

A priori and a posteriori error analysis in $\mathbf{H}(\text{curl})$: localization, minimal regularity, and p -optimality

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Inria Paris & Ecole des Ponts

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Outline

- 1 Introduction
- 2 Equilibration in $\mathbf{H}(\text{curl})$
- 3 A posteriori error estimates in $\mathbf{H}(\text{curl})$
- 4 A stable local commuting projector in $\mathbf{H}(\text{curl})$
- 5 Local–best–global–best equivalence in $\mathbf{H}(\text{curl})$
- 6 Approximation error estimates in $\mathbf{H}(\text{curl})$
- 7 Conclusions

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- a priori & a posteriori analysis
disconnected (different tools)

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 - go **discrete straight-away** and then use a posteriori tools in a priori analysis (equilibration)

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Equilibration in $\mathbf{H}(\text{curl})$

Previous contributions

- Braess & Schöberl (2008): lowest-order case $p = 0$
- Licht (2019): a conceptual discussion
- Gedicke, Geevers, & Perugia (2020): equilibrated-residual-style construction
- Gedicke, Geevers, Perugia, & Schöberl (2021): p -robust modification

Our construction

$$\underbrace{\iota_h \in \mathcal{N}_p(\mathcal{T}_h), j \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)}_{\text{???} = 0 \quad \forall \text{???}}$$

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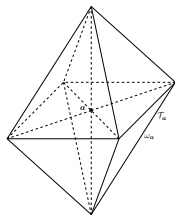
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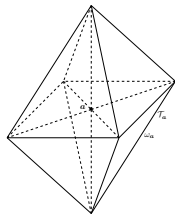
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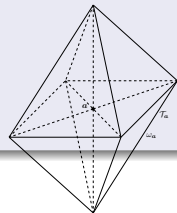
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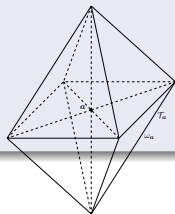
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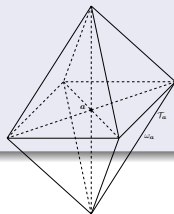
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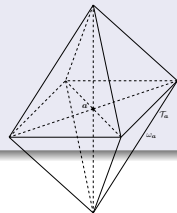
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- Take $\mathbf{h}^a := \psi^a(\nabla \times \mathbf{A}) \in \mathbf{H}_0(\text{curl}, \omega_a)$
and note that $\sum_{a \in \mathcal{V}_h} \mathbf{h}^a = \nabla \times \mathbf{A}.$

- Rewritten implicitly,

$$\mathbf{h}^a = \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v} = \mathbf{j}^a}} \|\psi^a(\nabla \times \mathbf{A}) - \mathbf{v}\|_{\omega_a}^2$$

with

$$\mathbf{j}^a := \psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}).$$

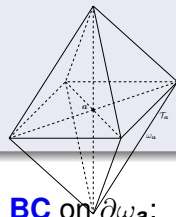
Definition (Chaumont-Frelet, Vohralík (2022))

For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\mathbf{h}_h^a := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v}_h = \psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}_h)}} \|\psi^a(\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_a}^2$$

and combine

$$\mathbf{h}_h := \sum_{a \in \mathcal{V}_h} \mathbf{h}_h^a.$$



Key points

- **homogeneous tangential BC** on $\partial \omega_a$:

$$\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$$

- **global equilibrium** $\nabla \times \mathbf{h}_h = \sum_{a \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^a$

$$= \sum_{a \in \mathcal{V}_h} (\psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}_h)) = \mathbf{j}$$

Patchwise equilibrated fluxes

Continuous level

- $\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies
 $(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$

- Thus $\nabla \times \mathbf{A} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ with
 $\nabla \times (\nabla \times \mathbf{A}) = \mathbf{j}.$

- Take $\mathbf{h}^a := \psi^a(\nabla \times \mathbf{A}) \in \mathbf{H}_0(\text{curl}, \omega_a)$
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- Rewritten implicitly,

$$\mathbf{h}^a = \arg \min_{\substack{\mathbf{v} \in \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v} = \mathbf{j}^a}} \|\psi^a(\nabla \times \mathbf{A}) - \mathbf{v}\|_{\omega_a}^2$$

with

$$\mathbf{j}^a := \psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}).$$

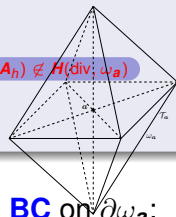
Definition (Chaumont-Frelet, Vohralík (2022))

For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

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► $\psi^a \mathbf{j} \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)$ but $\nabla \psi^a \times (\nabla \times \mathbf{A}_h) \notin \mathbf{H}(\text{div}, \omega_a)$

$$\mathbf{h}_h := \sum_{a \in \mathcal{V}_h} \mathbf{h}_h^a.$$



Key points

- **homogeneous tangential BC** on $\partial \omega_a$:
 $\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$

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Patchwise equilibrated fluxes

Continuous level

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with

$$\mathbf{j}^a := \psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}).$$

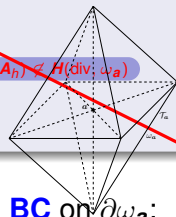
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For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

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► $\psi^a \mathbf{j} \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)$ but $\nabla \psi^a \times (\nabla \times \mathbf{A}_h) \notin \mathbf{H}(\text{div}, \omega_a)$

$$\mathbf{h}_h := \sum_{a \in \mathcal{V}_h} \mathbf{h}_h^a.$$



Key points

- **homogeneous tangential BC** on $\partial \omega_a$:

$$\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$$

- **global equilibrium** $\nabla \times \mathbf{h}_h = \sum_{a \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^a$

$$= \sum_{a \in \mathcal{V}_h} (\psi^a \mathbf{j} + \nabla \psi^a \times (\nabla \times \mathbf{A}_h)) = \mathbf{j}$$

Stage 1:

Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)$ to a Raviart–Thomas space

For all vertices $\mathbf{a} \in \mathcal{V}_h$, consider p' := min{p, 1}-degree patchwise minimizations:

$$\theta_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = -\nabla\psi^{\mathbf{a}} \cdot \mathbf{j}}} \|\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_{\mathbf{a}}}^2.$$

$$(\mathbf{v}_h, \mathbf{r}_h)_K = (\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h), \mathbf{r}_h)_K \quad \forall \mathbf{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\mathbf{a}}$$

