Polynomial-degree-robust a posteriori error estimates in a unified setting and applications

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in collaboration with

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Inria Paris & Ecole des Ponts

Linz, June 19, 2017

Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
   - Applications & numerics
4. Nonlinear Laplace: using adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

Optimal a posteriori error estimate

Guaranteed upper bound

- $\|u - u_h\|_{?,\Omega}^2 \le \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K(u_h) \le C_{\mathrm{eff}} \|u - u_h\|_{?,\text{neighbors of } K}$
- **local** error lower bound (optimal space mesh refinement)

Robustness

- C_{eff} independent of data, domain Ω, meshes, solution u, **polynomial degree** of u_h

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 / \|u - u_h\|_{?,\Omega}^2 \searrow 1$
- overestimation factor goes to one with meshes size

Small evaluation cost

- estimators $\eta_K(u_h)$ can be evaluated cheaply (locally)

Error components identification

- $\eta_K(u_h)$ can distinguish the different error components

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Laplace model problem

Model problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (primal variable constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (dual variable constraint)

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Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), . . . **)**

- *Let* $u \in H^1_0(\Omega)$ *be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$ *be arbitrary*
- $s_h \in H^1_0(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla \cdot \boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} \Bigg(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Bigg)^2$$

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Proof I

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\| = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s, definition of u:

$$\|\nabla(u - s)\| = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\| = 1}}_{\text{dual norm of the residual}}$$

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Proof (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \leq \|\nabla(u_h - s_h)\|$$

- adding and subtracting equilibrated flux, Green theorem:

$$(f, \varphi) - (\nabla u_h, \nabla \varphi) = (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla \varphi)$$

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$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) = -\sum_{K \in \mathcal{T}_h} (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)_K$$

$$\begin{aligned}(f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) &= \sum_{K \in \mathcal{T}_h} (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi - \varphi_K)_K \\ &\leq \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla{\cdot}\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K\end{aligned}$$

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Proof (continuation).

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Global potential and flux reconstructions

Ideally

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

- ✓ computable, discrete spaces $V_h \subset H^1_0(\Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$
- ✗ too expensive, **global** minimization problems (the hypercircle method . . .)

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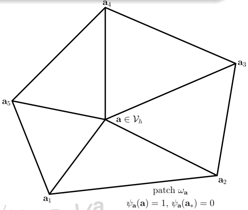
Local potential reconstruction

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013), EV (2015))

For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}.$$



Equivalent form

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

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Key ideas

- **local** minimizations
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- $V_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})$: homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}}$
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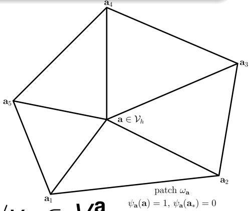

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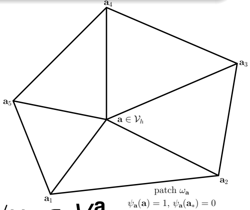

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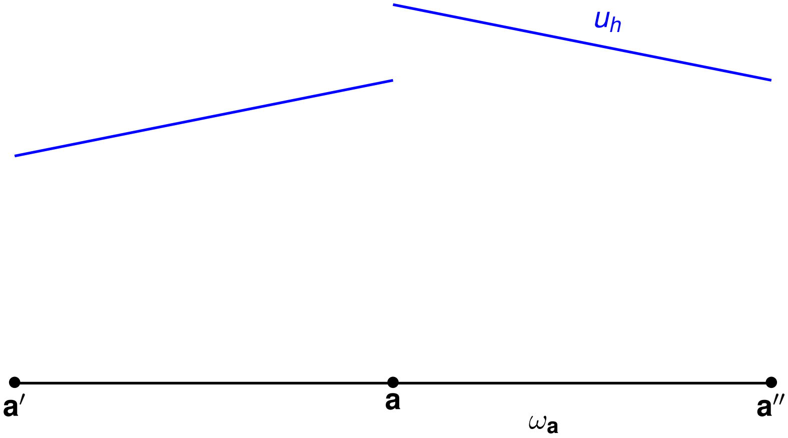
Potential reconstruction in 1D



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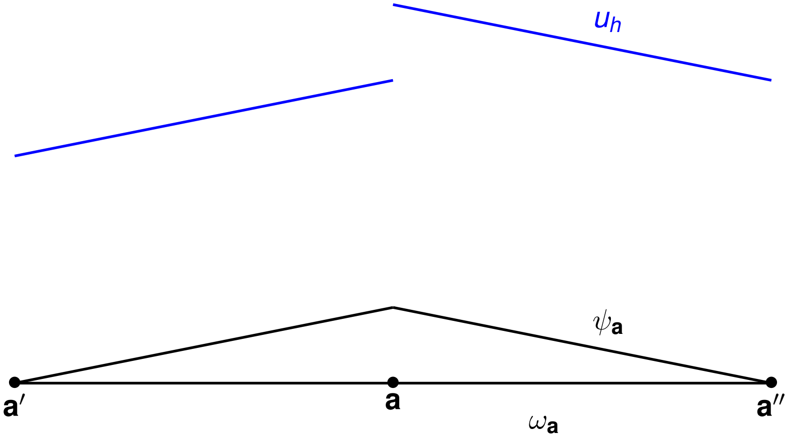

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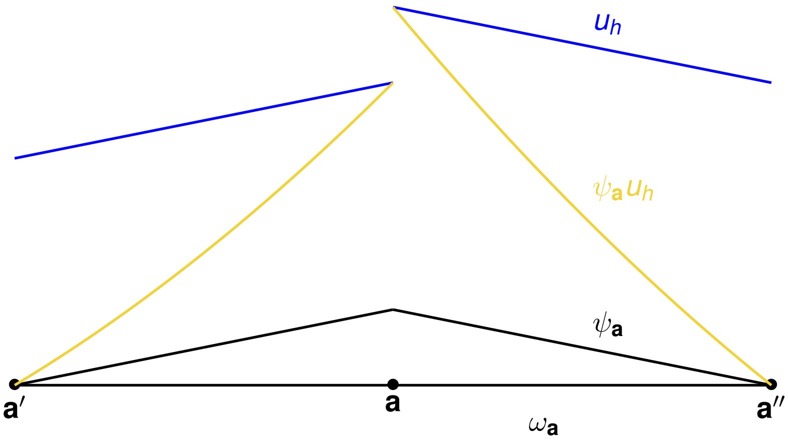

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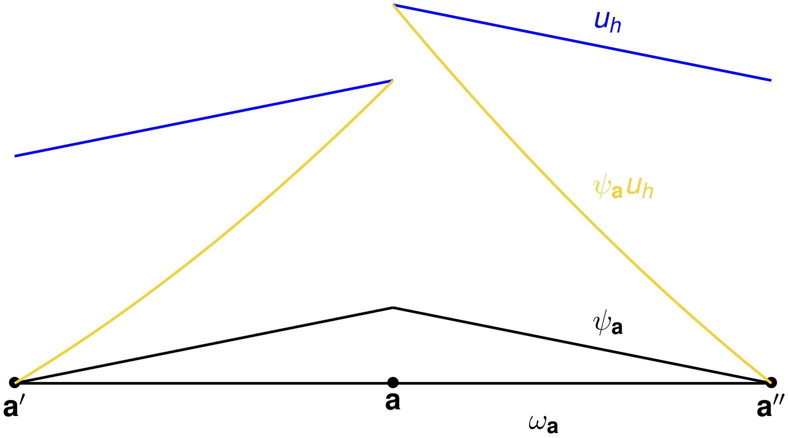
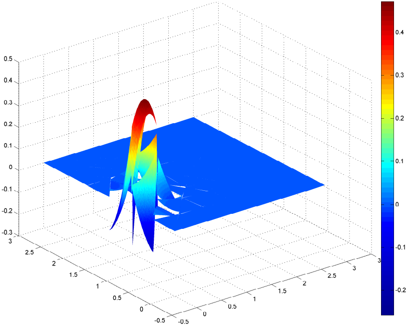


