A posteriori estimates: heat equation

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Outline

Model parabolic problem

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega \end{aligned}$$

Spaces

$$\begin{aligned} X &:= L^2(0,T;H_0^1(\Omega)), \\ \|v\|_X^2 &:= \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \\ Y &:= L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)), \\ \|v\|_Y^2 &:= \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2 + \|\nabla v\|^2 \,\mathrm{d}t + \|v(T)\|^2 \end{aligned}$$

Weak solution

Find $u \in Y$ with $u(0) = u_0$ such that

$$\int_0^T \langle \partial_t u, v\rangle + (\nabla u, \nabla v)\,\mathrm{d}t = \int_0^T (f,v)\,\mathrm{d}t \qquad \forall v \in X.$$

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An optimal a posteriori estimate for evolutive problems

Guaranteed upper bound

- $\|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{?,\text{neighbors of } K\times(t^{n-1},t^n)}$
- optimal space-time mesh refinement
- **local** in **time** and in **space** error lower bound

Robustness

- C_{eff} independent of data, domain Ω, **final time** T, meshes, solution u, **polynomial degrees** of $u_{h\tau}$ in space and in time

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|^2_{?,\Omega\times(0,T)} \searrow 1$
- overestimation factor goes to one with meshes size

Small evaluation cost

- estimators can be evaluated cheaply (locally)

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- Bieterman and Babuška (1982), introduction
- Picasso / Verfürth (1998), work with the energy norm X:
  - upper bound $\|u - u_{h\tau}\|_X^2 \leq C \sum_{n=1}^N \sum_{K\in\mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
  - constrained lower bound (h and τ strongly linked)
- Verfürth (2003) (cf. also Bergam, Bernardi, and Mghazli (2005)), work with the Y norm:
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  - robustness with respect to the final time
  - efficiency local in time but global in space
- Eriksson and Johnson (1991), duality techniques & Makridakis and Nochetto (2003), elliptic reconstruction: $L^2(L^2)$ / $L^\infty(L^2)$ / $L^\infty(L^\infty)$ / higher-order norms
- Makridakis and Nochetto (2006): τq estimates
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Equivalence between error and residual (steady case)

Laplace equation

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$, the misfit of u_h in the weak formulation:

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v) \qquad v \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{v \in H_0^1(\Omega);\, \|\nabla v\|=1} \langle \mathcal{R}(u_h), v \rangle$$

Energy error is the dual norm of the residual

$$\|\nabla(u - u_h)\| = \sup_{v \in H_0^1(\Omega);\, \|\nabla v\|=1} (\nabla(u - u_h), \nabla v) = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

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Bounding and localizing dual norms (steady case)

- $V := W_0^{1,p}(\Omega)$, $p > 1$, bounded linear functional $\mathcal{R} \in V'$
- norm $\|\mathcal{R}\|_{V'} := \sup_{v \in V;\, \|\nabla v\|_p = 1} \langle \mathcal{R}, v \rangle_{V',V}$
- localized energy space $V^{\mathbf{a}} := W_0^{1,p}(\omega_{\mathbf{a}})$ for $\mathbf{a} \in \mathcal{V}_h$
- restriction of $\mathcal{R}$ to $(V^{\mathbf{a}})'$ (zero extension of $v \in V^{\mathbf{a}}$),

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Theorem (Bounding and localizing $\|\mathcal{R}\|_{V'}$)

There holds

$$\|\mathcal{R}\|_{V'} \leq (d+1) C_{\rm cont,PF} \left\{ \frac{1}{(d+1)} \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \text{ if } \underbrace{\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle = 0}_{\text{orthogonality}} \ \forall \mathbf{a} \in \mathcal{V}_h^{\rm int};$$

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Outline

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Error and residual in the unsteady case

Theorem (Parabolic inf–sup identity)

For every $\varphi \in Y$*, we have*

$$\|\varphi\|_Y^2 = \left[\sup_{v \in X,\, \|v\|_X = 1} \int_0^T \langle \partial_t \varphi, v\rangle + (\nabla \varphi, \nabla v)\,\mathrm{d}t\right]^2 + \|\varphi(0)\|^2.$$