Comments

- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$
- remainder $\delta_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}}$
 - should be zero ($\sim$ partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_h} \{\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)\} = 0$), but is not
 - $\delta_h \in \mathcal{RT}_{p'}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ and $\nabla \cdot \delta_h = 0$
- additional orthogonality constraint
 - crucial for stage 2 below
 - only possible thanks to the lowest-order Galerkin orthogonality of $\mathbf{A}_h$
 - requests min{p, 1} (and not simply p)

Stage 1:

Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)$ to a Raviart–Thomas space

For all vertices $\mathbf{a} \in \mathcal{V}_h$, consider p' := min{ $p, 1$ }-degree patchwise minimizations:

$$\theta_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = -\nabla\psi^{\mathbf{a}} \cdot \mathbf{j}}} \|\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_{\mathbf{a}}}^2.$$

$$(\mathbf{v}_h, \mathbf{r}_h)_K = (\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h), \mathbf{r}_h)_K \quad \forall \mathbf{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\mathbf{a}}$$

Comments

- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$
- remainder $\delta_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}}$
 - should be zero ($\sim$ partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_h} \{\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A})\} = 0$), but is not
 - $\delta_h \in \mathcal{RT}_{p'}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ and $\nabla \cdot \delta_h = 0$
- additional orthogonality constraint
 - crucial for stage 2 below
 - only possible thanks to the lowest-order Galerkin orthogonality of $\mathbf{A}_h$
 - requests min{ $p, 1$ } (and not simply p)

Stage 1:

Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)$ to a Raviart–Thomas space

For all vertices $\mathbf{a} \in \mathcal{V}_h$, consider p' := min{p, 1}-degree patchwise minimizations:

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$$(\mathbf{v}_h, \mathbf{r}_h)_K = (\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h), \mathbf{r}_h)_K \quad \forall \mathbf{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\mathbf{a}}$$

Comments

- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$
- remainder $\delta_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}}$
 - should be zero ($\sim$ partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_h} \{\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A})\} = 0$), but is not
 - $\delta_h \in \mathcal{RT}_{p'}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ and $\nabla \cdot \delta_h = 0$
- additional orthogonality constraint
 - crucial for stage 2 below
 - only possible thanks to the lowest-order Galerkin orthogonality of $\mathbf{A}_h$
 - requests min{p, 1} (and not simply p)

Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h)$ to a Raviart–Thomas space

For all vertices $\mathbf{a} \in \mathcal{V}_h$, consider p' := min{ $p, 1$ }–degree patchwise minimizations:

$$\theta_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = -\nabla\psi^{\mathbf{a}} \cdot \mathbf{j} \\ (\mathbf{v}_h, \mathbf{r}_h)_K = (\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h), \mathbf{r}_h)_K \quad \forall \mathbf{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\mathbf{a}}}} \|\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_{\mathbf{a}}}^2.$$

Comments

- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}})$
- remainder $\delta_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \theta_h^{\mathbf{a}}$
 - should be zero ($\sim$ partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_h} \{\nabla\psi^{\mathbf{a}} \times (\nabla \times \mathbf{A})\} = 0$), but is not
 - $\delta_h \in \mathcal{RT}_{p'}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ and $\nabla \cdot \delta_h = 0$
- additional orthogonality constraint
 - crucial for stage 2 below
 - only possible thanks to the lowest-order Galerkin orthogonality of $\mathbf{A}_h$
 - requests min{ $p, 1$ } (and not simply p)

Stage 2: **divergence-free decomposition** of the given divergence-free Raviart–Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$ -degree elementwise minimizations:

$$\delta_h^a|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_1(K) \\ \nabla \cdot \mathbf{v}_h = 0 \\ \mathbf{v}_h \cdot \mathbf{n}_K = I_{\mathcal{RT}}^1(\psi^a \delta_h) \cdot \mathbf{n}_K \text{ on } \partial K}} \|\mathbf{v}_h - I_{\mathcal{RT}}^1(\psi^a \delta_h)\|_K^2 \quad \forall \mathbf{a} \in \mathcal{V}_K \text{ when } p = 0,$$

$$\delta_h^a|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{p+1}(K) \\ \nabla \cdot \mathbf{v}_h = 0 \\ \mathbf{v}_h \cdot \mathbf{n}_K = \psi^a \delta_h \cdot \mathbf{n}_K \text{ on } \partial K}} \|\mathbf{v}_h - \psi^a \delta_h\|_K^2 \quad \forall \mathbf{a} \in \mathcal{V}_K \text{ when } p \geq 1.$$

Comments

- patchwise contributions

$$\delta_h^a \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a) \quad \text{and} \quad \nabla \cdot \delta_h^a = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h$$

Stage 2: divergence-free decomposition of the given divergence-free Raviart–Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$ -degree elementwise minimizations:

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Comments

- patchwise contributions

$$\delta_h^{\mathbf{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \quad \text{and} \quad \nabla \cdot \delta_h^{\mathbf{a}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h$$

- $\delta_h^{\mathbf{a}}$ form a divergence-free decomposition of δ_h , $\delta_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \delta_h^{\mathbf{a}}$

Stage 2: **divergence-free decomposition** of the given divergence-free Raviart–Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$ -degree elementwise minimizations:

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Comments

- patchwise contributions

$$\delta_h^{\mathbf{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \quad \text{and} \quad \nabla \cdot \delta_h^{\mathbf{a}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h$$

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Stage 2: **divergence-free decomposition** of the given divergence-free Raviart–Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$ -degree elementwise minimizations:

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Comments

- patchwise contributions

$$\delta_h^{\mathbf{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{a}}) \quad \text{and} \quad \nabla \cdot \delta_h^{\mathbf{a}} = 0 \quad \forall \mathbf{a} \in \mathcal{V}_h$$

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Stage 2: divergence-free decomposition of the given divergence-free current density $\mathbf{j}$

Divergence-free decomposition of the current density $\mathbf{j}$

Set

$$\mathbf{j}_h^{\mathbf{a}} := \psi^{\mathbf{a}} \mathbf{j} + \boldsymbol{\theta}_h^{\mathbf{a}} - \boldsymbol{\delta}_h^{\mathbf{a}}.$$

Then

$$\begin{aligned} \mathbf{j}_h^{\mathbf{a}} &\in \mathcal{RT}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\operatorname{div}, \omega_{\mathbf{a}}), \\ \nabla \cdot \mathbf{j}_h^{\mathbf{a}} &= 0, \\ \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{j}_h^{\mathbf{a}} &= \mathbf{j}. \end{aligned}$$