Potential reconstruction in 2D



Potential u_h

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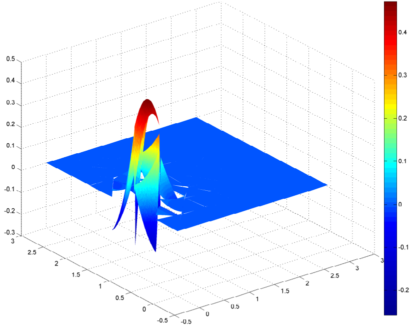

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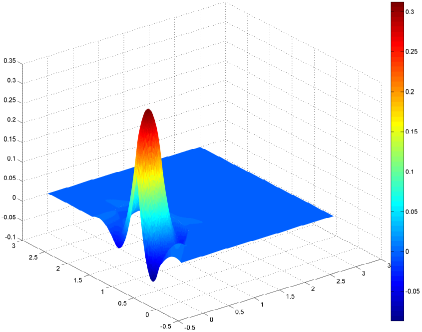

Potential reconstruction s_h

Local flux reconstructions

Assumption A (Galerkin orthogonality wrt hat functions)

There holds

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

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Key points

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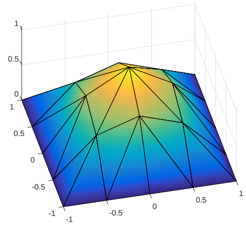

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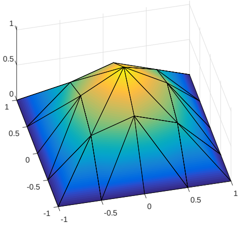
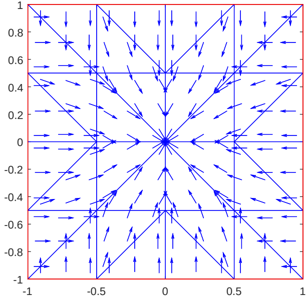
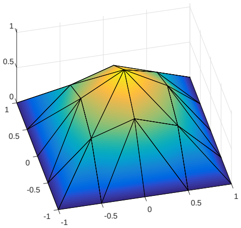
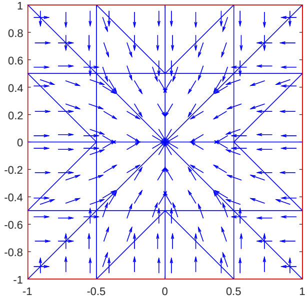




Flux $-\nabla u_h$

Equilibrated flux reconstruction





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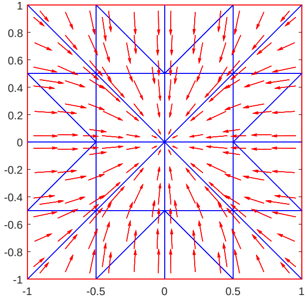

Flux reconstruction $\boldsymbol{\sigma}_h$

Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
   - Applications & numerics
4. Nonlinear Laplace: using adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

Polynomial-degree-robust efficiency

Assumption B (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The degrees of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.

Theorem (Polynomial-degree-robust efficiency Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015)**)**

Let u be the weak solution and let Assumptions A and B hold. Then there exists constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

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Remarks

- equivalence error–estimate
- maximal overestimation factor guaranteed

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Existing results

Fundamental results on a reference tetrahedron

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- Demkowicz, Gopalakrishnan, Schöberl (2009, 2012): polynomial extensions in H^1 and $\mathbf{H}(\mathrm{div})$ for polynomial boundary data

Stable broken $\mathbf{H}(\mathrm{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
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Stable broken H^1 polynomial extensions on a patch

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Potentials (any BCs, physical tetrahedron)

Lemma (H^1 polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{D}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{D}})$ be continuous on $\mathcal{E}_K^{\mathrm{D}}$. Then for C only depending on the shape regularity of K,

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_e && \text{on all } e \in \mathcal{E}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } e \in \mathcal{E}_K \setminus \mathcal{E}_K^{\mathrm{D}}. \end{aligned}$$

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Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{D}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{D}})$ be continuous on $\mathcal{E}_K^{\mathrm{D}}$. Then for C only depending on the shape regularity of K,

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \leq C \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = C\|\nabla \zeta_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_e && \text{on all } e \in \mathcal{E}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } e \in \mathcal{E}_K \setminus \mathcal{E}_K^{\mathrm{D}}. \end{aligned}$$

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Lemma ($\mathbf{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

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$$\min_{\substack{\mathbf{v}_h \in \mathbf{RTN}_p(K) \\ \mathbf{v}_h \cdot \mathbf{n}_K = r_e \ \forall e \in \mathcal{E}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v}_h = r_K}} \|\mathbf{v}_h\|_K \leq C \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, K) \\ \mathbf{v} \cdot \mathbf{n}_K = r_e \ \forall e \in \mathcal{E}_K^{\mathrm{N}} \\ \nabla \cdot \mathbf{v} = r_K}} \|\mathbf{v}\|_K \qquad .$$

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A graph result for patch enumerations in 3D (shellability of polytopes, e.g. Ziegler, Lectures on Polytopes)

Two families of faces

- already visited faces: $\mathcal{E}_i^\sharp := \{e \in \mathcal{E}_{\mathbf{a}}^{\mathrm{int}},\ e = \partial K_i \cap \partial K_j,\ j < i\}$
- yet unvisited faces: $\mathcal{E}_i^\flat := \mathcal{E}_{\mathbf{a}}^{\mathrm{int}} \cap \mathcal{E}_{K_i} \setminus \mathcal{E}_i^\sharp$
- $|\mathcal{E}_i^\flat| + |\mathcal{E}_i^\sharp| = 3$, $\mathcal{E}_1^\sharp = \emptyset$, and $\mathcal{E}^\flat_{|\mathcal{T}_{\mathbf{a}}|} = \emptyset$

Lemma (Interior patch enumeration)

There exists an enumeration of the patch $\mathcal{T}_{\mathbf{a}}$ so that

(i) *If $|\mathcal{E}_i^\sharp| \geq 2$ with $\{e_i^1, e_i^2\} \subset \mathcal{E}_i^\sharp$, then $K_j \in \mathcal{T}_{e_i^1 \cap e_i^2} \setminus \{K_i\}$ implies $j < i$.*

(ii) *For all $1 < i < |\mathcal{T}_{\mathbf{a}}|$, $|\mathcal{E}_i^\sharp| \in \{1, 2\}$.*

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$$r_e := \psi_{\mathbf{a}} [\![u_h]\!] |_e,$$

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Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
   - Applications & numerics
4. Nonlinear Laplace: using adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

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Find $u_h \in V_h$ such that

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Find $u_h \in V_h$ such that

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Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
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Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d}u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.01

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d}u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.01

Uniform refinement: **asymptotic exactness**

h	p	$\lVert\nabla_{\mathrm{d}}(u-u_h)\rVert$	$\lVert\nabla_{\mathrm{d}}u_h+\sigma_h\rVert$	η_{osc}	$\lVert\nabla_{\mathrm{d}}(u_h-s_h)\rVert$	η	I^{eff}
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.01

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\mathrm{d}}(u-u_h)\|$	$\|\nabla_{\mathrm{d}} u_h+\sigma_h\|$	η_{osc}	$\|\nabla_{\mathrm{d}}(u_h-s_h)\|$	η	I^{eff}
h_0	**1**	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	**1.17**
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	**1.09**
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	**1.06**
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	**1.04**
h_0	**2**	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	**1.06**
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	**1.04**
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	**1.03**
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	**1.03**
h_0	**3**	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	**1.03**
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	**1.01**
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	**1.01**
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	**1.01**
h_0	**4**	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	**1.02**
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	**1.01**
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	**1.01**
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	**1.01**
h_0	**5**	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	**1.02**
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	**1.01**
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	**1.00**
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	**1.00**
h_0	**6**	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	**1.02**
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	**1.01**
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	**1.01**

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|u-u_h\|_{\rm DG}$	$\|\nabla_{\rm d}u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$\eta_{\rm DG}$	$I^{\rm eff}$	$I^{\rm eff}_{\rm DG}$
h_0	1	1.07E-00	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
$\approx h_0/2$		5.56E-01	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
$\approx h_0/4$		2.92E-01	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
$\approx h_0/8$		1.39E-01	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
h_0	2	1.54E-01	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
$\approx h_0/2$		4.07E-02	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
$\approx h_0/4$		1.10E-02	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
$\approx h_0/8$		2.50E-03	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
h_0	3	1.37E-02	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
h_0	4	9.87E-04	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
$\approx h_0/2$		6.92E-05	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
$\approx h_0/8$		2.58E-07	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
h_0	5	5.64E-05	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
$\approx h_0/4$		7.74E-08	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
h_0	6	2.85E-06	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