Residual of $u_{h\tau} \in Y$

- $\mathcal{R}(u_{h\tau}) \in X'$, the misfit of $u_{h\tau}$ in the weak formulation:

$$\langle \mathcal{R}(u_{h\tau}), v\rangle := \int_0^T (f, v) - \langle \partial_t u_{h\tau}, v\rangle - (\nabla u_{h\tau}, \nabla v)\,\mathrm{d}t$$

- dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{v \in X,\, \|v\|_X = 1} \langle \mathcal{R}(u_{h\tau}), v\rangle$$

Y norm error is the dual X norm of the residual + IC error

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Outline

High-order time discretization & Radau reconstruction

CG of degree *p* in space & DG of degree *q* in time

Find $u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n}(I_n; V_h^n)$ with $u_{h\tau}(0) = \Pi_h u_0$ such that

$$\int_{I_n} (\partial_t u_{h\tau}, v_{h\tau}) + (\nabla u_{h\tau}, \nabla v_{h\tau})\,\mathrm{d}t - ((\!| u_{h\tau} |\!)_{n-1}, v_{h\tau}(t_{n-1}^+))$$
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- *p*-degree **continuous** finite elements in **space**
$$V_h^n := \left\{ v_h \in H_0^1(\Omega),\ v_h|_K \in \mathcal{P}_{p_K}(K) \quad \forall K \in \mathcal{T}^n \right\}$$
- *q*-degree **discontinuous** polynomials in **time**
$$\mathcal{Q}_{q_n}(I_n; V) := \left\{ V\text{-valued pols of degree at most } q_n \text{ over } I_n \right\}$$

Radau reconstruction $\mathcal{I}u_{h\tau} \in Y$, $\mathcal{I}u_{h\tau}|_{I_n} \in \mathcal{Q}_{q_n+1}(I_n; \widetilde{V_h^n})$

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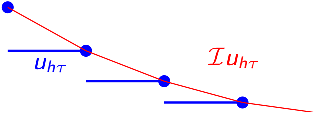

High-order time discretization & Radau reconstruction

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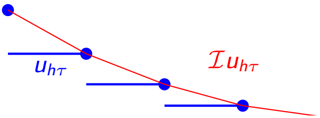

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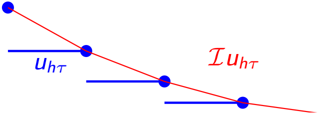

Outline

Missing Galerkin orthogonality

Situation

- $u_{h\tau} \notin Y \Rightarrow$ impossible to estimate $\|u - u_{h\tau}\|_Y$
- $\mathcal{I}u_{h\tau} \in Y \Rightarrow$ **error** $\|u - \mathcal{I}u_{h\tau}\|_Y$
- but $\mathcal{I}u_{h\tau}$ misses the Galerkin orthogonality:

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Remedy

Remedy

- augment the norm: $\|v\|^2_{\mathcal{E}_Y} := \|\mathcal{I}v\|^2_Y + \|v - \mathcal{I}v\|^2_X$, $v \in Y + V_{h\tau}$
- $\mathcal{I}u = u \Rightarrow$

$$\|u - u_{h\tau}\|^2_{\mathcal{E}_Y} = \|u - \mathcal{I}u_{h\tau}\|^2_Y + \|u_{h\tau} - \mathcal{I}u_{h\tau}\|^2_X$$

- we are **adding** to Y norm the **time jumps** in X norm:

$$\int_{I_n} \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2 \,\mathrm{d}t = \frac{\tau_n(q_n+1)}{(2q_n+1)\,(2q_n+3)} \|\nabla [\![u_{h\tau}]\!]_{n-1}\|^2$$

Equivalence between the Y and $\mathcal{E}_Y$ norms

Global equivalence

$$\|u - \mathcal{I}u_{h\tau}\|_Y \leq \|u - u_{h\tau}\|_{\mathcal{E}_Y} \leq 3\|u - \mathcal{I}u_{h\tau}\|_Y$$

- holds if there is no source term oscillation or no coarsening
- otherwise an additional source term oscillation or coarsening term
- the two norms still may **differ locally**

Outline

Equilibrated flux reconstruction

Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\boldsymbol{v}_h = g_{h\tau}^{\mathbf{a},n}}} \int_{I_n} \|\boldsymbol{v}_h + \boldsymbol{\tau}_{h\tau}^{\mathbf{a},n}\|_{\omega_{\mathbf{a}}}^2 \,\mathrm{d}t,$$

