla\cdot\boldsymbol{v})$ *commuting,*

$P_p \boldsymbol{v} = \boldsymbol{v}$ if $\boldsymbol{v} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ projector,

$$\|P_p \boldsymbol{v}\| \lesssim_p \|\boldsymbol{v}\| + \left\{ \sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2 \right\}^{1/2}$$ *stable up to osc.*

Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p \boldsymbol{v} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) =$ flux reconstruction *of*

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with $p' = p$ *is locally defined,*

$$\nabla\cdot(P_p \boldsymbol{v}) = \Pi_p(\nabla\cdot\boldsymbol{v}) \qquad \textit{commuting},$$

$$P_p \boldsymbol{v} = \boldsymbol{v} \text{ if } \boldsymbol{v} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \qquad \textit{projector},$$

$$\|P_p \boldsymbol{v}\| \lesssim_p \|\boldsymbol{v}\| + \left\{ \sum_{K \in \mathcal{T}} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2 \right\}^{1/2} \qquad \textit{stable up to osc.}$$

Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019)**)**

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p\boldsymbol{v} := \boldsymbol{\sigma}_h$ $\in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) =$ ▸ flux reconstruction of $\boldsymbol{\xi}_h|_K := \arg\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K),\, \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$ for all $K \in \mathcal{T}$ with $p' = p$ is locally defined,

$\nabla\cdot(P_p\boldsymbol{v}) = \Pi_p(\nabla\cdot\boldsymbol{v})$ *commuting*,

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$$\|P_p\boldsymbol{v}\| \lesssim_p \|\boldsymbol{v}\| + \left\{\sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2\right\}^{1/2}$$ *stable up to osc.*

Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

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Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then, $P_p\boldsymbol{v} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) =$ flux reconstruction *of*

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$$\|P_p\boldsymbol{v}\| \lesssim_p \|\boldsymbol{v}\| + \left\{\sum_{K\in\mathcal{T}} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K^2\right\}^{1/2} \textit{stable up to osc.}$$

Comments

- P_p defined on the entire $\boldsymbol{H}(\mathrm{div},\Omega)$ (no regularity)
- $\lesssim_p$: only depends on d, shape-regularity of $\mathcal{T}$, and p
- $h_K\|\nabla\cdot\boldsymbol{v} - \Pi_p(\nabla\cdot\boldsymbol{v})\|_K/(p+1)$: data oscillation term, disappears when $\nabla\cdot\boldsymbol{v}$ is a pw p-degree polynomial

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Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), . . . **)**

- *Let* $u \in H^1_0(\Omega)$ *be the weak solution;*
- $u_h \in \mathbb{P}_p(\mathcal{T})$, $p \geq 1$, *be arbitrary subject to*

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}^{\mathrm{int}};$$

- $\xi_h := u_h$: $s_h \in \mathbb{P}_{p+1}(\mathcal{T}) \cap H^1_0(\Omega)$ potential reconstruction;
- $\xi_h := -\nabla_h u_h, f$: $\boldsymbol{\sigma}_h \in \mathbf{RTN}_p(\mathcal{T}) \cap \mathbf{H}(\mathrm{div}, \Omega)$ flux reconstruction.

Then

$$\|\nabla_h (u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}} \Big(\underbrace{\|\nabla_h u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \Big)^2 + \sum_{K \in \mathcal{T}} \underbrace{\|\nabla_h (u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (A guaranteed *a posteriori* error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), …)

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity Braess, Pillwein, and Schöberl (2009), EV (2015, 2016))

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F\in\mathcal{F}} h_F^{-1} \|\Pi_0^F [\![u_h]\!]\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \boldsymbol{\sigma}_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of ▸ H^1 stability and ▸ $H(\mathrm{div})$ stability with $p' = p + 1$
- p-robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathbb{P}_{p-1}(\mathcal{T})$ for simplicity Braess, Pillwein, and Schöberl (2009), EV (2015, 2016)**)**

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

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$$\|\nabla_h u_h + \boldsymbol{\sigma}_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of ▸ H^1 stability and ▸ $\boldsymbol{H}(\mathrm{div})$ stability with $p' = p + 1$
- p-robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

Outline

1 Introduction

2 Potential reconstruction

3 Flux reconstruction

4 A priori estimates
- Global-best – local-best equivalence in H^1
- Constrained global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
- Stable commuting local projector in $\boldsymbol{H}(\mathrm{div})$

5 A posteriori estimates
- Guaranteed upper bound
- Polynomial-degree-robust local efficiency
- Applications and numerical results

6 Tools

7 Conclusions and outlook

Applications

Discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ mixed finite elements

Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method: $u_h \notin H_0^1(\Omega)$
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Numerics: smooth case with localized features

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega := (-1,1)^2, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(x,y) = (x^2-1)(y^2-1)\exp\left(-100(x^2+y^2)\right)$$

Discretization

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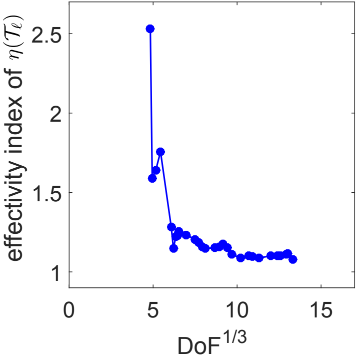
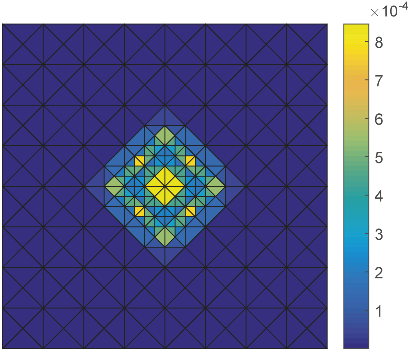
How precise are the estimates?