Numerics: singular case

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

Numerics: singular case

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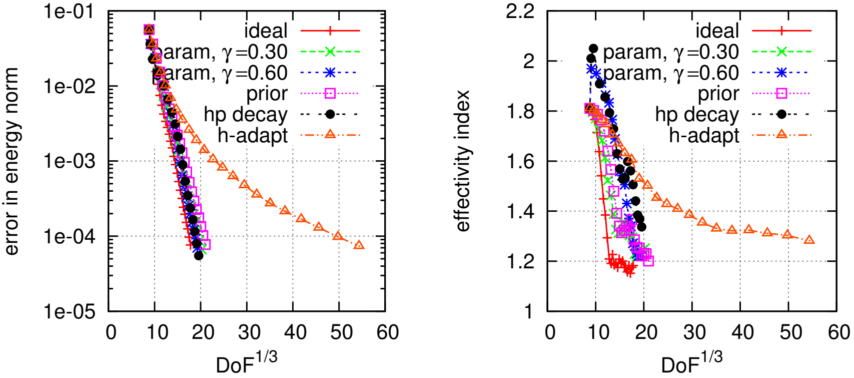
Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

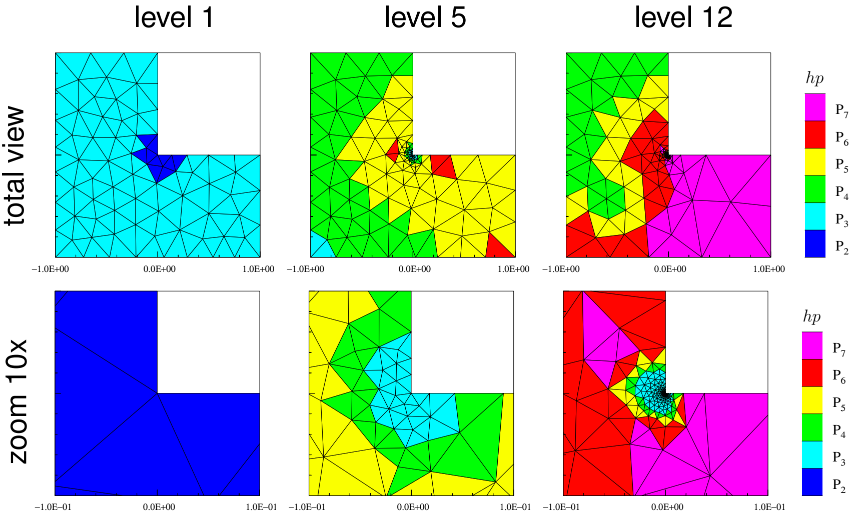
Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

hp-adaptive refinement: **exponential convergence**



hp-refinement grids



Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
   - Applications & numerics
4. Nonlinear Laplace: using adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

Setting

Laplace problem
Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation
Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Linear algebraic system
Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)
On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h^i = F_h - R_h^i \qquad (R_h^i := F_h - \mathbb{A}_h U_h^i)$$

Setting

Laplace problem

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

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Setting

Laplace problem

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Linear algebraic system

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)

On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h^i = F_h - R_h^i \qquad (R_h^i := F_h - \mathbb{A}_h U_h^i)$$

Setting

Laplace problem

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Linear algebraic system

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)

On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h^i = F_h - R_h^i \qquad (R_h^i := F_h - \mathbb{A}_h U_h^i)$$

Goals

Algebraic error

$$\|\nabla(u_h - u_h^i)\|$$

Total error

$$\|\nabla(u - u_h^i)\|$$

Discretization error

$$\|\nabla(u - u_h)\|$$

Goals: find **a posteriori estimates** for **any** $i \geq 1$

Algebraic error

$$\underline{\eta}^i_{\mathrm{alg}} \leq \|\nabla(u_h - u_h^i)\| \leq \eta^i_{\mathrm{alg}}$$

Total error

$$\underline{\eta}^i_{\mathrm{tot}} \leq \|\nabla(u - u_h^i)\| \leq \eta^i_{\mathrm{tot}}$$

Discretization error

$$\underline{\eta}^i_{\mathrm{dis}} \leq \|\nabla(u - u_h)\| \leq \eta^i_{\mathrm{dis}}$$

Goals: find **a posteriori estimates** for **any** $i \geq 1$

Algebraic error

$$\underline{\eta}_{\mathrm{alg}}^{i} \leq \|\nabla(u_h - u_h^i)\| \leq \eta_{\mathrm{alg}}^{i}$$

Total error

$$\underline{\eta}_{\mathrm{tot}}^{i} \leq \|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{tot}}^{i}$$

Discretization error

$$\underline{\eta}_{\mathrm{dis}}^{i} \leq \|\nabla(u - u_h)\| \leq \eta_{\mathrm{dis}}^{i}$$

Further goals

- estimate the **distribution** of the errors (local efficiency)
- design reliable (local) **stopping criteria**

The pathway

Algebraic residual representer

- $r_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ represents R_h^i
- gives equivalent form of residual equation: $u_h^i \in V_h$ s.t.

$$(\nabla u_h^i, \nabla \psi_l) = (f, \psi_l) - (r_h^i, \psi_l) \qquad \forall l = 1, \ldots, N$$

- $(r_h^i, \psi_l) = (R_h^i)_l$, $l = 1, \ldots, N$
- consequence of equations for u_h and u_h^i:

$$(\nabla(u_h - u_h^i), \nabla v_h) = (r_h^i, v_h) \qquad \forall v_h \in V_h$$

Tools

- flux and potential reconstructions
- local Neumann MFE & local Dirichlet FE problems
- separate components for algebraic & discretization errors
- multilevel hierarchy (algebraic components)

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Algebraic error upper bound

Theorem (Upper bound via algebraic error flux reconstruction)

Let $\boldsymbol{\sigma}^i_{h,\mathrm{alg}} \in \mathbf{H}(\mathrm{div},\Omega)$ *be such that* $\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$. *Then*

$$\underbrace{\|\nabla(u_h - u^i_h)\|}_{\text{algebraic error}} \leq \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{upper algebraic est.}} .$$

Proof.

$$\|\nabla(u_h - u^i_h)\| = \sup_{v_h\in V_h, \|\nabla v_h\|=1} (\nabla(u_h - u^i_h), \nabla v_h);$$

$$(\nabla(u_h - u^i_h), \nabla v_h) = (r^i_h, v_h) = (\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}}, v_h) = -(\boldsymbol{\sigma}^i_{h,\mathrm{alg}}, \nabla v_h).$$

Previous cheap constructions of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$

1. sequential sweep trough $\mathcal{T}_h$, local min. (JSV (2010))
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Definition (Coarse grid Riesz representer)

Find $\rho^i_{H,\mathrm{alg}} \in V_H := \mathbb{P}_1(\mathcal{T}_H) \cap H^1_0(\Omega)$ such that

$$(\nabla \rho^i_{H,\mathrm{alg}}, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (r^i_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_H$$

- $\mathbb{P}_1$ FEs on $\mathcal{T}_H$ (no need for multigrid w/o post-smoothing)
- gives hat function orthogonality on $\mathcal{T}_H$

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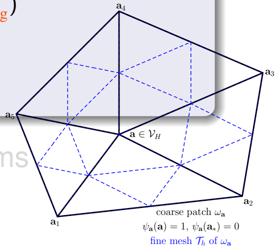

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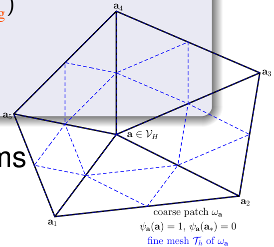

Divergence of the algebraic error flux reconstruction

Lemma (Divergence of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$)

There holds $\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$.

Proof.

