A posteriori error estimates: Laplace equation

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Inria Paris & Ecole des Ponts

Kanpur, July 17–21, 2017

Outline

1. Introduction
2. A posteriori estimates based on potential & flux reconstruction
   - Guaranteed upper bound in a unified framework
   - Potential and flux reconstructions
   - Polynomial-degree-robust local efficiency
   - Applications
   - Numerical illustration
3. Algebraic estimates and stopping criteria for iterative solvers
   - Upper and lower bounds on the algebraic error
   - Bounds on the total error
   - Stopping criteria
   - Numerical illustration
4. Conclusions and outlook

Optimal a posteriori error estimate

Guaranteed upper bound

- $\|u - u_h\|^2_{?,\Omega} \leq \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K(u_h) \leq C_{\mathrm{eff}} \|u - u_h\|_{?,\text{neighbors of } K}$
- **local** error lower bound (optimal space mesh refinement)

Robustness

- C_{eff} independent of data, domain Ω, meshes, solution u, **polynomial degree** of u_h

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 / \|u - u_h\|^2_{?,\Omega} \searrow 1$
- overestimation factor goes to one with meshes size

Small evaluation cost

- estimators $\eta_K(u_h)$ can be evaluated cheaply (locally)

Error components identification

- $\eta_K(u_h)$ can distinguish the different error components

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Laplace model problem

Model problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (primal variable constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
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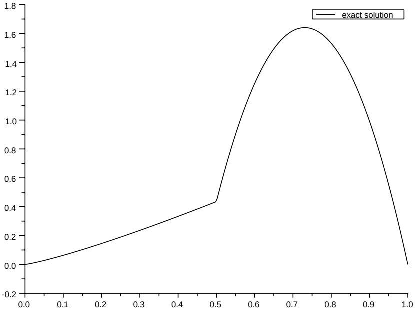
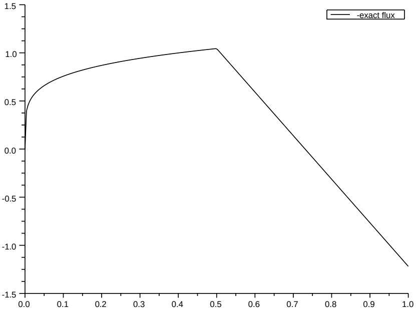
Exact solution and flux

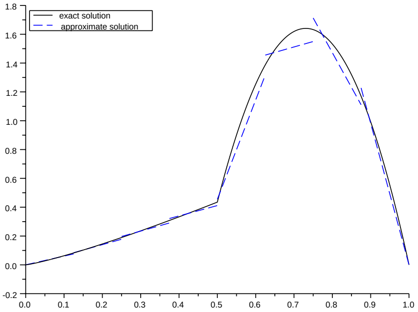


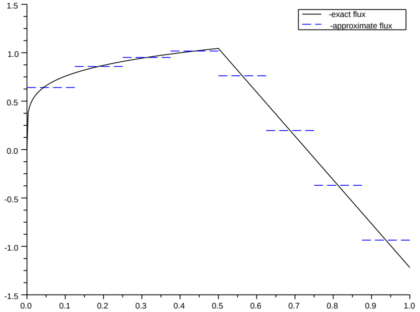


Solution u is continuous

Flux $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u$ is continuous

Approximate solution and flux

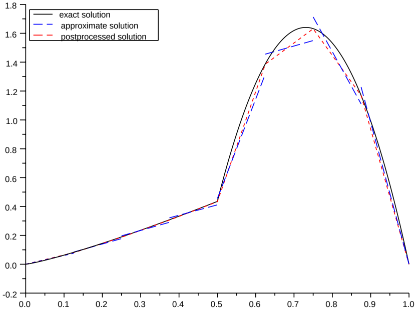




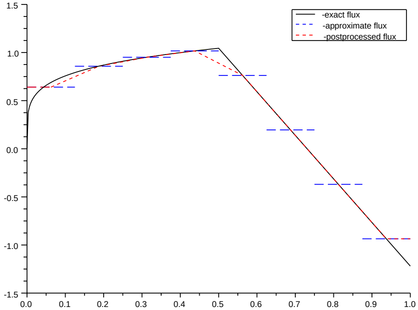
Approximate solution u_h is not necessarily continuous

Approximate flux $-\underline{\mathbf{K}}\nabla u_h$ is not necessarily continuous

Potential and flux reconstructions



Potential reconstruction



Flux reconstruction

Outline

Guaranteed a posteriori error estimate

Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega), v|_K \in H^1(K)\, \forall K \in \mathcal{T}_h\}$ *be arbitrary*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u-u_h)\|^2 \le \sum_{K\in\mathcal{T}_h}\Big(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}}\Big)^2 + \sum_{K\in\mathcal{T}_h}\underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

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Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\| = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s, definition of u:

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Proof II

Proof (continuation).

- nonconformity upper bound:

$$\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\| \le \|\nabla(u_h - s_h)\|$$

- adding and subtracting equilibrated flux, Green theorem:

$$(f,\varphi) - (\nabla u_h, \nabla\varphi) = (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)$$

$$(f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) = \sum_{K\in\mathcal{T}_h} (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi \quad)_K$$

$$\le \sum_{K\in\mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K$$

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Proof II

Proof (continuation).

- nonconformity upper bound:

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Outline

Global potential and flux reconstructions

Ideally

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

✓ computable, discrete spaces $V_h \subset H_0^1(\Omega)$, $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$

✗ too expensive, **global** minimization problems (the hypercircle method . . .)

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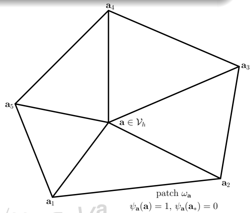
Local potential reconstruction

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013), EV (2015))

For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}.$$



Equivalent form

Find $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that

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Key ideas

- **local** minimizations
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- $V_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})$: homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}}$
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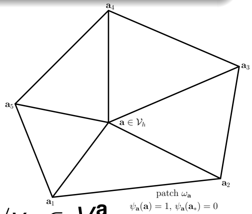

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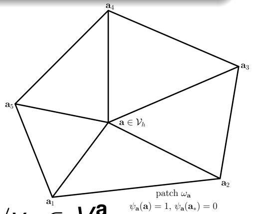

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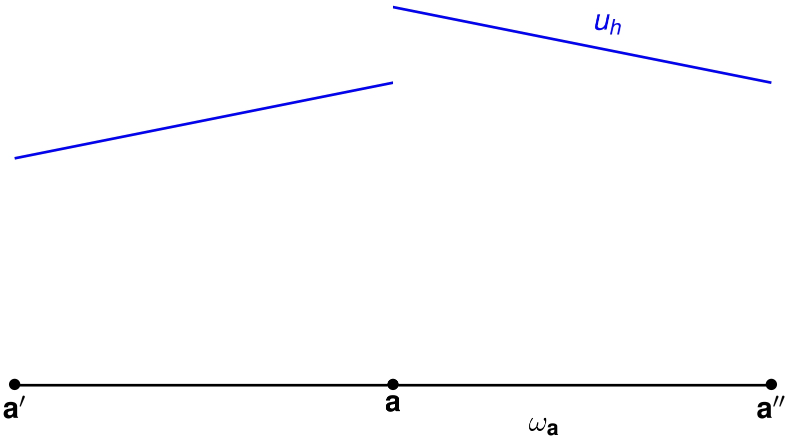