Stage 3: discrete patchwise equilibrated fluxes

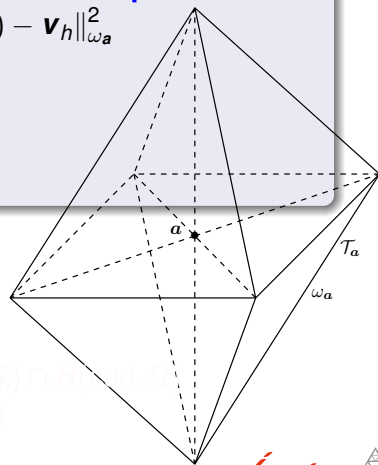
Definition (Chaumont-Frelet, Vohralík (2021))

For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\mathbf{h}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{curl}, \omega_a) \\ \nabla \times \mathbf{v}_h = \mathbf{j}_h^{\mathbf{a}}}} \|\psi^{\mathbf{a}}(\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_a}^2$$

and combine

$$\mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^{\mathbf{a}}.$$



Key points

- homogeneous tangential BC on $\partial\omega_a$: $\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}_0(\text{curl}, \Omega)$
- global equilibrium $\nabla \times \mathbf{h}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^{\mathbf{a}} = \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{j}_h^{\mathbf{a}} = \mathbf{j}$

Stage 3: discrete patchwise equilibrated fluxes

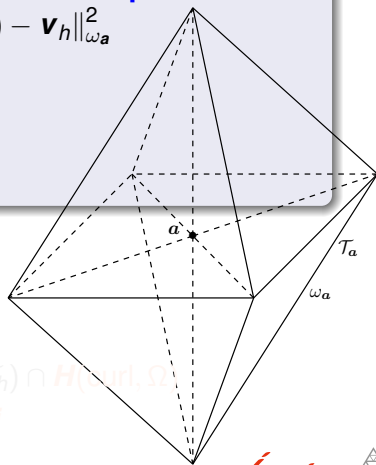
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For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

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and combine

$$\mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^{\mathbf{a}}.$$



Key points

- **homogeneous tangential BC** on $\partial\omega_{\mathbf{a}}$: $\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$
- **global equilibrium** $\nabla \times \mathbf{h}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \nabla \times \mathbf{h}_h^{\mathbf{a}} = \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{j}_h^{\mathbf{a}} = \mathbf{j}$

Stage 3: discrete patchwise equilibrated fluxes

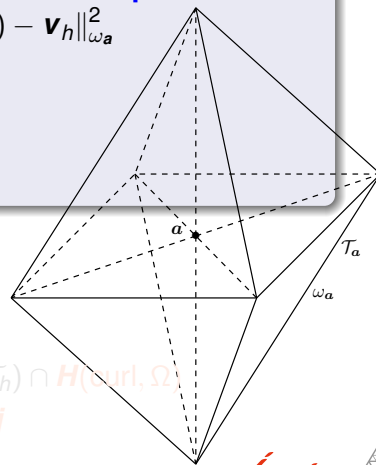
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For each vertex $\mathbf{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\mathbf{h}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{curl}, \omega_{\mathbf{a}}) \\ \nabla \times \mathbf{v}_h = \mathbf{j}_h^{\mathbf{a}}}} \|\psi^{\mathbf{a}}(\nabla \times \mathbf{A}_h) - \mathbf{v}_h\|_{\omega_{\mathbf{a}}}^2$$

and combine

$$\mathbf{h}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \mathbf{h}_h^{\mathbf{a}}.$$



Key points

- **homogeneous tangential BC** on $\partial\omega_{\mathbf{a}}$: $\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$
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Stage 3: discrete patchwise equilibrated fluxes

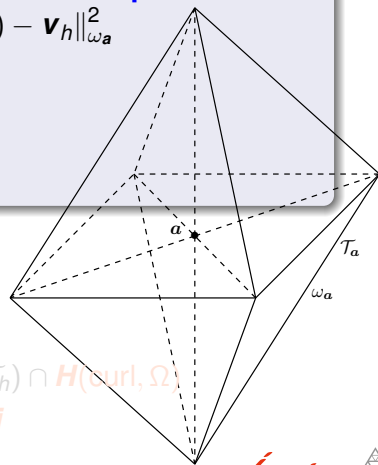
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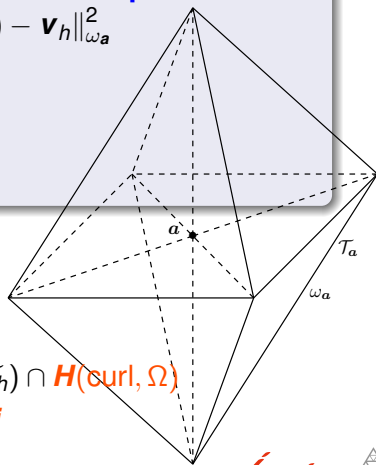
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A posteriori error estimates

Weak formulation (consequence)

$\mathbf{A} \in \mathbf{H}_{0,D}(\text{curl}, \Omega)$ satisfies

$$(\nabla \times \mathbf{A}, \nabla \times \mathbf{v}) = (\mathbf{j}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{0,D}(\text{curl}, \Omega).$$

Primal Nédélec finite element approximation

$\mathbf{A}_h \in \mathbf{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,D}(\text{curl}, \Omega)$, $p \geq 0$, satisfies

$$(\nabla \times \mathbf{A}_h, \nabla \times \mathbf{v}_h) = (\mathbf{j}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

$\mathbf{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)$ s.t. $\nabla \times \mathbf{h}_h = \mathbf{j}$: local equilibrated flux reconstruction

Theorem (Guaranteed upper bound, efficiency, and p -robustness)

$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla \times \mathbf{A}_h - \mathbf{h}_h\|}_{\text{computable estimator}} \leq C(\kappa_{\mathcal{T}_h}) \underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}}$$