- every fine grid element $K \in \mathcal{T}_h$ lies exactly in $(d+1)$ coarse patches $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_H$
- partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \psi^{\mathbf{a}} = 1|_K$
-

$$\begin{aligned}\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}}|_K &= \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \nabla\cdot\boldsymbol{\sigma}^{\mathbf{a},i}_{h,\mathrm{alg}}|_K \\ &= \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \Pi_{Q_h}(\psi_{\mathbf{a}} r^i_h - \nabla\psi_{\mathbf{a}}\cdot\nabla\rho^i_{H,\mathrm{alg}})|_K = r^i_h|_K\end{aligned}$$

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$$\begin{aligned}
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\end{aligned}$$

Algebraic error lower bound

Theorem (Lower bound via algebraic residual liftings)

Let $\rho^i_{h,\mathrm{alg}} \in V_h$ be arbitrary. Then

$$\underbrace{\|\nabla(u_h - u_h^i)\|}_{\text{algebraic error}} \geq \underbrace{\frac{(r_h^i, \rho^i_{h,\mathrm{alg}})}{\|\nabla \rho^i_{h,\mathrm{alg}}\|}}_{\text{lower algebraic est.}}.$$

Proof.

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\| = 1} (r_h^i, v_h) \geq \frac{(r_h^i, \rho^i_{h,\mathrm{alg}})}{\|\nabla \rho^i_{h,\mathrm{alg}}\|}.$$

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Algebraic residual lifting, two-level setting

Definition (Algebraic residual lifting, ≈ Bank & Smith (1993), Oswald (1993), Rüde (1993), …, Ern & V. (2015))

Find $\rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\omega_{\mathbf{a}})$ such that

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Set

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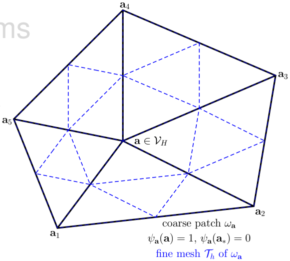

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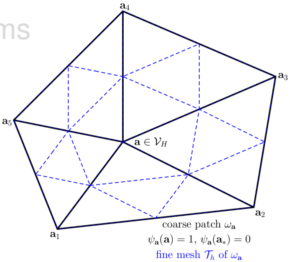

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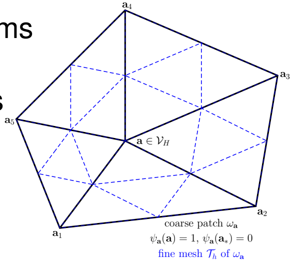

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Numerical illustration

Peak

$$\Omega = (0,1) \times (0,1),$$

$$u(x,y) = x(x-1)y(y-1)e^{-100(x-0.5)^2-100(y-117/1000)^2}$$

L-shape

$$\Omega = (-1,1) \times (-1,1) \setminus \ [0,1] \times [-1,0],$$

$$u(r,\theta) = r^{2/3} \sin(2\theta/3)$$

Discretization

- conforming finite elements, $p = 1, \ldots, 4$
- unstructured triangular meshes
- 4 uniform refinements

Multigrid

- geometric multigrid V-cycle
- 5 pre-smoothing steps of Gauss–Seidel

PCG

- incomplete Cholesky with drop-off tolerance 1e-4 prec.

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Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^3)	1	7.00×10^{-3}	1.15	1.28^{-1}	9.25×10^{-3}	1.62	1.16^{-1}	6.06×10^{-3}	2.31	—
	2	2.87×10^{-4}	1.14	1.32^{-1}	6.06×10^{-3}	1.10	1.05^{-1}		1.10	1.05^{-1}
2 (1.03×10^4)	1	7.81×10^{-3}	1.18	1.83^{-1}	7.82×10^{-3}	1.74	1.16^{-1}	3.87×10^{-4}	3.33×10^1	—
	2	4.04×10^{-4}	1.19	1.09^{-1}	5.59×10^{-4}	1.69	1.17^{-1}		2.25	—
	3	8.48×10^{-6}	1.19	1.07^{-1}	3.87×10^{-4}	1.05	1.03^{-1}		1.05	1.03^{-1}
3 (2.34×10^4)	1	4.49×10^{-3}	1.15	2.02^{-1}	4.49×10^{-3}	1.61	1.23^{-1}	1.89×10^{-5}	3.65×10^2	—
	2	3.22×10^{-4}	1.22	1.58^{-1}	3.23×10^{-4}	1.87	1.07^{-1}		2.99×10^1	—
	3	1.20×10^{-5}	1.17	1.19^{-1}	2.24×10^{-5}	1.53	1.29^{-1}		1.74	1.86^{-1}
	4	5.46×10^{-7}	1.16	1.12^{-1}	1.89×10^{-5}	1.05	1.09^{-1}		1.05	1.09^{-1}
4 (4.17×10^4)	1	5.56×10^{-3}	1.22	1.55^{-1}	5.56×10^{-3}	1.84	1.17^{-1}	8.15×10^{-7}	1.17×10^4	—
	3	5.87×10^{-5}	1.17	1.35^{-1}	5.87×10^{-5}	1.74	1.06^{-1}		1.13×10^2	—
	5	7.36×10^{-7}	1.15	1.18^{-1}	1.10×10^{-6}	1.57	1.26^{-1}		1.96	3.60^{-1}
	7	1.02×10^{-8}	1.14	1.12^{-1}	8.15×10^{-7}	1.03	1.14^{-1}		1.03	1.14^{-1}

Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^{3})	1	7.00×10^{-3}	1.15	1.28^{-1}	9.25×10^{-3}	1.62	1.16^{-1}	6.06×10^{-3}	2.31	—
	2	2.87×10^{-4}	1.14	1.32^{-1}	6.06×10^{-3}	1.10	1.05^{-1}		1.10	1.05^{-1}
2 (1.03×10^{4})	1	7.81×10^{-3}	1.18	1.83^{-1}	7.82×10^{-3}	1.74	1.16^{-1}	3.87×10^{-4}	3.33×10^{1}	—
	2	4.04×10^{-4}	1.19	1.09^{-1}	5.59×10^{-4}	1.69	1.17^{-1}		2.25	—
	3	8.48×10^{-6}	1.19	1.07^{-1}	3.87×10^{-4}	1.05	1.03^{-1}		1.05	1.03^{-1}
3 (2.34×10^{4})	1	4.49×10^{-3}	1.15	2.02^{-1}	4.49×10^{-3}	1.61	1.23^{-1}	1.89×10^{-5}	3.65×10^{2}	—
	2	3.22×10^{-4}	1.22	1.58^{-1}	3.23×10^{-4}	1.87	1.07^{-1}		2.99×10^{1}	—
	3	1.20×10^{-5}	1.17	1.19^{-1}	2.24×10^{-5}	1.53	1.29^{-1}		1.74	1.86^{-1}
	4	5.46×10^{-7}	1.16	1.12^{-1}	1.89×10^{-5}	1.05	1.09^{-1}		1.05	1.09^{-1}
4 (4.17×10^{4})	1	5.56×10^{-3}	1.22	1.55^{-1}	5.56×10^{-3}	1.84	1.17^{-1}	8.15×10^{-7}	1.17×10^{4}	—
	3	5.87×10^{-5}	1.17	1.35^{-1}	5.87×10^{-5}	1.74	1.06^{-1}		1.13×10^{2}	—
	5	7.36×10^{-7}	1.15	1.18^{-1}	1.10×10^{-6}	1.57	1.26^{-1}		1.96	3.60^{-1}
	7	1.02×10^{-8}	1.14	1.12^{-1}	8.15×10^{-7}	1.03	1.14^{-1}		1.03	1.14^{-1}