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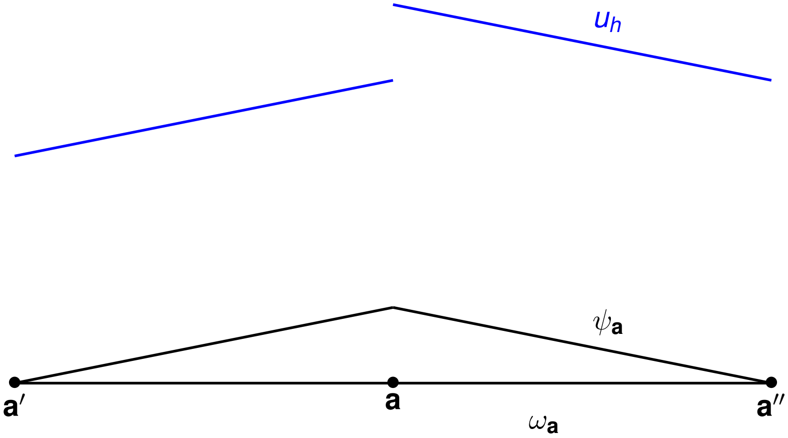

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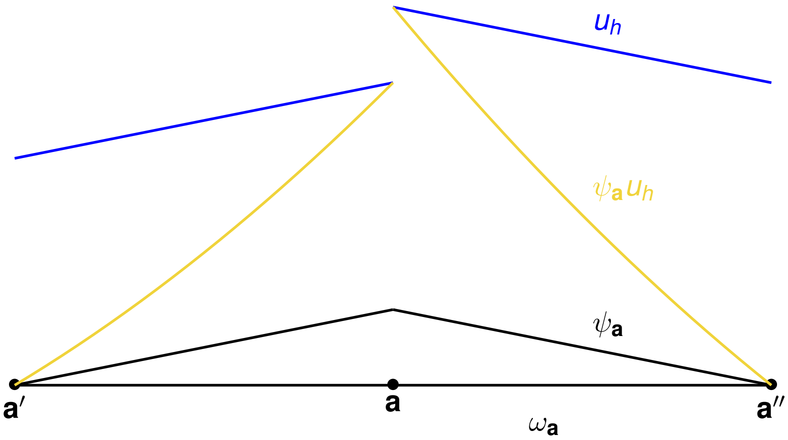

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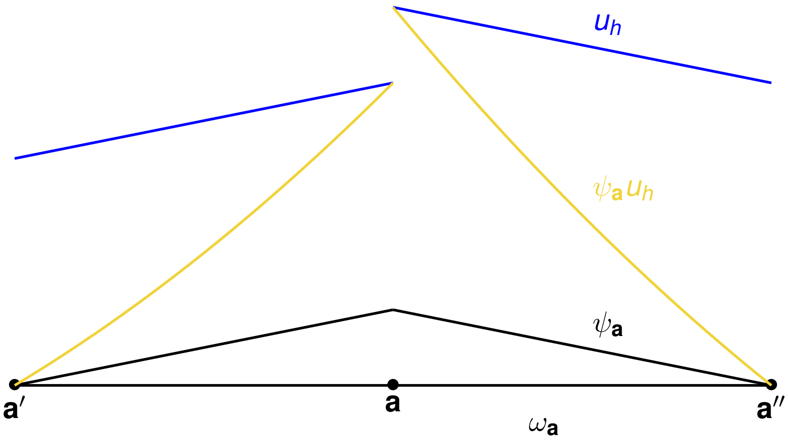
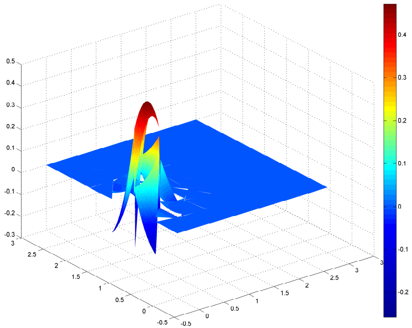


Potential reconstruction in 2D



Potential u_h

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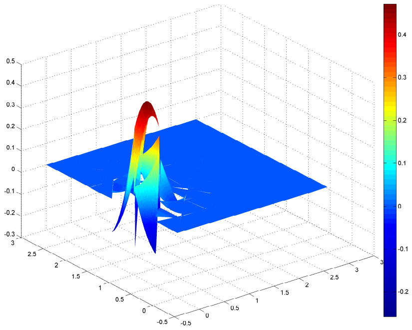

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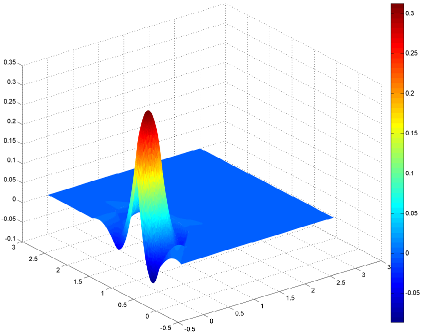

Potential reconstruction s_h

Local flux reconstructions

Assumption A (Galerkin orthogonality wrt hat functions)

There holds

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

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For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local mixed FE problem**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}.$$

Key points

- $\mathbf{V}_h^{\mathbf{a}} \subset \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$: homogeneous Neumann BC on $\partial\omega_{\mathbf{a}}$
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$\mathbf{H}(\mathrm{div},\Omega)$-conformity

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Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs $(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0 \Rightarrow$

Divergence

- Neumann compatibility condition gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

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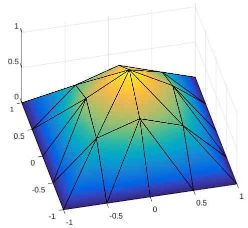

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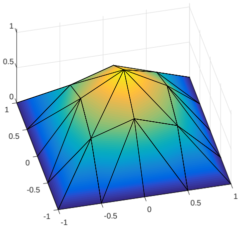

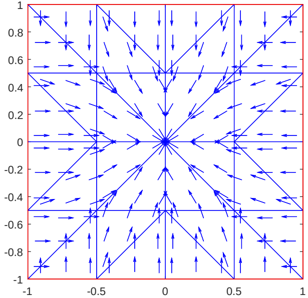

Flux $-\nabla u_h$

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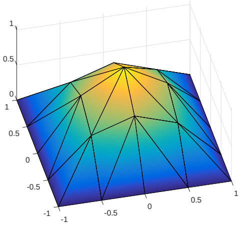

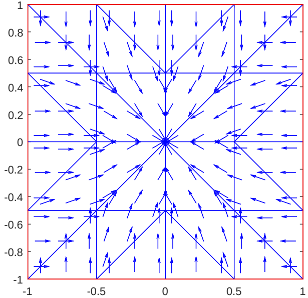

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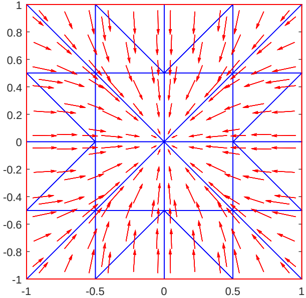

Flux reconstruction σ_h

Outline

1. Introduction
2. A posteriori estimates based on potential & flux reconstruction
   - Guaranteed upper bound in a unified framework
   - Potential and flux reconstructions
   - Polynomial-degree-robust local efficiency
   - Applications
   - Numerical illustration
3. Algebraic estimates and stopping criteria for iterative solvers
   - Upper and lower bounds on the algebraic error
   - Bounds on the total error
   - Stopping criteria
   - Numerical illustration
4. Conclusions and outlook

Polynomial-degree-robust efficiency

Assumption B (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The degrees of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.

Theorem (Polynomial-degree-robust efficiency Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015)**)**

Let u be the weak solution and let Assumptions A and B hold. Then there exists constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

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Remarks

- equivalence error–estimate
- maximal overestimation factor guaranteed

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- equivalence error–estimate
- maximal overestimation factor guaranteed

Polynomial-degree-robust efficiency

Assumption B (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The degrees of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.