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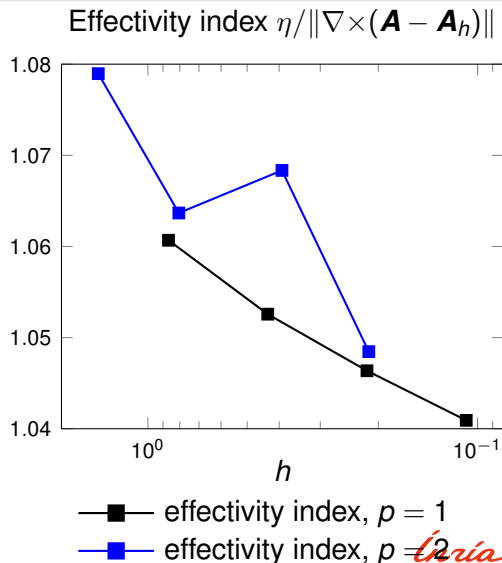
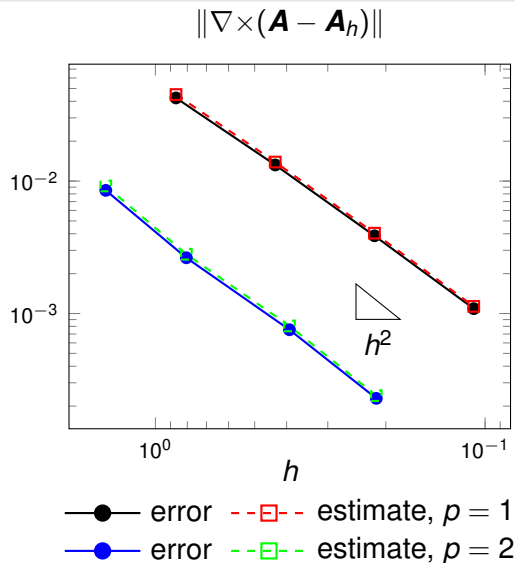
$$(\nabla \times \mathbf{A}_h, \nabla \times \mathbf{v}_h) = (\mathbf{j}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

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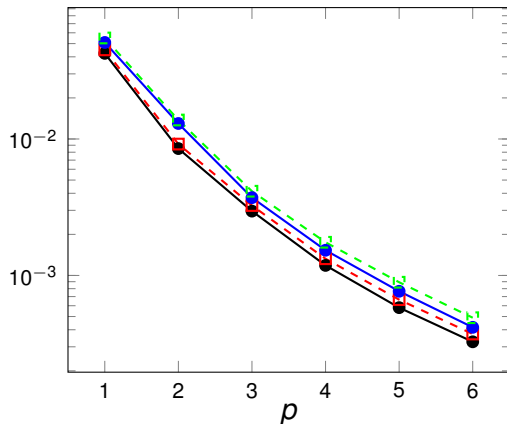
$$\underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla \times \mathbf{A}_h - \mathbf{h}_h\|}_{\text{computable estimator}} \leq C(\kappa_{\mathcal{T}_h}) \underbrace{\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|}_{\text{unknown error}}$$

H^3 solution, uniform h -refinement

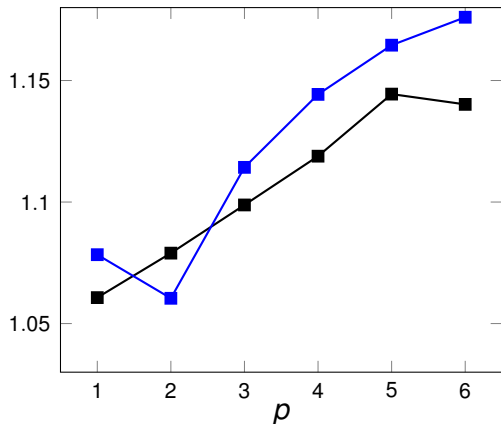


H^3 solution, uniform p -refinement

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|$$



$$\text{Effectivity index } \eta / \|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|$$

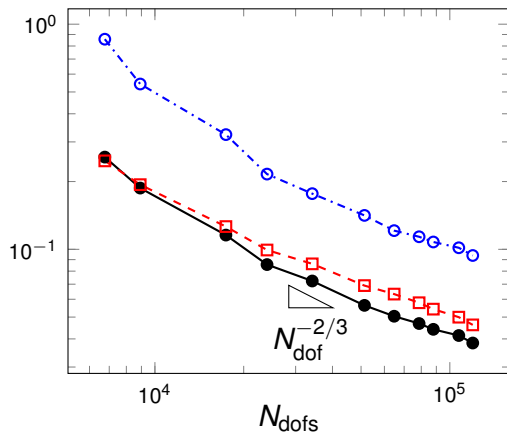


—●— error - - □ - - estimate, struct. mesh
 —●— error - - □ - - estimate, unstruct. mesh

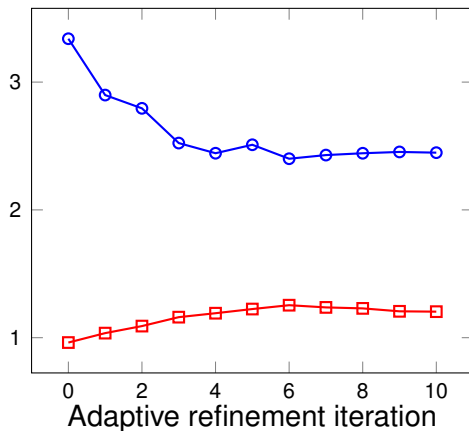
—■— effectivity index, struct. mesh
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Singular solution, adaptive mesh refinement ($p = 2$)

$$\|\nabla \times (\mathbf{A} - \mathbf{A}_h)\|$$

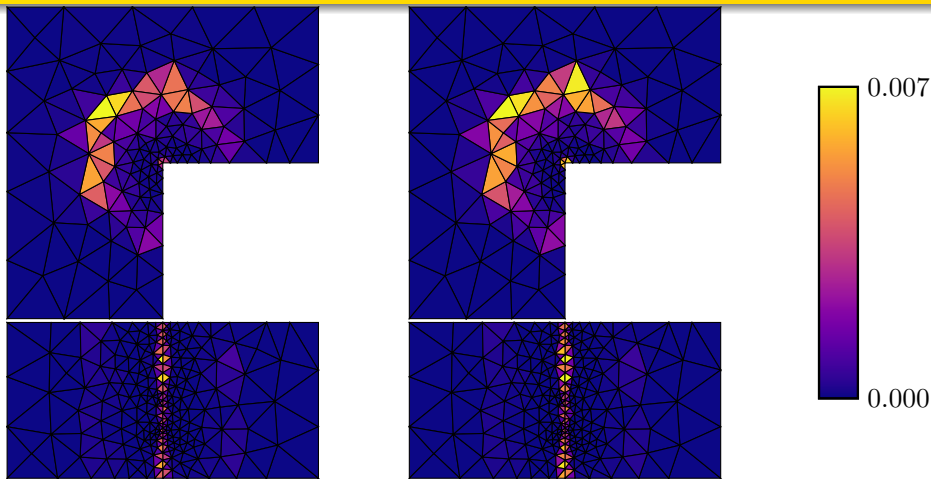


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—●— error - - □ - - estimate - - ○ - - est.+d. osc. - □ - no d. osc. - □ - d. osc.