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	2	2.87×10^{-4}	1.14	1.32^{-1}	6.06×10^{-3}	1.10	1.05^{-1}		1.10	1.05^{-1}
2 (1.03×10^4)	1	7.81×10^{-3}	1.18	1.83^{-1}	7.82×10^{-3}	1.74	1.16^{-1}	3.87×10^{-4}	3.33×10^1	—
	2	4.04×10^{-4}	1.19	1.09^{-1}	5.59×10^{-4}	1.69	1.17^{-1}		2.25	—
	3	8.48×10^{-6}	1.19	1.07^{-1}	3.87×10^{-4}	1.05	1.03^{-1}		1.05	1.03^{-1}
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	2	2.87×10^{-4}	1.14	1.32^{-1}	6.06×10^{-3}	1.10	1.05^{-1}		1.10	1.05^{-1}
2 (1.03×10^4)	1	7.81×10^{-3}	1.18	1.83^{-1}	7.82×10^{-3}	1.74	1.16^{-1}	3.87×10^{-4}	3.33×10^1	—
	2	4.04×10^{-4}	1.19	1.09^{-1}	5.59×10^{-4}	1.69	1.17^{-1}		2.25	—
	3	8.48×10^{-6}	1.19	1.07^{-1}	3.87×10^{-4}	1.05	1.03^{-1}		1.05	1.03^{-1}
3 (2.34×10^4)	1	4.49×10^{-3}	1.15	2.02^{-1}	4.49×10^{-3}	1.61	1.23^{-1}	1.89×10^{-5}	3.65×10^2	—
	2	3.22×10^{-4}	1.22	1.58^{-1}	3.23×10^{-4}	1.87	1.07^{-1}		2.99×10^1	—
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	4	5.46×10^{-7}	1.16	1.12^{-1}	1.89×10^{-5}	1.05	1.09^{-1}		1.05	1.09^{-1}
4 (4.17×10^4)	1	5.56×10^{-3}	1.22	1.55^{-1}	5.56×10^{-3}	1.84	1.17^{-1}	8.15×10^{-7}	1.17×10^4	—
	3	5.87×10^{-5}	1.17	1.35^{-1}	5.87×10^{-5}	1.74	1.06^{-1}		1.13×10^2	—
	5	7.36×10^{-7}	1.15	1.18^{-1}	1.10×10^{-6}	1.57	1.26^{-1}		1.96	3.60^{-1}
	7	1.02×10^{-8}	1.14	1.12^{-1}	8.15×10^{-7}	1.03	1.14^{-1}		1.03	1.14^{-1}

L-shape problem, PCG

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.87×10^{-1}	1.25	1.06^{-1}	2.90×10^{-1}	1.38	6.15^{-1}	3.55×10^{-2}	8.23	—
	4	1.21×10^{-3}	1.24	1.04^{-1}	3.56×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.06×10^{-1}	1.14	1.08^{-1}	2.07×10^{-1}	1.26	6.03^{-1}	1.44×10^{-2}	1.23×10^1	—
	6	2.46×10^{-3}	1.18	1.12^{-1}	1.47×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.23×10^{-6}	1.17	1.09^{-1}	1.44×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.26	1.06	1.10^{-1}	1.26	1.10	10.8^{-1}	8.56×10^{-3}	9.00×10^1	—
	8	9.95×10^{-2}	1.10	1.27^{-1}	9.98×10^{-2}	1.24	6.02^{-1}		1.12×10^1	—
	12	1.25×10^{-2}	1.10	1.26^{-1}	1.51×10^{-2}	1.71	2.67^{-1}		2.79	—
	16	8.23×10^{-4}	1.10	1.26^{-1}	8.60×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.67×10^{-1}	1.24	1.38^{-1}	1.67×10^{-1}	1.42	3.35^{-1}	6.16×10^{-3}	3.29×10^1	—
	10	2.41×10^{-3}	1.22	1.29^{-1}	6.61×10^{-3}	1.78	1.83^{-1}		1.89	2.93^{-1}
	15	2.29×10^{-5}	1.27	1.41^{-1}	6.16×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}

L-shape problem, PCG

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.87×10^{-1}	1.25	1.06^{-1}	2.90×10^{-1}	1.38	6.15^{-1}	3.55×10^{-2}	8.23	—
	4	1.21×10^{-3}	1.24	1.04^{-1}	3.56×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.06×10^{-1}	1.14	1.08^{-1}	2.07×10^{-1}	1.26	6.03^{-1}	1.44×10^{-2}	1.23×10^1	—
	6	2.46×10^{-3}	1.18	1.12^{-1}	1.47×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.23×10^{-6}	1.17	1.09^{-1}	1.44×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.26	1.06	1.10^{-1}	1.26	1.10	10.8^{-1}	8.56×10^{-3}	9.00×10^1	—
	8	9.95×10^{-2}	1.10	1.27^{-1}	9.98×10^{-2}	1.24	6.02^{-1}		1.12×10^1	—
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	16	8.23×10^{-4}	1.10	1.26^{-1}	8.60×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.67×10^{-1}	1.24	1.38^{-1}	1.67×10^{-1}	1.42	3.35^{-1}	6.16×10^{-3}	3.29×10^1	—
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	15	2.29×10^{-5}	1.27	1.41^{-1}	6.16×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}

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	6	2.46×10^{-3}	1.18	1.12^{-1}	1.47×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
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	12	1.25×10^{-2}	1.10	1.26^{-1}	1.51×10^{-2}	1.71	2.67^{-1}		2.79	—
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	6	2.46×10^{-3}	1.18	1.12^{-1}	1.47×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.23×10^{-6}	1.17	1.09^{-1}	1.44×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
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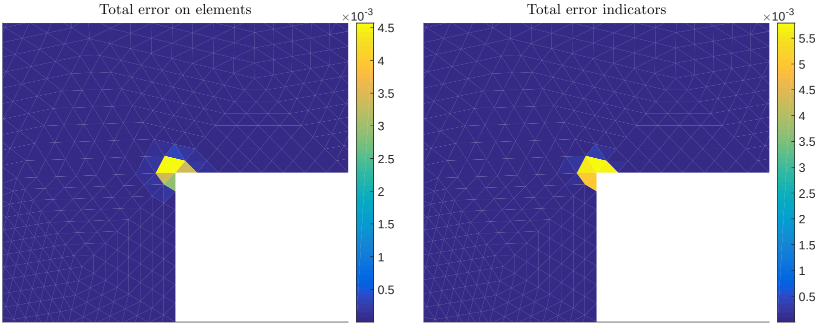
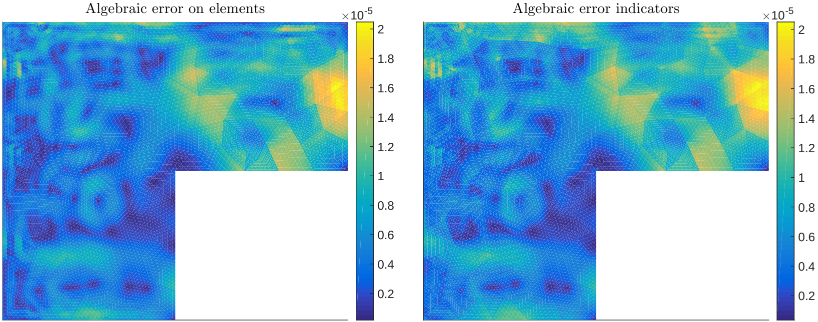
L-shape problem, $p = 3$, total error, 16th PCG iteration



L-shape problem, $p = 3$, alg. error, 16th PCG iteration



Outline

1. Introduction
2. Laplace equation: potential & flux reconstructions
   - Guaranteed upper bound in a unified framework
   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
   - Applications & numerics
4. Nonlinear Laplace: using adaptive stopping criteria
   - Adaptive inexact Newton method
   - Applications & numerical results
5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

Inexact iterative linearization

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*

   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do an algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k + 1$, back to 2.*

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   2. *Do an algebraic solver step* $\Rightarrow U^{k,i}$ *s.t.* ($R^{k,i}$ *algebraic res.*)
      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK* $\Rightarrow U^k := U^{k,i}$. *KO* $\Rightarrow i := i + 1$, *back to 3.2.*
4. *Convergence? OK* $\Rightarrow$ *finish. KO* $\Rightarrow k := k + 1$, *back to 2.*

Inexact iterative linearization

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      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK* $\Rightarrow U^k := U^{k,i}$. *KO* $\Rightarrow i := i + 1$, *back to 3.2.*
4. *Convergence? OK* $\Rightarrow$ *finish. KO* $\Rightarrow k := k + 1$, *back to 2.*

Inexact iterative linearization

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector* U^0. *Set* $k := 1$.
2. $U^{k-1} \Rightarrow$ *matrix* $\mathbb{A}^{k-1}$ *and vector* F^{k-1}*: find* U^k *s.t.*

   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

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Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria Eisenstat and Walker (1990's), Becker, Johnson, and Rannacher (1995), Deuflhard (2004 book), Arioli (2000's)**)**