Theorem (Polynomial-degree-robust efficiency Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015))

Let u be the weak solution and let Assumptions A and B hold. Then there exists constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}},$$

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Existing results

Fundamental results on a reference tetrahedron

- Costabel & McIntosh (2010): bounded right inverse of the divergence operator for polynomial volume data
- Demkowicz, Gopalakrishnan, Schöberl (2009, 2012): polynomial extensions in H^1 and $\mathbf{H}(\mathrm{div})$ for polynomial boundary data

Stable broken $\mathbf{H}(\mathrm{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
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Potentials (any BCs, physical tetrahedron)

Lemma (H^1 polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}_h$, $\mathcal{E}_K^{\mathrm{D}} \subset \mathcal{E}_K$. Let $r \in \mathbb{P}_p(\mathcal{E}_K^{\mathrm{D}})$ be continuous on $\mathcal{E}_K^{\mathrm{D}}$. Then for C only depending on the shape regularity of K,

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \leq C \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_e \text{ on all } e \in \mathcal{E}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} .$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_e && \text{on all } e \in \mathcal{E}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \mathbf{n}_K &= 0 && \text{on all } e \in \mathcal{E}_K \setminus \mathcal{E}_K^{\mathrm{D}}. \end{aligned}$$

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A graph result for patch enumerations in 3D (shellability of polytopes, e.g. Ziegler, Lectures on Polytopes)

Two families of faces

- already visited faces: $\mathcal{E}_i^\sharp := \{e \in \mathcal{E}_{\mathbf{a}}^{\mathrm{int}},\ e = \partial K_i \cap \partial K_j,\ j < i\}$
- yet unvisited faces: $\mathcal{E}_i^\flat := \mathcal{E}_{\mathbf{a}}^{\mathrm{int}} \cap \mathcal{E}_{K_i} \setminus \mathcal{E}_i^\sharp$
- $|\mathcal{E}_i^\flat| + |\mathcal{E}_i^\sharp| = 3$, $\mathcal{E}_1^\sharp = \emptyset$, and $\mathcal{E}_{|\mathcal{T}_{\mathbf{a}}|}^\flat = \emptyset$

Lemma (Interior patch enumeration)

There exists an enumeration of the patch $\mathcal{T}_{\mathbf{a}}$ so that

(i) *If $|\mathcal{E}_i^\sharp| \geq 2$ with $\{e_i^1, e_i^2\} \subset \mathcal{E}_i^\sharp$, then $K_j \in \mathcal{T}_{e_i^1 \cap e_i^2} \setminus \{K_i\}$ implies $j < i$.*

(ii) *For all $1 < i < |\mathcal{T}_{\mathbf{a}}|$, $|\mathcal{E}_i^\sharp| \in \{1, 2\}$.*

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Flux case

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$$r_K := \psi_{\mathbf{a}} (f + \Delta u_h)|_K$$

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Outline

1. Introduction
2. A posteriori estimates based on potential & flux reconstruction
   - Guaranteed upper bound in a unified framework
   - Potential and flux reconstructions
   - Polynomial-degree-robust local efficiency
   - Applications
   - Numerical illustration
3. Algebraic estimates and stopping criteria for iterative solvers
   - Upper and lower bounds on the algebraic error
   - Bounds on the total error
   - Stopping criteria
   - Numerical illustration
4. Conclusions and outlook

Conforming finite elements

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Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

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$$\nabla_{\mathrm{d}} u_h := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

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- jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

$$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

- $\Rightarrow$ modified Galerkin orthogonality

$$(\nabla_{\mathrm{d}} u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+\sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$

✓ Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

$$\nabla_{\rm d} u_h := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

$$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

$$(\nabla_{\rm d} u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\rm int}$$

Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$
\begin{aligned}
(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\
(\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad && \forall q_h \in Q_h.
\end{aligned}
$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \forall e \in \mathcal{E}_h$$

✓ Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization

✓ no jumps

Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\boldsymbol{\sigma}_h, q_h) = (f, q_h) \qquad \forall q_h \in Q_h.$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

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Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\boldsymbol{\sigma}_h, q_h) = (f, q_h) \qquad \forall q_h \in Q_h.$$

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$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \forall e \in \mathcal{E}_h$$

✓ Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization

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Outline

Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform h refinement

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Discretization

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform h refinement

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d}u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.01

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d} u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.17
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	1.09
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	1.06
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.04
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.06
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
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$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
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Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_d(u-u_h)\|$	$\|\nabla_d u_h+\sigma_h\|$	η_{osc}	$\|\nabla_d(u_h-s_h)\|$	η	I^{eff}
h_0	1	1.07E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	**1.17**
$\approx h_0/2$		5.56E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	**1.09**
$\approx h_0/4$		2.92E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	**1.06**
$\approx h_0/8$		1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	**1.04**
h_0	2	1.54E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	**1.06**
$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	**1.04**
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	**1.03**
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	**1.03**
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	**1.03**
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	**1.01**
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	**1.01**
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	**1.01**
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	**1.02**
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	**1.01**
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	**1.01**
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	**1.01**
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	**1.02**
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	**1.01**
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	**1.00**
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	**1.00**
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	**1.02**
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	**1.01**
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	**1.01**

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|\nabla_{\rm d} u_h+\sigma_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$I^{\rm eff}$
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$\approx h_0/2$		4.07E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	1.04
$\approx h_0/4$		1.10E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.03
$\approx h_0/8$		2.50E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	1.03
h_0	3	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	1.01
h_0	4	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.02
$\approx h_0/2$		6.92E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	1.01
$\approx h_0/8$		2.58E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	1.01
h_0	5	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	1.01
$\approx h_0/4$		7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.00
h_0	6	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.01

Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\mathrm{d}}(u-u_h)\|$	$\|u-u_h\|_{\mathrm{DG}}$	$\|\nabla_{\mathrm{d}} u_h+\sigma_h\|$	η_{osc}	$\|\nabla_{\mathrm{d}}(u_h-s_h)\|$	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
h_0	1	1.07E-00	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
$\approx h_0/2$		5.56E-01	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
$\approx h_0/4$		2.92E-01	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
$\approx h_0/8$		1.39E-01	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
h_0	2	1.54E-01	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
$\approx h_0/2$		4.07E-02	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
$\approx h_0/4$		1.10E-02	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
$\approx h_0/8$		2.50E-03	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
h_0	3	1.37E-02	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
h_0	4	9.87E-04	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
$\approx h_0/2$		6.92E-05	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
$\approx h_0/8$		2.58E-07	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
h_0	5	5.64E-05	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
$\approx h_0/4$		7.74E-08	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
h_0	6	2.85E-06	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

Numerics: singular case

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3}\sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