Singular solution, adaptive mesh refinement ($p = 2$)



Estimators (left) and actual error (right), adaptive mesh refinement iteration #10.
Top view (top) and side view (bottom)

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Commuting de Rham diagram with operator $P_h^{p,\text{curl}}$

Commuting de Rham diagram

$$\begin{array}{ccccccc}
 H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & L_*^2(\Omega) \\
 \downarrow P_h^{p+1,\text{grad}} & & \downarrow P_h^{p,\text{curl}} & & \downarrow P_h^{p,\text{div}} & & \downarrow \Pi_h^p \\
 \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
 \end{array}$$

Commuting de Rham diagram with operator $\mathbf{P}_h^{p,\text{curl}}$

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 \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
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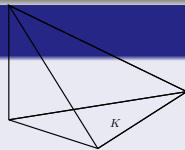
Properties of $\mathbf{P}_h^{p,\text{curl}}$

- 1 is defined over the **entire** $\mathbf{H}_{0,N}(\text{curl}, \Omega)$ (**minimal regularity**)
- 2 is defined **locally** (in neighborhood of mesh elements)
- 3 is defined **simply** (starting from the **elementwise L^2 orthogonal projection**)
- 4 has **optimal hp approximation properties**, that of **elementwise curl-unconstrained L^2 -orthogonal projector** (local-global equivalence)
- 5 is **stable in $L^2(\Omega)$** (up to data oscillation)
- 6 satisfies the **commuting properties** expressed by the arrows
- 7 is **projector**, i.e., leaves intact piecewise polynomials

A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$

Definition (A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ be given (minimal regularity).



- 1 For each $K \in \mathcal{T}_h$, prepare the datum $\tau_h|_K$

$$\tau_h|_K := \arg \min_{\substack{\mathbf{w}_h \in \mathcal{RT}_p(K) \\ \nabla \cdot \mathbf{w}_h = 0}} \|\nabla \times \mathbf{v} - \mathbf{w}_h\|_K$$

and define $\iota_h|_K$ by the **elementwise (constrained) projection**

$$\iota_h|_K := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(K) \\ \nabla \times \mathbf{v}_h = \tau_h}} \|\mathbf{v} - \mathbf{v}_h\|_K$$

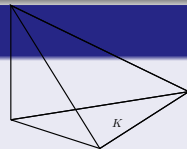
(discrete but nonconforming (tangential-trace discontinuous)).

- 2 Obtain $\mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)$ by applying the **flux equilibration procedure** to ι_h ; in particular, $\mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) := \mathbf{h}_h := \sum_{a \in \mathcal{V}_h} \mathbf{h}_h^a$, where $\mathbf{h}_h^a$ are obtained by **local energy minimizations** on the patch subdomains ω_a .

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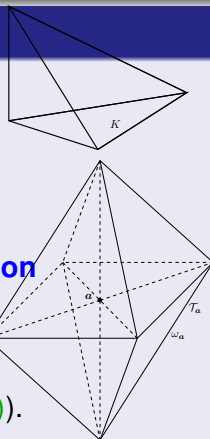
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A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$

Theorem (A stable local commuting projector $\mathbf{P}_h^{p,\text{curl}}$)

$\mathbf{P}_h^{p,\text{curl}}$ is a **commuting projector** since

$$\begin{aligned}\nabla \times \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) &= \mathbf{P}_h^{p,\text{div}}(\nabla \times \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{0,\text{N}}(\text{curl}, \Omega), \\ \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}) &= \mathbf{v} & \forall \mathbf{v} \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,\text{N}}(\text{curl}, \Omega).\end{aligned}$$

Moreover, it has **local-best approximation properties** and is **L^2 stable** up to data oscillation, since, for all $\mathbf{v} \in \mathbf{H}_{0,\text{N}}(\text{curl}, \Omega)$ and $K \in \mathcal{T}_h$,

$$\begin{aligned}\|\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 &+ \left(\frac{h_K}{p+1} \|\nabla \times (\mathbf{v} - \mathbf{P}_h^{p,\text{curl}}(\mathbf{v}))\|_K \right)^2 \\ &\lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \min_{\mathbf{v}_h \in \mathcal{N}_p(K')} \|\mathbf{v} - \mathbf{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\}, \\ \|\mathbf{P}_h^{p,\text{curl}}(\mathbf{v})\|_K^2 &\lesssim_p \sum_{K' \in \mathcal{T}_K} \left\{ \|\mathbf{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_{K'} \right)^2 \right\}.\end{aligned}$$

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Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

bigger $\approx$ smaller

Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

$$\min_{\text{smaller space}} \text{with curl constraints} \approx \min_{\text{bigger space}} \text{without curl constraints}$$

Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

$$\min_{\text{conforming Nédélec space with curl constraints}} \approx \min_{\text{broken Nédélec space without curl constraints}}$$

Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) \\ \nabla \times \mathbf{v}_h = \mathbf{P}_h^{p,\text{div}}(\nabla \times \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \text{curl constraint}}} \\ \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\left[\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2 \right]}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}}.$$

- $\approx_p$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and p

Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

Theorem (Constrained equivalence in $\mathbf{H}(\text{curl})$)

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$$\underbrace{\min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) \\ \nabla \times \mathbf{v}_h = \mathbf{P}_h^{p,\text{div}}(\nabla \times \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \text{curl constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\left[\min_{\mathbf{v}_h \in \mathcal{N}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \mathbf{v} - \Pi_{\mathcal{RT}}^p(\nabla \times \mathbf{v})\|_K \right)^2 \right]}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}}.$$

- $\approx$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and p

Global-best approximation $\approx$ local-best approximation in $\mathbf{H}(\text{curl})$

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- 4 A stable local commuting projector in $\mathbf{H}(\text{curl})$
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- 6 Approximation error estimates in $\mathbf{H}(\text{curl})$**
- 7 Conclusions

Approximation error estimates

Theorem (Local hp -optimal approximation under minimal Sobolev regularity)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{curl}, \Omega)$ with

$$\mathbf{v}|_K \in \mathbf{H}^{s_K}(K), \quad (\nabla \times \mathbf{v})|_K \in [\mathcal{P}_{p-1}(K)]^3 \quad \forall K \in \mathcal{T}_h$$