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error Verfürth (1994), Carstensen and Klose (2003), Chaillou and Suri (2006), Kim (2007)**)**

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Abstract assumptions

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *such that*

$$\nabla\cdot\boldsymbol{\sigma}_h^{k,i} = f.$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\boldsymbol{\sigma}_{h,\mathrm{dis}}^{k,i}, \boldsymbol{\sigma}_{h,\mathrm{lin}}^{k,i}, \boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\boldsymbol{\sigma}_h^{k,i} = \boldsymbol{\sigma}_{h,\mathrm{dis}}^{k,i} + \boldsymbol{\sigma}_{h,\mathrm{lin}}^{k,i} + \boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i}$;

(ii) *as the linear solver converges,* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}^{k,i}\|_q \to 0$;

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Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds (up to quadrature and data oscillation)

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i}.$$

Stopping criteria: error components of similar size

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{\mathrm{alg},K}^{k,i} \leq \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \leq \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

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Comments

- ✓ same physical units (fluxes)
- ✓ naturally relative
- ✓ proper $[L^q(\Omega)]^d$ framework × l_2 norms of algebraic vectors

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Applications

Discretization methods

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin
- ✓ various finite volumes
- ✓ mixed finite elements

Linearizations

- ✓ fixed point
- ✓ Newton

Linear solvers

- ✓ independent of the linear solver

…all Assumptions A to D verified

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Numerical experiment I

Model problem

- p-Laplacian

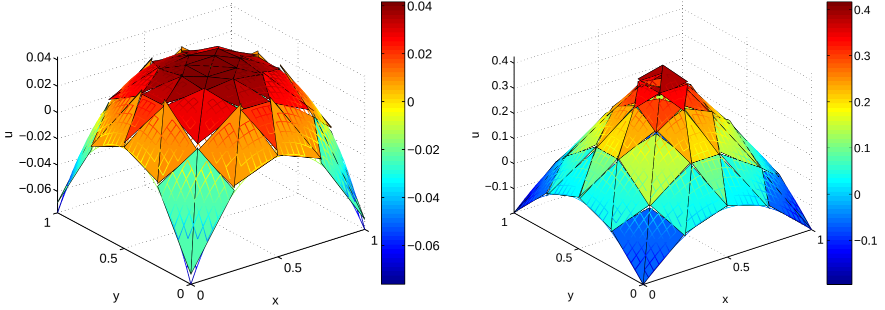
$$\begin{aligned} \nabla\cdot(|\nabla u|^{p-2}\nabla u) &= f && \text{in } \Omega,\\ u &= u_{\mathrm{D}} && \text{on } \partial\Omega \end{aligned}$$

- weak solution (used to impose the Dirichlet BC)

$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

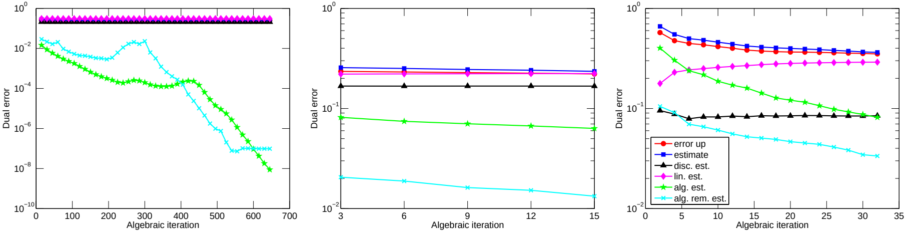
- tested values $p = 1.5$ and 10
- Crouzeix–Raviart nonconforming finite elements

Analytical and approximate solutions



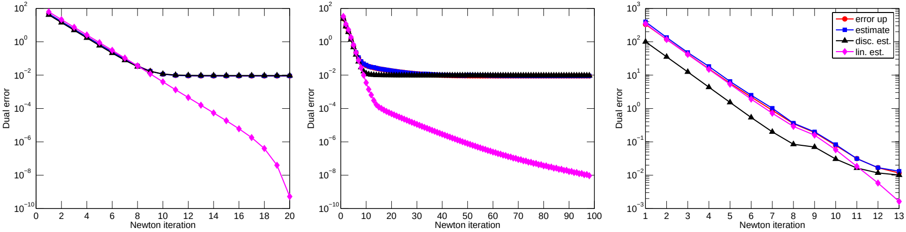
Case $p = 1.5$ Case $p = 10$

Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step



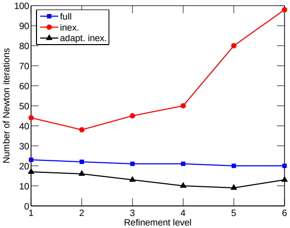
Newton inexact Newton ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh

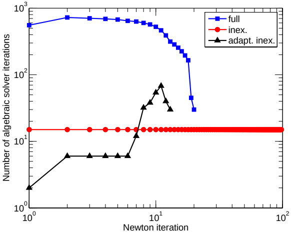


Newton inexact Newton ad. inexact Newton

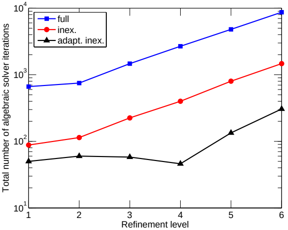
Newton and algebraic iterations, $p = 10$



Newton it. / refinement



alg. it. / Newton step



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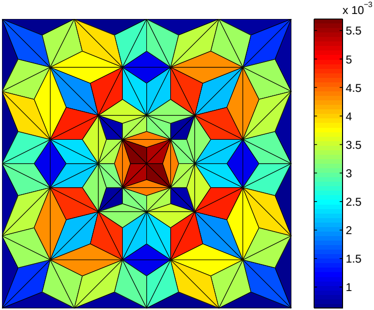
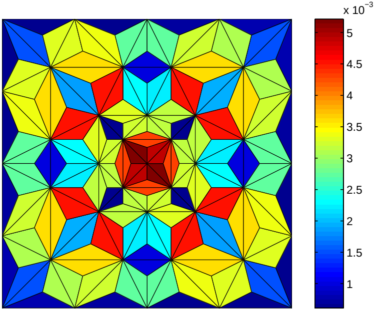
Error distribution, $p = 10$

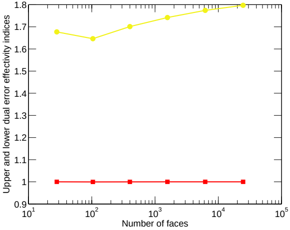


Estimated error distribution

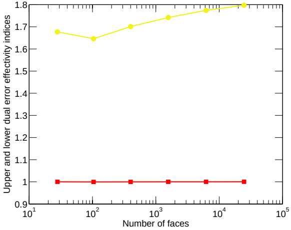


Exact error distribution

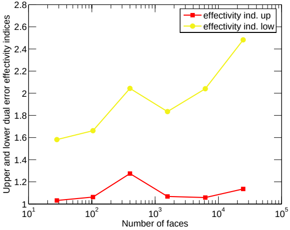
Effectivity indices, $p = 10$



Newton



inexact Newton



ad. inexact Newton

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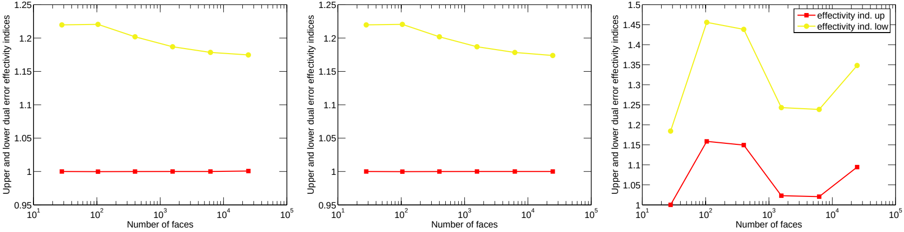

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Numerical experiment II

Model problem

- *p*-Laplacian

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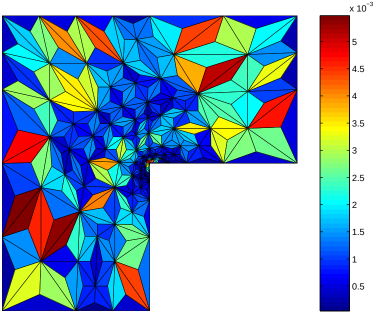
- weak solution (used to impose the Dirichlet BC)