Effectivity indices on *hp* meshes

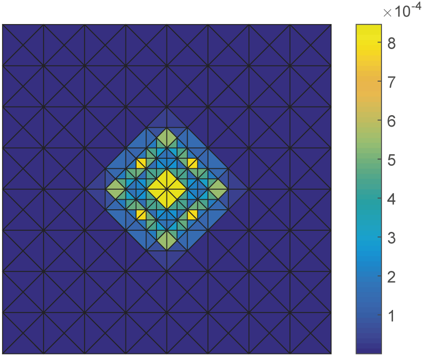
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Where (in space) is the error localized?



Estimated error distribution

$\eta_K(u_h)$



Exact error distribution

$\|\nabla(u - u_h)\|_K$

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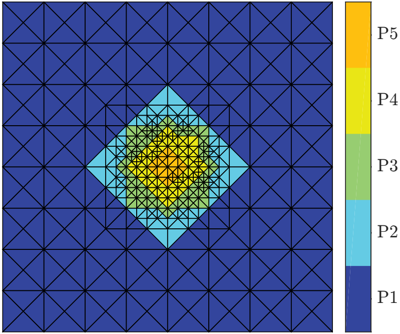
Can we decrease the error efficiently?



Mesh $\mathcal{T}$ and pol. degrees p_K

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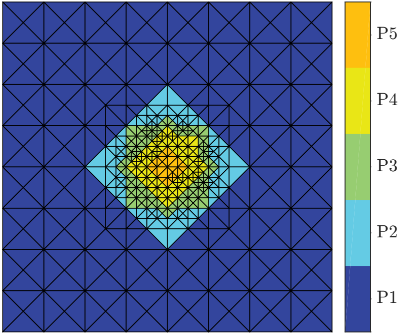

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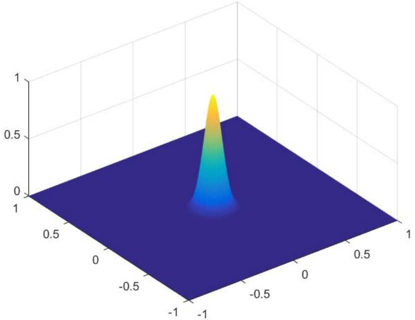

Exact solution

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Numerics: singular case

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- conforming finite elements: $u_h \in H^1(\Omega)$
- unstructured nested triangular grids
- *hp*-adaptive refinement

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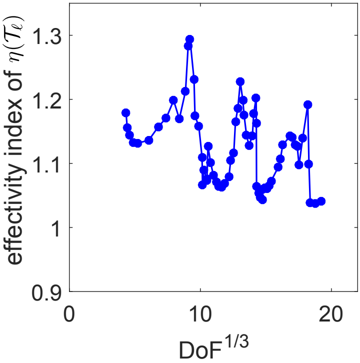
How precise are the estimates?



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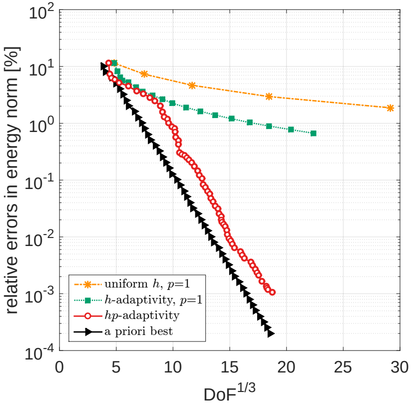

Relative error as a function of no. of unknowns

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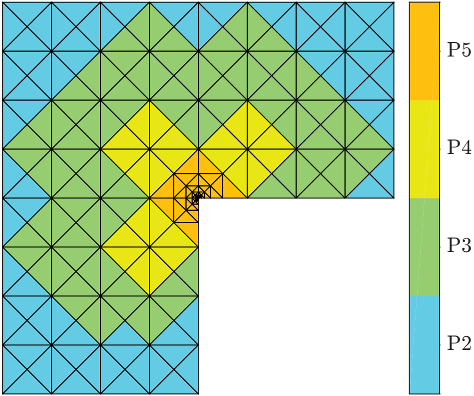
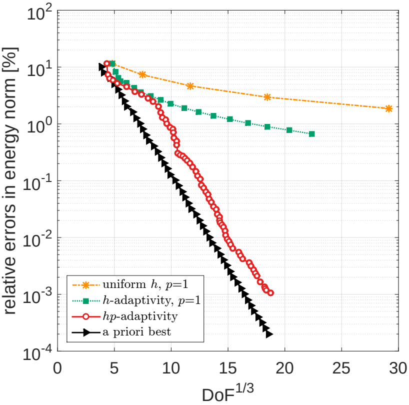


Mesh $\mathcal{T}$ and polynomial degrees p_K



Relative error as a function of no. of unknowns

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials: patch

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2016))

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility

$$
\begin{aligned}
r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}, \\
\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.
\end{aligned}
$$

Then

$$
\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}}) \\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.
$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

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$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

Context

$$-\Delta \zeta_K = r_K \quad \text{in } K,$$

$$-\nabla \zeta_K \cdot \boldsymbol{n}_K = r_F \quad \text{on all } F \in \mathcal{F}_K^{\mathrm{N}},$$

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Set $\varphi_K := -\nabla \zeta_K$.

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For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

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- optimal *hp* *a priori* error estimates
- *p*-robust *a posteriori* error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, and others carried out

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Thank you for your attention!