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Then set

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Definition (Equilibrated flux reconstruction)

For each time-step interval I_n and for each vertex $\mathbf{a} \in \mathcal{V}^n$, let

$$\boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_{h\tau}^{\mathbf{a},n} \\ \nabla\cdot\boldsymbol{v}_h = g_{h\tau}^{\mathbf{a},n}}} \int_{I_n} \|\boldsymbol{v}_h + \boldsymbol{\tau}_{h\tau}^{\mathbf{a},n}\|_{\omega_{\mathbf{a}}}^2 \,\mathrm{d}t,$$

where

$$\boldsymbol{\tau}_{h\tau}^{\mathbf{a},n} := \psi_{\mathbf{a}} \nabla u_{h\tau}|_{\omega_{\mathbf{a}} \times I_n},$$
$$g_{h\tau}^{\mathbf{a},n} := \psi_{\mathbf{a}} \left(\Pi_{h\tau}^{\mathbf{a},n} f - \partial_t \mathcal{I} u_{h\tau}\right)|_{\omega_{\mathbf{a}} \times I_n} - \nabla\psi_{\mathbf{a}} \cdot \nabla u_{h\tau}|_{\omega_{\mathbf{a}} \times I_n}.$$

Then set

$$\boldsymbol{\sigma}_{h\tau} := \sum_{n=1}^{N} \sum_{\mathbf{a} \in \mathcal{V}^n} \boldsymbol{\sigma}_{h\tau}^{\mathbf{a},n}.$$

Comment

- a priori a local space-time problem, $\boldsymbol{V}_{h\tau}^{\mathbf{a},n} := \mathcal{Q}_{q_n}(I_n; \boldsymbol{V}_h^{\mathbf{a},n})$
- can be uncoupled to q_n elliptic problems posed in $\boldsymbol{V}_h^{\mathbf{a},n}$

Guaranteed upper bound

Theorem (Guaranteed upper bound)

In the absence of data oscillation, there holds

$$\|u - u_{h\tau}\|_{\mathcal{E}_Y}^2 \leq \sum_{n=1}^{N} \sum_{K \in \mathcal{T}^n} \int_{I_n} \|\boldsymbol{\sigma}_{h\tau} + \nabla \mathcal{I} u_{h\tau}\|_K^2 + \|\nabla (u_{h\tau} - \mathcal{I} u_{h\tau})\|_K^2 \,\mathrm{d}t.$$

Outline

Local space-time efficiency and robustness

Local error contributions

$$|u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}} = \int_{I_n} \|\partial_t(u - \mathcal{I}u_{h\tau})\|^2_{H^{-1}(\omega_{\mathbf{a}})} + \|\nabla(u - \mathcal{I}u_{h\tau})\|^2_{\omega_{\mathbf{a}}} \,\mathrm{d}t + \frac{\tau_n(q_n+1)}{(2q_n+1)\,(2q_n+3)} \|\nabla(\!| u_{h\tau} |\!)_{n-1}\|^2_{\omega_{\mathbf{a}}}$$

Theorem (Local space-time efficiency and robustness)

For each time-step interval I_n and for each element $K \in \mathcal{T}^n$, there holds, in the absence of data oscillation,

$$\int_{I_n} \|\sigma_{h\tau} + \nabla \mathcal{I}u_{h\tau}\|^2_K + \|\nabla(u_{h\tau} - \mathcal{I}u_{h\tau})\|^2_K \mathrm{d}t \le C^2_{\mathrm{eff}} \sum_{\mathbf{a}\in\mathcal{V}_K} |u - u_{h\tau}|^2_{\mathcal{E}_Y^{\mathbf{a},n}}.$$

Comments

- **local** in **space** and **time**
- C_{eff} only depends on shape regularity $\Rightarrow$ **robustness**
- **no requirement on coarsening** between $\mathcal{T}^{n-1}$ and $\mathcal{T}^n$

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Conclusions and future directions

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- local **space-time** efficiency is possible (adding the time jumps to the Y-norm error)
- **robustness** with respect to both spatial and temporal degree
- arbitrarily large **coarsening** allowed

Future directions

- estimates in the X norm
- nonlinear problems

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Thank you for your attention!

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