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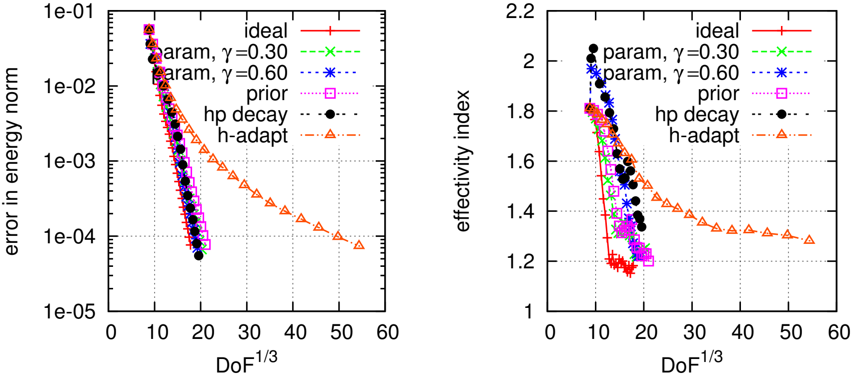
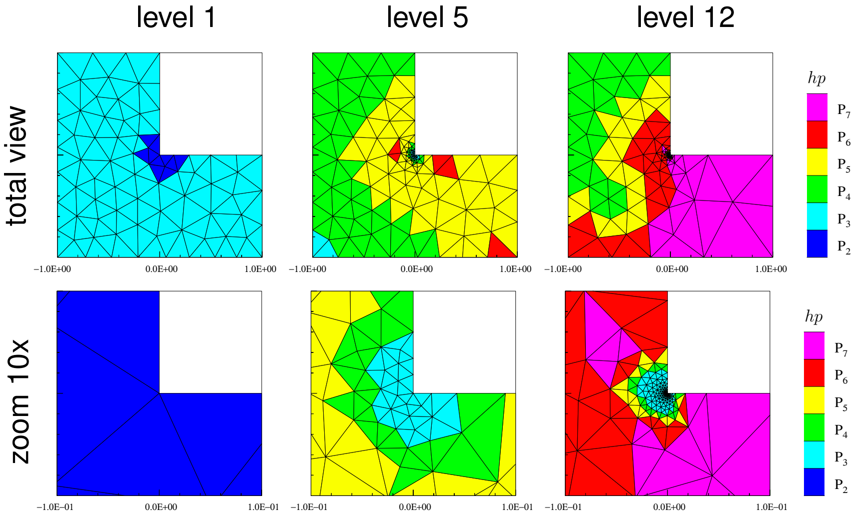
hp-adaptive refinement: **exponential convergence**



hp-refinement grids



Outline

Setting

Laplace problem

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Finite element approximation

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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Linear algebraic system

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)

On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h^i = F_h - R_h^i \qquad (R_h^i := F_h - \mathbb{A}_h U_h^i)$$

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Setting

Laplace problem

Find $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega)$$

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Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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Goals

Algebraic error

$$\|\nabla(u_h - u_h^i)\|$$

Total error

$$\|\nabla(u - u_h^i)\|$$

Discretization error

$$\|\nabla(u - u_h)\|$$

Goals: find **a posteriori estimates** for **any** $i \geq 1$

Algebraic error

$$\underline{\eta}_{\mathrm{alg}}^{i} \leq \|\nabla(u_h - u_h^i)\| \leq \eta_{\mathrm{alg}}^{i}$$

Total error

$$\underline{\eta}_{\mathrm{tot}}^{i} \leq \|\nabla(u - u_h^i)\| \leq \eta_{\mathrm{tot}}^{i}$$

Discretization error

$$\underline{\eta}_{\mathrm{dis}}^{i} \leq \|\nabla(u - u_h)\| \leq \eta_{\mathrm{dis}}^{i}$$

Goals: find **a posteriori estimates** for **any** $i \geq 1$

Algebraic error

$$\underline{\eta}^i_{\mathrm{alg}} \leq \|\nabla(u_h - u_h^i)\| \leq \eta^i_{\mathrm{alg}}$$

Total error

$$\underline{\eta}^i_{\mathrm{tot}} \leq \|\nabla(u - u_h^i)\| \leq \eta^i_{\mathrm{tot}}$$

Discretization error

$$\underline{\eta}^i_{\mathrm{dis}} \leq \|\nabla(u - u_h)\| \leq \eta^i_{\mathrm{dis}}$$

Further goals

- estimate the **distribution** of the errors (local efficiency)
- design reliable (local) **stopping criteria**

The pathway

Algebraic residual representer

- $r_h^i \in \mathbb{P}_p(\mathcal{T}_h)$ represents R_h^i
- gives equivalent form of residual equation: $u_h^i \in V_h$ s.t.

$$(\nabla u_h^i, \nabla \psi_l) = (f, \psi_l) - (r_h^i, \psi_l) \qquad \forall l = 1, \ldots, N$$

- $(r_h^i, \psi_l) = (R_h^i)_l$, $l = 1, \ldots, N$
- consequence of equations for u_h and u_h^i:

$$(\nabla(u_h - u_h^i), \nabla v_h) = (r_h^i, v_h) \qquad \forall v_h \in V_h$$

Tools

- flux and potential reconstructions
- local Neumann MFE & local Dirichlet FE problems
- separate components for algebraic & discretization errors
- multilevel hierarchy (algebraic components)

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Outline

Algebraic error upper bound

Theorem (Upper bound via algebraic error flux reconstruction)

Let $\boldsymbol{\sigma}^i_{h,\mathrm{alg}} \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that* $\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$. *Then*

$$\underbrace{\|\nabla(u_h - u^i_h)\|}_{\text{algebraic error}} \leq \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{upper algebraic est.}}.$$

Proof.

$$\|\nabla(u_h - u^i_h)\| = \sup_{v_h \in V_h, \|\nabla v_h\|=1} (\nabla(u_h - u^i_h), \nabla v_h);$$

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Previous cheap constructions of $\sigma^i_{h,\mathrm{alg}}$

1. sequential sweep trough $\mathcal{T}_h$, local min. (JSV (2010))
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Algebraic error flux reconstruction, two-level setting

Definition (Coarse grid Riesz representer)

Find $\rho^i_{H,\mathrm{alg}} \in V_H := \mathbb{P}_1(\mathcal{T}_H) \cap H^1_0(\Omega)$ such that

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- $\mathbb{P}_1$ FEs on $\mathcal{T}_H$ (no need for multigrid w/o post-smoothing)
- gives hat function orthogonality on $\mathcal{T}_H$

Definition (Algebraic error flux reconstruction)

$$\sigma^{\mathbf{a},i}_{h,\mathrm{alg}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}^{\mathbf{a}}_h,\ \nabla\cdot\mathbf{v}_h = \Pi_{Q^{\mathbf{a}}_h}(\psi_{\mathbf{a}} r^i_h - \nabla\psi_{\mathbf{a}}\cdot\nabla\rho^i_{H,\mathrm{alg}})} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}},$$

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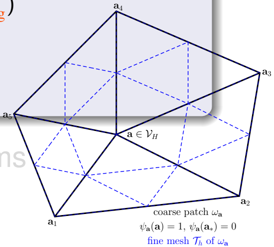

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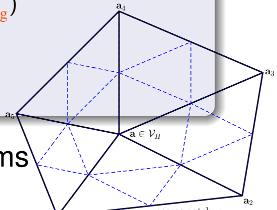

Divergence of the algebraic error flux reconstruction

Lemma (Divergence of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$)

There holds $\nabla{\cdot}\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$.

Proof.

- every fine grid element $K \in \mathcal{T}_h$ lies exactly in $(d+1)$ coarse patches $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_H$
- partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \psi^{\mathbf{a}} = 1|_K$
-

$$\nabla{\cdot}\boldsymbol{\sigma}^i_{h,\mathrm{alg}}|_K = \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \nabla{\cdot}\boldsymbol{\sigma}^{\mathbf{a},i}_{h,\mathrm{alg}}|_K$$

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Algebraic error lower bound

Theorem (Lower bound via algebraic residual liftings)

Let $\rho^i_{h,\mathrm{alg}} \in V_h$ be arbitrary. Then

$$\underbrace{\|\nabla(u_h - u_h^i)\|}_{\text{algebraic error}} \geq \underbrace{\frac{(r_h^i, \rho^i_{h,\mathrm{alg}})}{\|\nabla \rho^i_{h,\mathrm{alg}}\|}}_{\text{lower algebraic est.}}.$$

Proof.