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for $s_K \geq 0$. Then

$$\begin{aligned} & \min_{\substack{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) \\ \nabla \times \mathbf{v}_h = \nabla \times \mathbf{v}}} \|\mathbf{v} - \mathbf{v}_h\|^2 \\ & \leq C(\kappa_{\mathcal{T}_h}, s) \sum_{K \in \mathcal{T}_h} \left(\frac{h_K^{\min\{p+1, s_K\}}}{(p+1)^{s_K}} \|\mathbf{v}\|_{\mathbf{H}^{s_K}(K)} \right)^2. \end{aligned}$$

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for $s_K \geq 0$ and $t_K \geq 0$. Then

$$\begin{aligned} & \min_{\mathbf{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega)} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times (\mathbf{v} - \mathbf{v}_h)\|_K \right)^2 \right] \\ & \leq C(\kappa_{\mathcal{T}_h}, s, t) \sum_{K \in \mathcal{T}_h} \left[\left(\frac{h_K^{\min\{p+1, s_K\}}}{(p+1)^{s_K}} \|\mathbf{v}\|_{\mathbf{H}^{s_K}(K)} \right)^2 + \left(\frac{h_K}{p+1} \frac{h_K^{\min\{p+1, t_K\}}}{(p+1)^{t_K}} \|\nabla \times \mathbf{v}\|_{\mathbf{H}^{t_K}(K)} \right)^2 \right]. \end{aligned}$$

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 CHAUMONT-FRELET T., VOHRALÍK M. Equivalence of local-best and global-best approximations in $\mathbf{H}(\text{curl})$. *Calcolo* **58** (2021), 53.

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Thank you for your attention!

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Three key Sobolev spaces

 $H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $L^2(\Omega)$,
 $H^1(\Omega) := \{\mathbf{v} \in L^2(\Omega); \nabla \mathbf{v} \in L^2(\Omega)\}$

 $H(\text{curl}, \Omega)$

vector-valued $L^2(\Omega)$ functions with weak curls in $L^2(\Omega)$,
 $H(\text{curl}, \Omega) := \{\mathbf{v} \in L^2(\Omega); \nabla \times \mathbf{v} \in L^2(\Omega)\}$

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Three key Sobolev spaces with BCs

$$H_{0,N}^1(\Omega)$$

$$H_{0,N}^1(\Omega) := \{ \mathbf{v} \in H^1(\Omega); \mathbf{v} = 0 \text{ on } \Gamma_N \}$$

$$\mathbf{H}_{0,N}(\text{curl}, \Omega)$$

$$\mathbf{H}_{0,N}(\text{curl}, \Omega) := \{ \mathbf{v} \in \mathbf{H}(\text{curl}, \Omega); \mathbf{v} \times \mathbf{n}_\Omega = 0 \text{ on } \Gamma_N \text{ in appropriate sense} \}$$

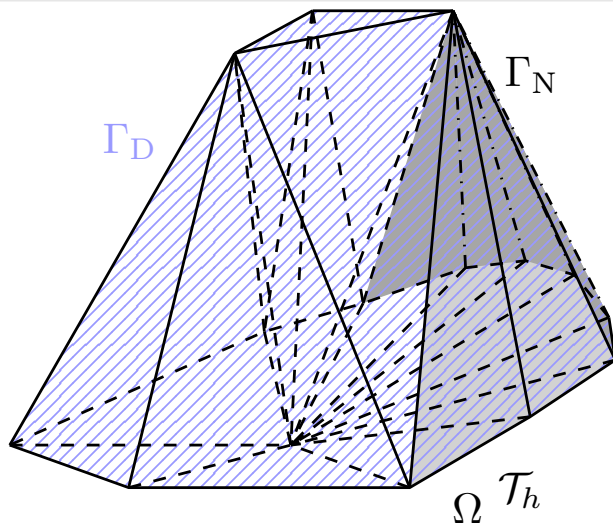
$$\mathbf{H}_{0,N}(\text{div}, \Omega)$$

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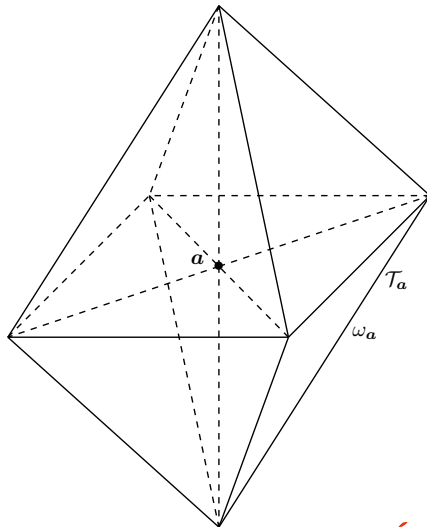
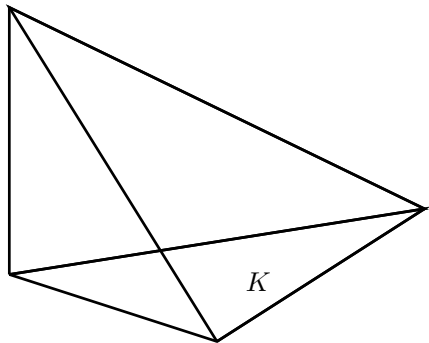
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Meshes, elements, and patches



Meshes, elements, and patches

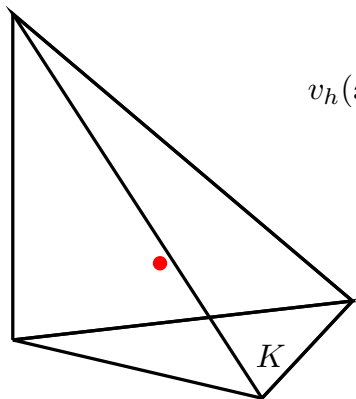


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Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

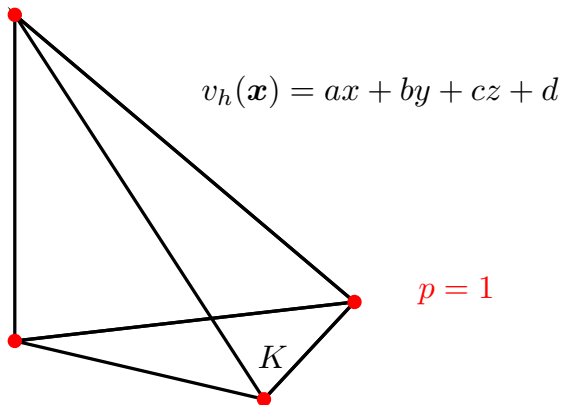
Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$