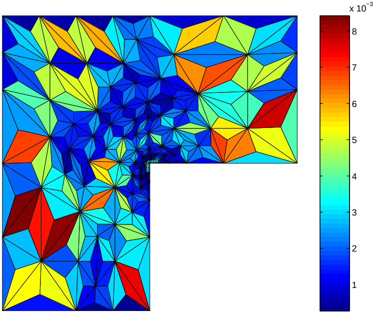
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- Crouzeix–Raviart nonconforming finite elements

Error distribution on an adaptively refined mesh

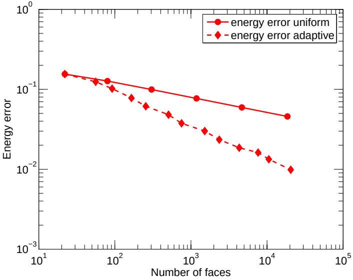


Estimated error distribution

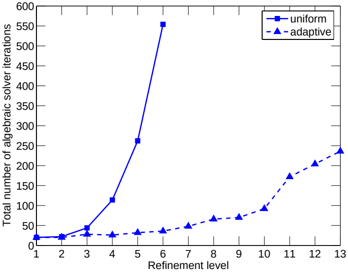


Exact error distribution

Energy error and overall performance



Energy error



Overall performance

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Laplace eigenvalue problem

Problem

Find eigenvector & eigenvalue pair (u, λ) such that

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Weak formulation

Find $(u_i, \lambda_i) \in V \times \mathbb{R}^+$, $i \geq 1$, with $\|u_i\| = 1$, such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \qquad \forall v \in V.$$

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Main results (conforming setting)

Assumption A (Conforming variational solution)

There holds

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $\quad (\Rightarrow \lambda_{1h} \geq \lambda_1)$

We bound

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

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$$\lambda_{ih} - \lambda_i \leq \eta_i(u_{ih}, \lambda_{ih})^2$$

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$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih}) \leq C_{\mathrm{eff},i} \|\nabla(u_i - u_{ih})\|$$

✓ $C_{\mathrm{eff},i}$ only depends on mesh shape regularity and on $\max\left\{\left(\frac{\lambda_i}{\lambda_{i-1}} - 1\right)^{-1}, \left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right)^{-1}\right\} \frac{\lambda_i}{\lambda_1}$

✓ we give computable upper bounds on $C_{\mathrm{eff},i}$

Main results (conforming setting)

Assumption A (Conforming variational solution)

There holds

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $(\Rightarrow \lambda_{1h} \geq \lambda_1)$

We bound

i-th eigenvalue upper and lower bounds

$$\lambda_{ih} - \eta_i(u_{ih}, \lambda_{ih})^2 \leq \lambda_i \leq \lambda_{ih} - \tilde{\eta}_i(u_{ih}, \lambda_{ih})^2$$

2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

The pathway (conforming setting)

1. estimate the $L^2(\Omega)$ error:

$$\|u_i - u_{ih}\| \leq \alpha_{ih}$$

2. prove equivalence of the eigenvalue & eigenvector errors:

$$C\|\nabla(u_i - u_{ih})\|^2 \leq \lambda_{ih} - \lambda_i \leq \|\nabla(u_i - u_{ih})\|^2$$

3. prove equivalence of the eigenvector error & of the dual norm of the residual:

$$\underline{C}\|\mathcal{R}(u_{ih}, \lambda_{ih})\|_{-1} \leq \|\nabla(u_i - u_{ih})\| \leq \overline{C}\|\mathcal{R}(u_{ih}, \lambda_{ih})\|_{-1},$$

where

$$\langle \mathcal{R}(u_{ih}, \lambda_{ih}), v\rangle_{V',V} := \lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v) \qquad v \in V$$

$$\|\mathcal{R}(u_{ih}, \lambda_{ih})\|_{-1} := \sup_{v\in V,\, \|\nabla v\|=1} \langle \mathcal{R}(u_{ih}, \lambda_{ih}), v\rangle_{V',V}$$

4. prove equivalence of the dual residual norm & its estimate:

$$\overline{C}\|\mathcal{R}(u_{ih}, \lambda_{ih})\|_{-1} \leq \eta_i(u_{ih}, \lambda_{ih}) \leq \tilde{C}\|\mathcal{R}(u_{ih}, \lambda_{ih})\|_{-1}$$

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Nonconforming discretizations

Nonconforming setting

- $u_{ih} \notin V$, $\|u_{ih}\| \neq 1$
- $\|\nabla u_{ih}\|^2 \neq \lambda_{ih}$

Main tool

- conforming eigenvector reconstruction

$$s_{ih}^{\mathbf{a}} := \arg \min_{v_h \in W_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_{ih} - v_h)\|_{\omega_{\mathbf{a}}}, \qquad s_{ih} := \sum_{\mathbf{a} \in \mathcal{V}_h} s_{ih}^{\mathbf{a}}$$

Unified framework

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin elements
- mixed finite elements

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   - Polynomial-degree-robust local efficiency
   - Applications & numerical results
3. Numerical linear algebra: taking into account solver error
   - Upper and lower bounds on the algebraic error
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5. Laplace eigenvalues and eigenvectors: guaranteed bounds
   - Upper and lower bounds
   - Applications & numerical results
6. Stokes equation: extension to systems
7. Heat equation: robustness wrt final time & local efficiency
8. Conclusions and outlook

Unit square

Setting

- $\Omega = (0,1)^2$
- $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$ known explicitly
- $u_1(x,y) = \sin(\pi x)\sin(\pi y)$ known explicitly

Effectivity indices

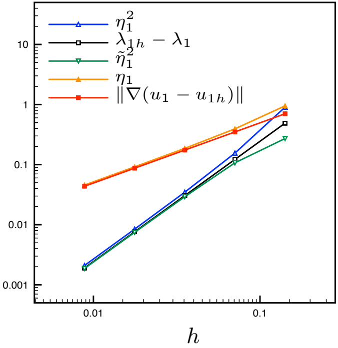
- recall $\tilde{\eta}_i^2 \leq \lambda_{ih} - \lambda_i \leq \eta_i^2$

$$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}} := \frac{\lambda_{ih} - \lambda_i}{\tilde{\eta}_i^2}, \qquad I^{\mathrm{ub}}_{\lambda,\mathrm{eff}} := \frac{\eta_i^2}{\lambda_{ih} - \lambda_i}$$

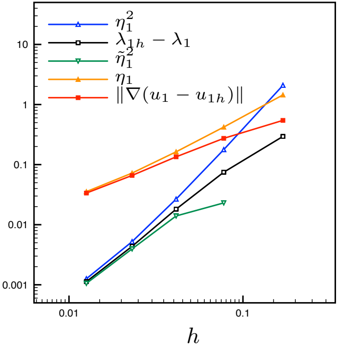
- recall $\|\nabla(u_i - u_{ih})\| \leq \eta_i$

$$I^{\mathrm{ub}}_{u,\mathrm{eff}} := \frac{\eta_i}{\|\nabla(u_i - u_{ih})\|}$$

Conforming finite elements



Structured meshes



Unstructured meshes

Conforming finite elements

N	h	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Structured meshes

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10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Unstructured meshes

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N	h	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda,\mathrm{eff}}^{\mathrm{lb}}$	$I_{\lambda,\mathrm{eff}}^{\mathrm{ub}}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
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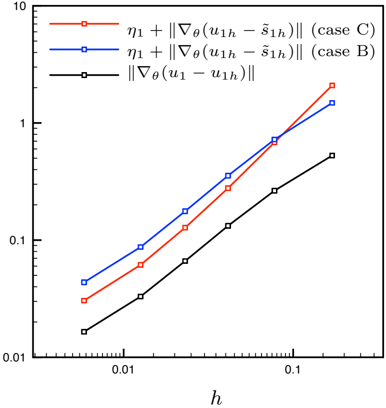
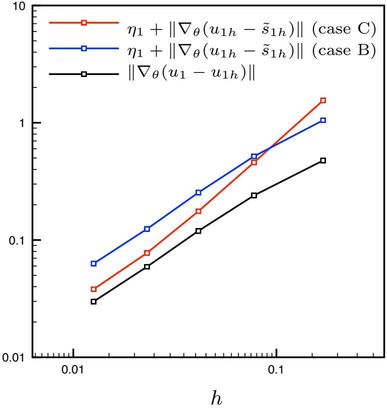
Unstructured meshes