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\|=1} (r_h^i, v_h) \geq \frac{(r_h^i, \rho^i_{h,\mathrm{alg}})}{\|\nabla \rho^i_{h,\mathrm{alg}}\|}.$$

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Algebraic residual lifting, two-level setting

Definition (Algebraic residual lifting, $\approx$ Bank & Smith (1993), Oswald (1993), Rüde (1993), ..., Ern & V. (2015)**)**

Find $\rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\omega_{\mathbf{a}})$ such that

$$(\nabla \rho_{h,\mathrm{alg}}^{\mathbf{a},i}, \nabla v_h)_{\omega_{\mathbf{a}}} = (r_h^i, v_h)_{\omega_{\mathbf{a}}} - (\nabla \rho_{H,\mathrm{alg}}^i, \nabla v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

Set

$$\rho_{h,\mathrm{alg}}^i := \rho_{H,\mathrm{alg}}^i + \sum_{\mathbf{a} \in \mathcal{V}_H} \rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in V_h.$$

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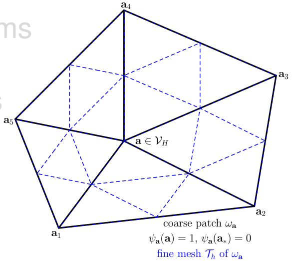

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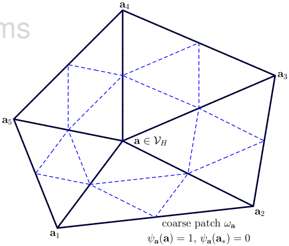


Algebraic residual lifting, two-level setting

Definition (Algebraic residual lifting, ≈ Bank & Smith (1993), Oswald (1993), Rüde (1993), . . . , Ern & V. (2015)**)**

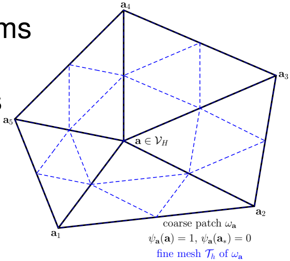
Find $\rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\omega_{\mathbf{a}})$ such that

$$(\nabla \rho_{h,\mathrm{alg}}^{\mathbf{a},i}, \nabla v_h)_{\omega_{\mathbf{a}}} = (r_h^i, v_h)_{\omega_{\mathbf{a}}} - (\nabla \rho_{H,\mathrm{alg}}^i, \nabla v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

Set

$$\rho_{h,\mathrm{alg}}^i := \rho_{H,\mathrm{alg}}^i + \sum_{\mathbf{a} \in \mathcal{V}_H} \rho_{h,\mathrm{alg}}^{\mathbf{a},i} \in V_h.$$

- local homogeneous Dirichlet FE problems
- fine meshes of coarse patches $\omega_{\mathbf{a}}$
- ✓ extends to an arbitrary number of levels



Outline

Discretization flux reconstruction

Definition (Discretization flux reconstruction, Braes & Schöberl (2008), EV (2013))

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^{\mathbf{a},i} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\ \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(f\psi^{\mathbf{a}} - \nabla u_h^i \cdot \nabla \psi_{\mathbf{a}} - r_h^i \psi^{\mathbf{a}})} \|\psi^{\mathbf{a}} \nabla u_h^i + \mathbf{v}_h\|_{\omega_{\mathbf{a}}},$$

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i} := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{h,\mathrm{dis}}^{\mathbf{a},i}$$

Neumann compatibility condition satisfied:

$$(\nabla u_h^i, \nabla \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} - (r_h^i, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Lemma (Divergence of $\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i}$)

There holds

$$\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i} = \Pi_{Q_h} f - r_h^i.$$

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Upper bound on the total error

Theorem (Total error upper bound)

On each iteration $i \geq 1$, there holds

$$\underbrace{\|\nabla(u-u_h^i)\|}_{\text{total error}} \leq \underbrace{\|\nabla u_h^i + \boldsymbol{\sigma}_{h,\mathrm{dis}}^i\|}_{\text{discretization est.}} + \underbrace{\|\boldsymbol{\sigma}_{h,\mathrm{alg}}^i\|}_{\text{algebraic est.}} + \underbrace{\left\{\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\pi^2}\|f-\Pi_{Q_h} f\|_K^2\right\}^{1/2}}_{\text{data osc. est.}}.$$

Proof.

$$\|\nabla(u-u_h^i)\| = \sup_{v\in H_0^1(\Omega),\, \|\nabla v\|=1} (\nabla(u-u_h^i), \nabla v)$$

$$(\nabla(u-u_h^i), \nabla v) = (f,v) - (\nabla u_h^i, \nabla v) = (f - \overbrace{\nabla\cdot(\boldsymbol{\sigma}_{h,\mathrm{alg}}^i + \boldsymbol{\sigma}_{h,\mathrm{dis}}^i)}, v)$$
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Definition (Total residual lifting, ≈ Babuška and Strouboulis (2001), Repin (2008))

Find $\rho_{h,\mathrm{tot}}^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H^1(\omega_{\mathbf{a}})$ (together with mean value or value on $\partial\Omega$ zero) such that

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- local homogeneous Neumann FE problems

Theorem (Lower bound on the total error)

There holds

$$\underbrace{\|\nabla(u - u_h^i)\|}_{\text{total error}} \geq \underbrace{\frac{\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla \rho_{h,\mathrm{tot}}^{\mathbf{a},i}\|_{\omega_{\mathbf{a}}}^2}{\|\nabla \rho_{h,\mathrm{tot}}^i\|}}_{\text{lower total est.}}.$$

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Outline

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Galerkin orthogonality

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Discretization error upper and lower bounds

- upper bound on total error & lower bound on algebraic error ⇒ upper bound on the discretization error
- lower bound on total error & upper bound on algebraic error ⇒ lower bound on the discretization error

Safe stopping criterion ($\gamma_{\mathrm{alg}} \approx 0.1$)

algebraic error $\leq \gamma_{\mathrm{alg}}$ discretization error

✓ stopping criterion ⇒ efficiency & p-robustness

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Numerical illustration

Peak

$$\Omega = (0,1) \times (0,1),$$
$$u(x,y) = x(x-1)y(y-1)e^{-100(x-0.5)^2-100(y-117/1000)^2}$$

L-shape

$$\Omega = (-1,1) \times (-1,1) \setminus \ [0,1] \times [-1,0],$$
$$u(r,\theta) = r^{2/3}\sin(2\theta/3)$$

Discretization

- conforming finite elements, $p = 1, \ldots, 4$
- unstructured triangular meshes
- 4 uniform refinements

Multigrid

- geometric multigrid V-cycle
- 5 pre-smoothing steps of Gauss–Seidel

PCG

- incomplete Cholesky with drop-off tolerance 1e-4 prec.