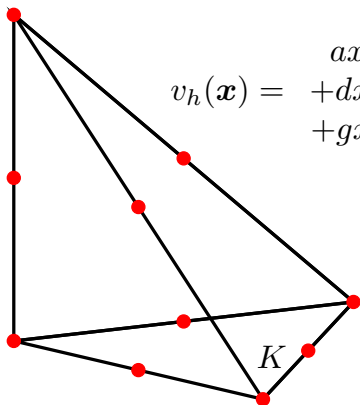
$$v_h(\mathbf{x}) = a$$

$$p = 0$$

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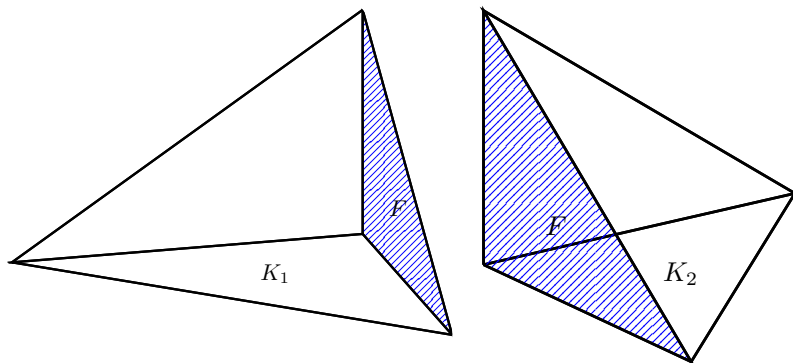


$$v_h(\mathbf{x}) = \begin{aligned} &ax^2 + by^2 + cz^2 \\ &+ dxy + eyz + fzx \\ &+ gx + hy + iz + j \end{aligned}$$

$p = 2$

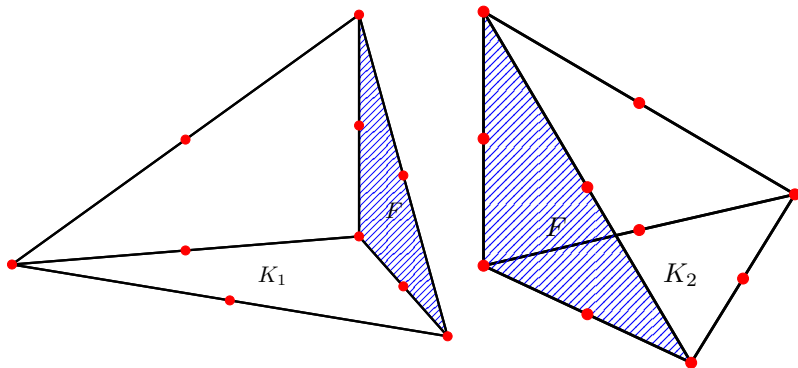
Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$



- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

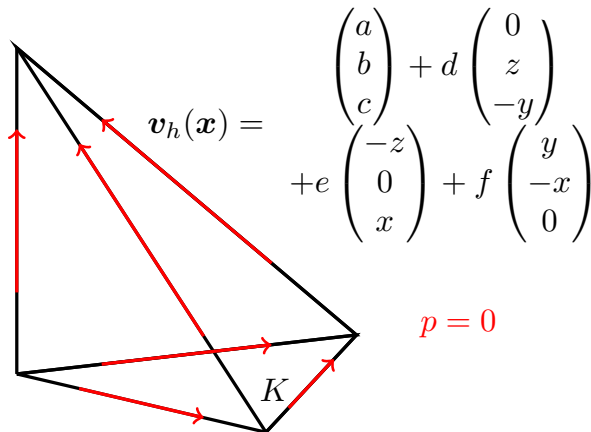
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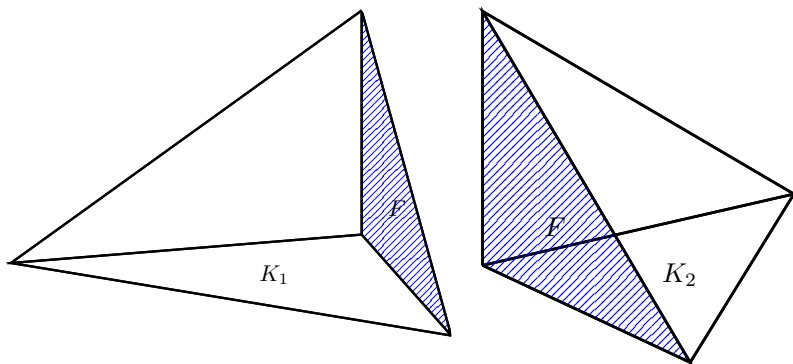
- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
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Nédélec space $\mathcal{N}_p(K) := [\mathcal{P}_p(K)]^3 + \mathbf{x} \times [\mathcal{P}_p(K)]^3, p \geq 0$

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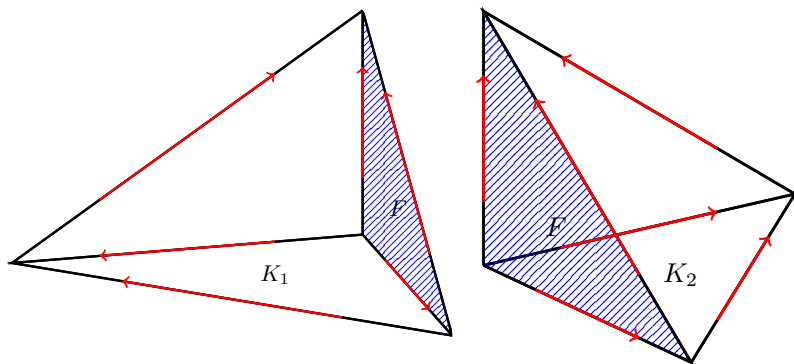


Nédélec piecewise polynomial space $\mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{curl}, \Omega)$, $p \geq 0$