Nonconforming finite elements & DG's



Nonconforming finite elements



Discontinuous Galerkin

Nonconforming finite elements & DG's

N	h	ndof	λ_1	λ_{1h}	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1414	320	19.7392	19.6850	18.8966	19.8262	4.80e-02	2.68
20	0.0707	1240	19.7392	19.7257	19.6495	19.7616	5.69e-03	2.11
40	0.0354	4880	19.7392	19.7358	19.7246	19.7448	1.02e-03	1.91
80	0.0177	19360	19.7392	19.7384	19.7361	19.7406	2.29e-04	1.85
160	0.0088	77120	19.7392	19.7390	19.7385	19.7396	5.53e-05	1.83
320	0.0044	307840	19.7392	19.7392	19.7390	19.7393	1.37e-05	1.83

Nonconforming finite elements

N	h	ndof	λ_1	λ_{1h}	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1698	732	19.7392	19.9432	17.8788	19.9501	1.10e-01	3.26
20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

SIP discontinuous Galerkin

Nonconforming finite elements & DG's

N	h	ndof	λ_1	λ_{1h}	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
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Nonconforming finite elements & DG's

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SIP discontinuous Galerkin

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Stokes problem

Stokes problem

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla{\cdot}\mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $\mathbf{f} \in [L^2(\Omega)]^d$
- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega) := \{q \in L^2(\Omega); (q, 1) = 0\}$

Weak formulation

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} (\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla{\cdot}\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) && \forall \mathbf{v} \in \mathbf{V}, \\ (\nabla{\cdot}\mathbf{u}, q) &= 0 && \forall q \in Q. \end{aligned}$$

Inf–sup condition

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q, \nabla{\cdot}\mathbf{v})}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta > 0$$

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Exact and approximate solutions

Properties of the weak solution

- $\mathbf{u} \in \mathbf{V}$
- $\underline{\boldsymbol{\sigma}} := \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$
- $\nabla \cdot \underline{\boldsymbol{\sigma}} = -\mathbf{f}$
- $\underline{\boldsymbol{\sigma}} \in [\mathbf{H}(\mathrm{div}, \Omega)]^d$

Approximate solution

- $\mathbf{u}_h \in [H^1(\mathcal{T}_h)]^d \not\subset \mathbf{V}$
- $p_h \in Q$
- $\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}} \not\in \mathbf{V}$
- $\nabla \cdot (\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}}) \neq -\mathbf{f}$

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Velocity and equilibrated stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$
- $\mathbf{s}_h$ constructed from $\mathbf{u}_h$

Equilibrated stress reconstruction

- $\underline{\boldsymbol{\sigma}}_h \in [\mathbf{H}(\mathrm{div}, \Omega)]^d$
- $-(\nabla \cdot \underline{\boldsymbol{\sigma}}_h, \mathbf{e}_i)_K = (\mathbf{f}, \mathbf{e}_i)_K \qquad i = 1, \ldots, d, \quad \forall K \in \mathcal{T}_h$
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A guaranteed a posteriori error estimate

Theorem (A guaranteed a posteriori error estimate)

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution &* $(\mathbf{u}_h, p_h) \in [H^1(\mathcal{T}_h)]^d \times Q$ *be arbitrary. Let* $\mathbf{s}_h$ *be a velocity reconstruction and* $\underline{\boldsymbol{\sigma}}_h$ *an equilibrated stress reconstruction. For any* $K \in \mathcal{T}_h$*, define*

$$\begin{aligned}
\eta_{\mathrm{R},K} &:= C_{\mathrm{P},K} h_K \|\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_K && \textit{residual est.},\\
\eta_{\mathrm{F},K} &:= \|\nabla\mathbf{u}_h - p_h\underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h\|_K && \textit{flux est.},\\
\eta_{\mathrm{NC},K} &:= \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_K && \textit{nonconformity est.},\\
\eta_{\mathrm{D},K} &:= \frac{\|\nabla\cdot\mathbf{s}_h\|_K}{\beta} && \textit{divergence est.}
\end{aligned}$$

Then

$$\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2 \le \sum_{K\in\mathcal{T}_h}(\eta_{\mathrm{R},K}+\eta_{\mathrm{F},K})^2 + \left\{\left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{D},K}^2\right\}^{1/2} + \left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{1/2}\right\}^2,$$

$$\|p-p_h\| \le \frac{1}{\beta}\left(\left\{\sum_{K\in\mathcal{T}_h}(\eta_{\mathrm{R},K}+\eta_{\mathrm{F},K})^2\right\}^{1/2} + \left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{D},K}^2\right\}^{1/2} + \left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{1/2}\right).$$

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Model parabolic problem

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega \end{aligned}$$

Spaces

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) \,\mathrm{d}t = \int_0^T (f, v) \,\mathrm{d}t \qquad \forall v \in X.$$

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Error and residual in the unsteady case

Theorem (Parabolic inf–sup identity)

For every $\varphi \in Y$, we have

$$\|\varphi\|_Y^2 = \left[\sup_{v \in X,\, \|v\|_X = 1} \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v)\,\mathrm{d}t \right]^2 + \|\varphi(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v \rangle - (\nabla u_{h\tau}, \nabla v)\,\mathrm{d}t$$

- dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v \rangle$$

Y norm error is the dual X norm of the residual + IC error

$$\|u - u_{h\tau}\|_Y^2 = \|\mathcal{R}(u_{h\tau})\|_{X'}^2 + \|u_0 - u_{h\tau}(0)\|^2$$

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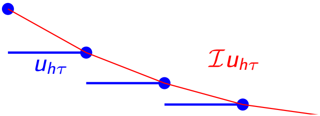
Radau reconstruction and error measure

Radau reconstruction of $u_{h\tau} \in X$, $u_{h\tau}|_{I_n} \in \mathbb{P}_{q_n}(I_n; V_h^n)$ and $u_{h\tau}(0) = \Pi_h u_0$: $\mathcal{I} u_{h\tau} \in Y$, $\mathcal{I} u_{h\tau}|_{I_n} \in \mathbb{P}_{q_n+1}(I_n; \widetilde{V_h^n})$

$$\int_{I_n} (\partial_t \mathcal{I} u_{h\tau}, v_{h\tau}) + (\nabla u_{h\tau}, \nabla v_{h\tau})\,\mathrm{d}t = \int_{I_n} (f, v_{h\tau})\,\mathrm{d}t \quad \forall v_{h\tau} \in \mathbb{P}_{q_n}(I_n; V_h^n)$$



Error measure

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} := \|u - \mathcal{I} u_{h\tau}\|_Y^2 + \|u_{h\tau} - \mathcal{I} u_{h\tau}\|_X^2$$

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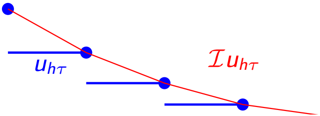

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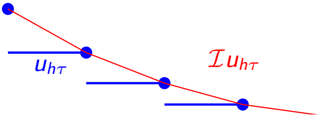

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A posteriori estimate

Guaranteed upper bound

- ✓ $\|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$
- ✓ no undetermined constant: **error control**

Local space-time efficiency

- ✓ $\eta_K^n(u_{h\tau}) \leq C_{\mathrm{eff}} \|u - u_{h\tau}\|_{\mathcal{E}_Y,\text{neighbors of } K\times(t^{n-1},t^n)}$
- ✓ optimal space-time mesh refinement
- ✓ **local** in **time** and in **space** error lower bound

Robustness

- ✓ C_{eff} independent of data, domain Ω, **final time** T, meshes, solution u, **polynomial degrees** of $u_{h\tau}$ in space and in time

Asymptotic exactness

- ✓ $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{\mathcal{E}_Y,\Omega\times(0,T)} \searrow 1$
- ✓ overestimation factor goes to one with meshes size

Small evaluation cost

- ✓ estimators can be evaluated cheaply (locally)

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Bibliography

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Thank you for your attention!