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Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^3)	1	8.1×10^{-3}	1.14	1.10^{-1}	1.0×10^{-2}	1.63	1.19^{-1}	6.1×10^{-3}	2.42	—
	2	4.3×10^{-4}	1.13	1.12^{-1}	6.1×10^{-3}	1.13	1.05^{-1}		1.13	1.06^{-1}
2 (1.03×10^4)	1	8.8×10^{-3}	1.17	1.08^{-1}	8.8×10^{-3}	1.72	1.18^{-1}	3.9×10^{-4}	3.28×10^1	—
	2	6.1×10^{-4}	1.19	1.03^{-1}	7.2×10^{-4}	1.75	1.12^{-1}		2.89	—
	3	2.0×10^{-5}	1.19	1.03^{-1}	3.9×10^{-4}	1.08	1.04^{-1}		1.08	1.04^{-1}
3 (2.34×10^4)	1	4.9×10^{-3}	1.14	1.06^{-1}	4.9×10^{-3}	1.59	1.26^{-1}	1.9×10^{-5}	3.33×10^2	—
	3	2.7×10^{-5}	1.17	1.04^{-1}	3.3×10^{-5}	1.69	1.17^{-1}		2.60	—
	5	1.6×10^{-7}	1.15	1.04^{-1}	1.9×10^{-5}	1.02	1.09^{-1}		1.02	1.09^{-1}
4 (4.17×10^4)	1	5.8×10^{-3}	1.22	1.05^{-1}	5.8×10^{-3}	1.83	1.17^{-1}	8.1×10^{-7}	1.12×10^4	—
	3	1.0×10^{-4}	1.16	1.03^{-1}	1.0×10^{-4}	1.71	1.08^{-1}		1.76×10^2	—
	5	2.4×10^{-6}	1.14	1.03^{-1}	2.5×10^{-6}	1.62	1.10^{-1}		4.12	—
	7	6.7×10^{-8}	1.13	1.03^{-1}	8.2×10^{-7}	1.10	1.16^{-1}		1.10	1.16^{-1}
5 (6.52×10^4)	1	4.8×10^{-3}	1.19	1.04^{-1}	4.8×10^{-3}	1.74	1.19^{-1}	3.1×10^{-8}	2.21×10^5	—
	3	2.1×10^{-4}	1.14	1.03^{-1}	2.1×10^{-4}	1.63	1.09^{-1}		8.78×10^3	—
	5	1.5×10^{-5}	1.11	1.02^{-1}	1.5×10^{-5}	1.55	1.07^{-1}		5.57×10^2	—
	7	1.4×10^{-6}	1.12	1.02^{-1}	1.4×10^{-6}	1.57	1.05^{-1}		5.34×10^1	—
	9	1.4×10^{-7}	1.14	1.01^{-1}	1.4×10^{-7}	1.65	1.06^{-1}		6.06	—
	11	1.3×10^{-8}	1.16	1.01^{-1}	3.4×10^{-8}	1.41	1.38^{-1}		1.47	1.62^{-1}
	13	1.2×10^{-9}	1.16	1.01^{-1}	3.1×10^{-8}	1.05	1.21^{-1}		1.05	1.21^{-1}

Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^3)	1	8.1×10^{-3}	1.14	1.10^{-1}	1.0×10^{-2}	1.63	1.19^{-1}	6.1×10^{-3}	2.42	—
	2	4.3×10^{-4}	1.13	1.12^{-1}	6.1×10^{-3}	1.13	1.05^{-1}		1.13	1.06^{-1}
2 (1.03×10^4)	1	8.8×10^{-3}	1.17	1.08^{-1}	8.8×10^{-3}	1.72	1.18^{-1}	3.9×10^{-4}	3.28×10^1	—
	2	6.1×10^{-4}	1.19	1.03^{-1}	7.2×10^{-4}	1.75	1.12^{-1}		2.89	—
	3	2.0×10^{-5}	1.19	1.03^{-1}	3.9×10^{-4}	1.08	1.04^{-1}		1.08	1.04^{-1}
3 (2.34×10^4)	1	4.9×10^{-3}	1.14	1.06^{-1}	4.9×10^{-3}	1.59	1.26^{-1}	1.9×10^{-5}	3.33×10^2	—
	3	2.7×10^{-5}	1.17	1.04^{-1}	3.3×10^{-5}	1.69	1.17^{-1}		2.60	—
	5	1.6×10^{-7}	1.15	1.04^{-1}	1.9×10^{-5}	1.02	1.09^{-1}		1.02	1.09^{-1}
4 (4.17×10^4)	1	5.8×10^{-3}	1.22	1.05^{-1}	5.8×10^{-3}	1.83	1.17^{-1}	8.1×10^{-7}	1.12×10^4	—
	3	1.0×10^{-4}	1.16	1.03^{-1}	1.0×10^{-4}	1.71	1.08^{-1}		1.76×10^2	—
	5	2.4×10^{-6}	1.14	1.03^{-1}	2.5×10^{-6}	1.62	1.10^{-1}		4.12	—
	7	6.7×10^{-8}	1.13	1.03^{-1}	8.2×10^{-7}	1.10	1.16^{-1}		1.10	1.16^{-1}
5 (6.52×10^4)	1	4.8×10^{-3}	1.19	1.04^{-1}	4.8×10^{-3}	1.74	1.19^{-1}	3.1×10^{-8}	2.21×10^5	—
	3	2.1×10^{-4}	1.14	1.03^{-1}	2.1×10^{-4}	1.63	1.09^{-1}		8.78×10^3	—
	5	1.5×10^{-5}	1.11	1.02^{-1}	1.5×10^{-5}	1.55	1.07^{-1}		5.57×10^2	—
	7	1.4×10^{-6}	1.12	1.02^{-1}	1.4×10^{-6}	1.57	1.05^{-1}		5.34×10^1	—
	9	1.4×10^{-7}	1.14	1.01^{-1}	1.4×10^{-7}	1.65	1.06^{-1}		6.06	—
	11	1.3×10^{-8}	1.16	1.01^{-1}	3.4×10^{-8}	1.41	1.38^{-1}		1.47	1.62^{-1}
	13	1.2×10^{-9}	1.16	1.01^{-1}	3.1×10^{-8}	1.05	1.21^{-1}		1.05	1.21^{-1}

Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^3)	1	8.1×10^{-3}	1.14	1.10^{-1}	1.0×10^{-2}	1.63	1.19^{-1}	6.1×10^{-3}	2.42	—
	2	4.3×10^{-4}	1.13	1.12^{-1}	6.1×10^{-3}	1.13	1.05^{-1}		1.13	1.06^{-1}
2 (1.03×10^4)	1	8.8×10^{-3}	1.17	1.08^{-1}	8.8×10^{-3}	1.72	1.18^{-1}	3.9×10^{-4}	3.28×10^1	—
	2	6.1×10^{-4}	1.19	1.03^{-1}	7.2×10^{-4}	1.75	1.12^{-1}		2.89	—
	3	2.0×10^{-5}	1.19	1.03^{-1}	3.9×10^{-4}	1.08	1.04^{-1}		1.08	1.04^{-1}
3 (2.34×10^4)	1	4.9×10^{-3}	1.14	1.06^{-1}	4.9×10^{-3}	1.59	1.26^{-1}	1.9×10^{-5}	3.33×10^2	—
	3	2.7×10^{-5}	1.17	1.04^{-1}	3.3×10^{-5}	1.69	1.17^{-1}		2.60	—
	5	1.6×10^{-7}	1.15	1.04^{-1}	1.9×10^{-5}	1.02	1.09^{-1}		1.02	1.09^{-1}
4 (4.17×10^4)	1	5.8×10^{-3}	1.22	1.05^{-1}	5.8×10^{-3}	1.83	1.17^{-1}	8.1×10^{-7}	1.12×10^4	—
	3	1.0×10^{-4}	1.16	1.03^{-1}	1.0×10^{-4}	1.71	1.08^{-1}		1.76×10^2	—
	5	2.4×10^{-6}	1.14	1.03^{-1}	2.5×10^{-6}	1.62	1.10^{-1}		4.12	—
	7	6.7×10^{-8}	1.13	1.03^{-1}	8.2×10^{-7}	1.10	1.16^{-1}		1.10	1.16^{-1}
5 (6.52×10^4)	1	4.8×10^{-3}	1.19	1.04^{-1}	4.8×10^{-3}	1.74	1.19^{-1}	3.1×10^{-8}	2.21×10^5	—
	3	2.1×10^{-4}	1.14	1.03^{-1}	2.1×10^{-4}	1.63	1.09^{-1}		8.78×10^3	—
	5	1.5×10^{-5}	1.11	1.02^{-1}	1.5×10^{-5}	1.55	1.07^{-1}		5.57×10^2	—
	7	1.4×10^{-6}	1.12	1.02^{-1}	1.4×10^{-6}	1.57	1.05^{-1}		5.34×10^1	—
	9	1.4×10^{-7}	1.14	1.01^{-1}	1.4×10^{-7}	1.65	1.06^{-1}		6.06	—
	11	1.3×10^{-8}	1.16	1.01^{-1}	3.4×10^{-8}	1.41	1.38^{-1}		1.47	1.62^{-1}
	13	1.2×10^{-9}	1.16	1.01^{-1}	3.1×10^{-8}	1.05	1.21^{-1}		1.05	1.21^{-1}