- $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{curl}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{curl}, K_2)$, and $(\mathbf{v}|_{K_1} \times \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \times \mathbf{n}_F)|_F$ in appropriate sense
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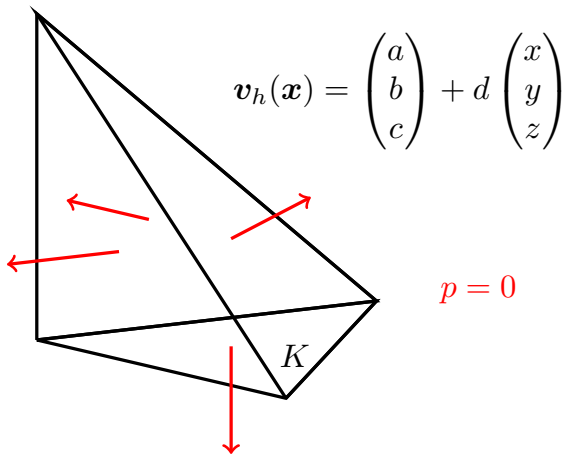
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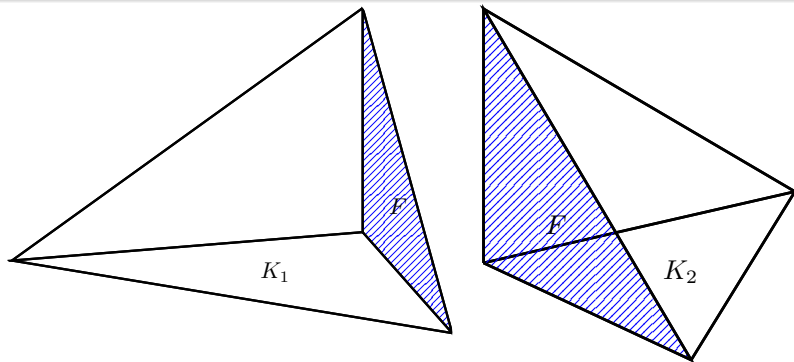
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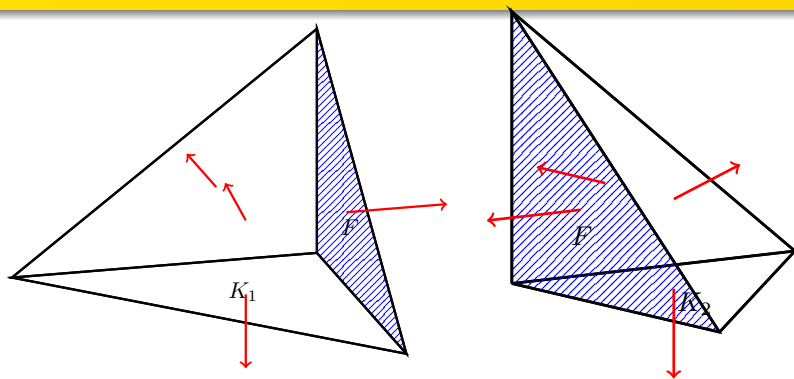


Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega)$, $p \geq 0$



- $\mathbf{v} \in \mathbf{H}(\text{div}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{div}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{div}, K_2)$, and $(\mathbf{v}|_{K_1} \cdot \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \cdot \mathbf{n}_F)|_F$ in appropriate sense
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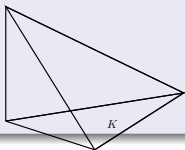
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$H(\text{curl})$ polynomial extensions on a tetrahedron

Theorem ($H(\text{curl})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010);

Demkowicz, Gopalakrishnan, & Schöberl (2009); Braess, Pillwein, & Schöberl (2009); Chaumont-Frelet, Ern, & Vohralík (2020))

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K . Then, for every polynomial degree $p \geq 0$, for all $\mathbf{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla \cdot \mathbf{r}_K = 0$, and for all $\mathbf{r}_{\mathcal{F}} \in \mathcal{N}_p^{\tau}(\Gamma_{\mathcal{F}})$ such that $\mathbf{r}_K \cdot \mathbf{n}_F = \text{curl}_F(\mathbf{r}_F)$ for all $F \in \mathcal{F}$, there holds



$$\min_{\substack{\mathbf{v}_p \in \mathcal{N}_p(K) \\ \nabla \times \mathbf{v}_p = \mathbf{r}_K \\ \mathbf{v}_p|_{\mathcal{F}} = \mathbf{r}_{\mathcal{F}}}} \|\mathbf{v}_p\|_K \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{curl}, K) \\ \nabla \times \mathbf{v} = \mathbf{r}_K \\ \mathbf{v}|_{\mathcal{F}} = \mathbf{r}_{\mathcal{F}}}} \|\mathbf{v}\|_K.$$

Comments

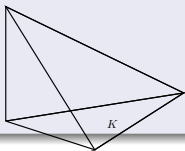
- C_{st} only depends on the **shape-regularity** of K
- **p -robustness**: for (pw) p -polynomial data $\mathbf{r}_K, \mathbf{r}_{\mathcal{F}}$, minimization over $\mathcal{N}_p(K)$ is up to C_{st} **as good as** minimization over the entire $\mathbf{H}(\text{curl}, K)$
- extension to an **edge patch**: Chaumont-Frelet, Ern, & Vohralík (2021)
- extension to a **vertex patch**: Chaumont-Frelet & Vohralík (2023)

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Comments

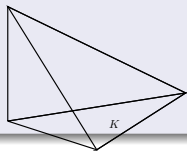
- C_{st} only depends on the **shape-regularity** of K
- **p -robustness**: for (pw) p -polynomial data $\mathbf{r}_K, \mathbf{r}_{\mathcal{F}}$, minimization over $\mathcal{N}_p(K)$ is up to C_{st} **as good as** minimization over the entire $\mathbf{H}(\text{curl}, K)$
- extension to an **edge patch**: Chaumont-Frelet, Ern, & Vohralík (2021)
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$H(\text{curl})$ polynomial extensions on a tetrahedron and on patches

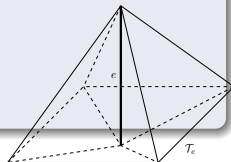
Theorem ($H(\text{curl})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010);

Demkowicz, Gopalakrishnan, & Schöberl (2009); Braess, Pillwein, & Schöberl (2009); Chaumont-Frelet, Ern, & Vohralík (2020))

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K . Then, for every polynomial degree $p \geq 0$, for all $\mathbf{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla \cdot \mathbf{r}_K = 0$, and for all $\mathbf{r}_{\mathcal{F}} \in \mathcal{N}_p^{\tau}(\Gamma_{\mathcal{F}})$ such that $\mathbf{r}_K \cdot \mathbf{n}_F = \text{curl}_F(\mathbf{r}_F)$ for all $F \in \mathcal{F}$, there holds



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Comments

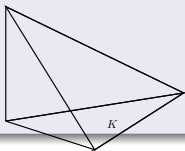
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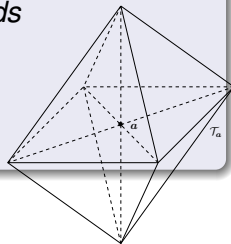
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