Peak problem, multigrid

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (2.55×10^3)	1	8.1×10^{-3}	1.14	1.10^{-1}	1.0×10^{-2}	1.63	1.19^{-1}	6.1×10^{-3}	2.42	—
	2	4.3×10^{-4}	1.13	1.12^{-1}	6.1×10^{-3}	1.13	1.05^{-1}		1.13	1.06^{-1}
2 (1.03×10^4)	1	8.8×10^{-3}	1.17	1.08^{-1}	8.8×10^{-3}	1.72	1.18^{-1}	3.9×10^{-4}	3.28×10^1	—
	2	6.1×10^{-4}	1.19	1.03^{-1}	7.2×10^{-4}	1.75	1.12^{-1}		2.89	—
	3	2.0×10^{-5}	1.19	1.03^{-1}	3.9×10^{-4}	1.08	1.04^{-1}		1.08	1.04^{-1}
3 (2.34×10^4)	1	4.9×10^{-3}	1.14	1.06^{-1}	4.9×10^{-3}	1.59	1.26^{-1}	1.9×10^{-5}	3.33×10^2	—
	3	2.7×10^{-5}	1.17	1.04^{-1}	3.3×10^{-5}	1.69	1.17^{-1}		2.60	—
	5	1.6×10^{-7}	1.15	1.04^{-1}	1.9×10^{-5}	1.02	1.09^{-1}		1.02	1.09^{-1}
4 (4.17×10^4)	1	5.8×10^{-3}	1.22	1.05^{-1}	5.8×10^{-3}	1.83	1.17^{-1}	8.1×10^{-7}	1.12×10^4	—
	3	1.0×10^{-4}	1.16	1.03^{-1}	1.0×10^{-4}	1.71	1.08^{-1}		1.76×10^2	—
	5	2.4×10^{-6}	1.14	1.03^{-1}	2.5×10^{-6}	1.62	1.10^{-1}		4.12	—
	7	6.7×10^{-8}	1.13	1.03^{-1}	8.2×10^{-7}	1.10	1.16^{-1}		1.10	1.16^{-1}
5 (6.52×10^4)	1	4.8×10^{-3}	1.19	1.04^{-1}	4.8×10^{-3}	1.74	1.19^{-1}	3.1×10^{-8}	2.21×10^5	—
	3	2.1×10^{-4}	1.14	1.03^{-1}	2.1×10^{-4}	1.63	1.09^{-1}		8.78×10^3	—
	5	1.5×10^{-5}	1.11	1.02^{-1}	1.5×10^{-5}	1.55	1.07^{-1}		5.57×10^2	—
	7	1.4×10^{-6}	1.12	1.02^{-1}	1.4×10^{-6}	1.57	1.05^{-1}		5.34×10^1	—
	9	1.4×10^{-7}	1.14	1.01^{-1}	1.4×10^{-7}	1.65	1.06^{-1}		6.06	—
	11	1.3×10^{-8}	1.16	1.01^{-1}	3.4×10^{-8}	1.41	1.38^{-1}		1.47	1.62^{-1}
	13	1.2×10^{-9}	1.16	1.01^{-1}	3.1×10^{-8}	1.05	1.21^{-1}		1.05	1.21^{-1}

L-shape problem, PCG

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.9×10^{-1}	1.25	4.08^{-1}	2.9×10^{-1}	1.38	6.15^{-1}	3.6×10^{-2}	1.11×10^1	—
	4	1.2×10^{-3}	1.24	4.17^{-1}	3.6×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.1×10^{-1}	1.14	3.62^{-1}	2.1×10^{-1}	1.26	6.03^{-1}	1.4×10^{-2}	1.76×10^1	—
	6	2.5×10^{-3}	1.18	3.17^{-1}	1.5×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.2×10^{-6}	1.17	3.53^{-1}	1.4×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.3	1.06	4.53^{-1}	1.3	1.10	$1.08 \times 10^{1^{-1}}$	8.6×10^{-3}	1.58×10^2	—
	8	9.9×10^{-2}	1.10	3.55^{-1}	10.0×10^{-2}	1.24	6.02^{-1}		1.41×10^1	—
	12	1.2×10^{-2}	1.10	3.58^{-1}	1.5×10^{-2}	1.71	2.67^{-1}		2.99	—
	16	8.2×10^{-4}	1.10	3.55^{-1}	8.6×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.7×10^{-1}	1.24	2.34^{-1}	1.7×10^{-1}	1.42	3.35^{-1}	6.2×10^{-3}	3.66×10^1	—
	10	2.4×10^{-3}	1.22	2.79^{-1}	6.6×10^{-3}	1.78	1.83^{-1}		1.90	2.93^{-1}
	15	2.3×10^{-5}	1.27	2.33^{-1}	6.2×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}
5 (2.02×10^5)	6	1.1	1.09	4.14^{-1}	1.1	1.16	7.42^{-1}	4.7×10^{-3}	2.71×10^2	—
	12	8.5×10^{-2}	1.11	3.75^{-1}	8.5×10^{-2}	1.23	5.77^{-1}		2.19×10^1	—
	18	7.5×10^{-3}	1.15	3.12^{-1}	8.9×10^{-3}	1.76	3.43^{-1}		3.31	—
	24	3.9×10^{-4}	1.15	3.17^{-1}	4.7×10^{-3}	1.56	1.80^{-1}		1.57	1.82^{-1}

L-shape problem, PCG

p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.9×10^{-1}	1.25	4.08^{-1}	2.9×10^{-1}	1.38	6.15^{-1}	3.6×10^{-2}	1.11×10^{1}	—
	4	1.2×10^{-3}	1.24	4.17^{-1}	3.6×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.1×10^{-1}	1.14	3.62^{-1}	2.1×10^{-1}	1.26	6.03^{-1}	1.4×10^{-2}	1.76×10^{1}	—
	6	2.5×10^{-3}	1.18	3.17^{-1}	1.5×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.2×10^{-6}	1.17	3.53^{-1}	1.4×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.3	1.06	4.53^{-1}	1.3	1.10	$1.08 \times 10^{1}{}^{-1}$	8.6×10^{-3}	1.58×10^{2}	—
	8	9.9×10^{-2}	1.10	3.55^{-1}	10.0×10^{-2}	1.24	6.02^{-1}		1.41×10^{1}	—
	12	1.2×10^{-2}	1.10	3.58^{-1}	1.5×10^{-2}	1.71	2.67^{-1}		2.99	—
	16	8.2×10^{-4}	1.10	3.55^{-1}	8.6×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.7×10^{-1}	1.24	2.34^{-1}	1.7×10^{-1}	1.42	3.35^{-1}	6.2×10^{-3}	3.66×10^{1}	—
	10	2.4×10^{-3}	1.22	2.79^{-1}	6.6×10^{-3}	1.78	1.83^{-1}		1.90	2.93^{-1}
	15	2.3×10^{-5}	1.27	2.33^{-1}	6.2×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}
5 (2.02×10^5)	6	1.1	1.09	4.14^{-1}	1.1	1.16	7.42^{-1}	4.7×10^{-3}	2.71×10^{2}	—
	12	8.5×10^{-2}	1.11	3.75^{-1}	8.5×10^{-2}	1.23	5.77^{-1}		2.19×10^{1}	—
	18	7.5×10^{-3}	1.15	3.12^{-1}	8.9×10^{-3}	1.76	3.43^{-1}		3.31	—
	24	3.9×10^{-4}	1.15	3.17^{-1}	4.7×10^{-3}	1.56	1.80^{-1}		1.57	1.82^{-1}

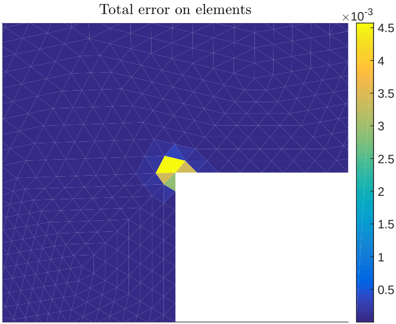
L-shape problem, PCG

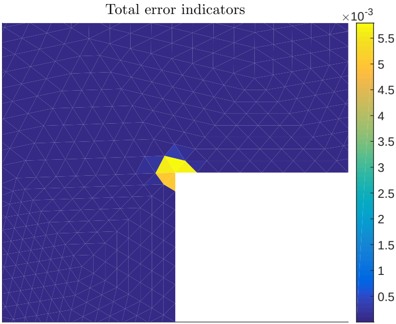
p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.9×10^{-1}	1.25	4.08^{-1}	2.9×10^{-1}	1.38	6.15^{-1}	3.6×10^{-2}	1.11×10^1	—
	4	1.2×10^{-3}	1.24	4.17^{-1}	3.6×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.1×10^{-1}	1.14	3.62^{-1}	2.1×10^{-1}	1.26	6.03^{-1}	1.4×10^{-2}	1.76×10^1	—
	6	2.5×10^{-3}	1.18	3.17^{-1}	1.5×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.2×10^{-6}	1.17	3.53^{-1}	1.4×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.3	1.06	4.53^{-1}	1.3	1.10	$1.08 \times 10^{1}{}^{-1}$	8.6×10^{-3}	1.58×10^2	—
	8	9.9×10^{-2}	1.10	3.55^{-1}	10.0×10^{-2}	1.24	6.02^{-1}		1.41×10^1	—
	12	1.2×10^{-2}	1.10	3.58^{-1}	1.5×10^{-2}	1.71	2.67^{-1}		2.99	—
	16	8.2×10^{-4}	1.10	3.55^{-1}	8.6×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.7×10^{-1}	1.24	2.34^{-1}	1.7×10^{-1}	1.42	3.35^{-1}	6.2×10^{-3}	3.66×10^1	—
	10	2.4×10^{-3}	1.22	2.79^{-1}	6.6×10^{-3}	1.78	1.83^{-1}		1.90	2.93^{-1}
	15	2.3×10^{-5}	1.27	2.33^{-1}	6.2×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}
5 (2.02×10^5)	6	1.1	1.09	4.14^{-1}	1.1	1.16	7.42^{-1}	4.7×10^{-3}	2.71×10^2	—
	12	8.5×10^{-2}	1.11	3.75^{-1}	8.5×10^{-2}	1.23	5.77^{-1}		2.19×10^1	—
	18	7.5×10^{-3}	1.15	3.12^{-1}	8.9×10^{-3}	1.76	3.43^{-1}		3.31	—
	24	3.9×10^{-4}	1.15	3.17^{-1}	4.7×10^{-3}	1.56	1.80^{-1}		1.57	1.82^{-1}

L-shape problem, PCG

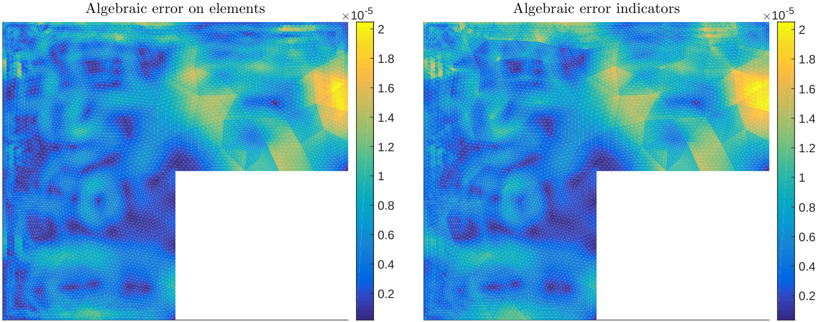
p	iter	**alg.** error	eff. UB	eff. LB	**tot.** error	eff. UB	eff. LB	**disc.** error	eff. UB	eff. LB
1 (7.97×10^3)	2	2.9×10^{-1}	1.25	4.08^{-1}	2.9×10^{-1}	1.38	6.15^{-1}	3.6×10^{-2}	1.11×10^1	—
	4	1.2×10^{-3}	1.24	4.17^{-1}	3.6×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.1×10^{-1}	1.14	3.62^{-1}	2.1×10^{-1}	1.26	6.03^{-1}	1.4×10^{-2}	1.76×10^1	—
	6	2.5×10^{-3}	1.18	3.17^{-1}	1.5×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.2×10^{-6}	1.17	3.53^{-1}	1.4×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.3	1.06	4.53^{-1}	1.3	1.10	$1.08 \times 10^{1}{}^{-1}$	8.6×10^{-3}	1.58×10^2	—
	8	9.9×10^{-2}	1.10	3.55^{-1}	10.0×10^{-2}	1.24	6.02^{-1}		1.41×10^1	—
	12	1.2×10^{-2}	1.10	3.58^{-1}	1.5×10^{-2}	1.71	2.67^{-1}		2.99	—
	16	8.2×10^{-4}	1.10	3.55^{-1}	8.6×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.7×10^{-1}	1.24	2.34^{-1}	1.7×10^{-1}	1.42	3.35^{-1}	6.2×10^{-3}	3.66×10^1	—
	10	2.4×10^{-3}	1.22	2.79^{-1}	6.6×10^{-3}	1.78	1.83^{-1}		1.90	2.93^{-1}
	15	2.3×10^{-5}	1.27	2.33^{-1}	6.2×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}
5 (2.02×10^5)	6	1.1	1.09	4.14^{-1}	1.1	1.16	7.42^{-1}	4.7×10^{-3}	2.71×10^2	—
	12	8.5×10^{-2}	1.11	3.75^{-1}	8.5×10^{-2}	1.23	5.77^{-1}		2.19×10^1	—
	18	7.5×10^{-3}	1.15	3.12^{-1}	8.9×10^{-3}	1.76	3.43^{-1}		3.31	—
	24	3.9×10^{-4}	1.15	3.17^{-1}	4.7×10^{-3}	1.56	1.80^{-1}		1.57	1.82^{-1}

L-shape problem, $p = 3$, total error, 16th PCG iteration





L-shape problem, $p = 3$, alg. error, 16th PCG iteration



Outline

Conclusions and outlook

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- guaranteed energy error estimates
- robustness (polynomial degree)
- unified framework for all classical numerical schemes

Ongoing work

- convergence and optimality
- nonlinear problems

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Thank you for your attention!