Mixed finite elements: a priori analysis, implementation, and use in a posteriori analysis

Martin Vohralík

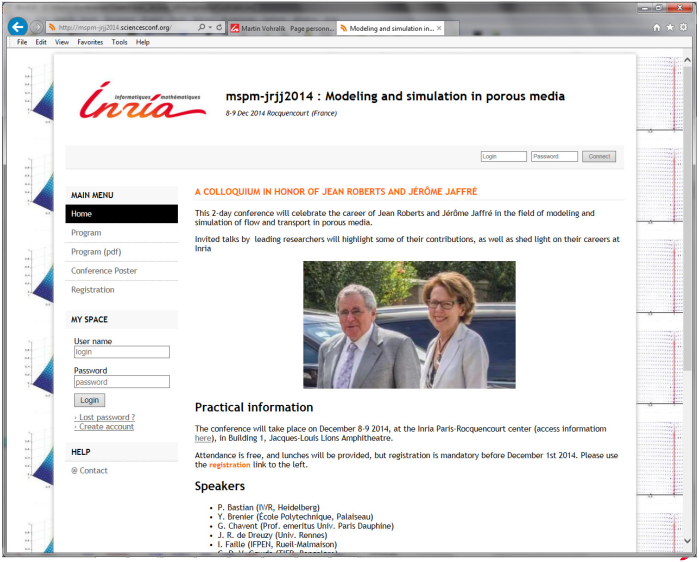
INRIA Paris-Rocquencourt

Rocquencourt, December 9, 2014

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Where does the picture comes from?



It comes from. . .

. . . my wedding

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Weak mixed formulation and mixed finite elements

Model problem

$$
\begin{aligned}
-\nabla\cdot(\underline{\mathbf{K}}\nabla p) &= f \qquad \text{in } \Omega,\\
p &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

Weak mixed formulation

Find $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
(\underline{\mathbf{K}}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) &= 0 \qquad \forall\, \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),\\
(\nabla\cdot\mathbf{u}, q) &= (f, q) \qquad \forall q \in L^2(\Omega)
\end{aligned}
$$

Mixed finite element approximation

Find $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ and $p_h \in Q_h \subset L^2(\Omega)$ such that

$$
\begin{aligned}
(\underline{\mathbf{K}}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h,\\
(\nabla\cdot\mathbf{u}_h, q_h) &= (f, q_h) \qquad \forall q_h \in Q_h
\end{aligned}
$$

Weak mixed formulation and mixed finite elements

Model problem

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla p) = f \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

Weak mixed formulation

Find $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$(\underline{\mathbf{K}}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
$$(\nabla\cdot\mathbf{u}, q) = (f, q) \quad \forall q \in L^2(\Omega)$$

Mixed finite element approximation

Find $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ and $p_h \in Q_h \subset L^2(\Omega)$ such that

$$(\underline{\mathbf{K}}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\mathbf{u}_h, q_h) = (f, q_h) \quad \forall q_h \in Q_h$$

Weak mixed formulation and mixed finite elements

Model problem

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla p) = f \quad \text{in } \Omega,$$
$$p = 0 \quad \text{on } \partial\Omega$$

Weak mixed formulation

Find $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \Omega)$ and $p \in L^2(\Omega)$ such that

$$(\underline{\mathbf{K}}^{-1}\mathbf{u}, \mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega),$$
$$(\nabla\cdot\mathbf{u}, q) = (f, q) \quad \forall q \in L^2(\Omega)$$

Mixed finite element approximation

Find $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$ and $p_h \in Q_h \subset L^2(\Omega)$ such that

$$(\underline{\mathbf{K}}^{-1}\mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla\cdot\mathbf{v}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla\cdot\mathbf{u}_h, q_h) = (f, q_h) \quad \forall q_h \in Q_h$$

Outline

1 Introduction

2 Primal-formulation-based a priori analysis
- Discrete existence and uniqueness
- A priori estimates for the fluxes
- A priori estimates for the pressures

3 Implementation
- One unknown per element and a link to the MPFA method
- General polygonal meshes

4 Use in a posteriori analysis
- Equilibration by local Neumann MFE problems
- Stability of MFEs for a posteriori efficiency
- Numerical experiment

5 Conclusions and future directions

Existence and uniqueness: afraid of the inf–sup condition?

Discrete existence and uniqueness

- the discrete system is square ⇒ enough to show that $f = 0$ implies $\mathbf{u}_h = \mathbf{0}$ and $p_h = 0$
- take $\mathbf{v}_h = \mathbf{u}_h$ in the first equation and $q_h = p_h$ in the second equation and sum

$$\begin{aligned} (\underline{\mathbf{K}}^{-1}\mathbf{u}_h, \mathbf{u}_h) - (p_h, \nabla\cdot\mathbf{u}_h) &= 0 \\ (\nabla\cdot\mathbf{u}_h, p_h) &= 0 \end{aligned}$$

$$\Rightarrow (\underline{\mathbf{K}}^{-1}\mathbf{u}_h, \mathbf{u}_h) = 0$$

- $\Rightarrow \mathbf{u}_h = \mathbf{0}$
- the first equation gives $(p_h, \nabla\cdot\mathbf{v}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{V}_h$
- $\nabla\cdot\mathbf{V}_h = Q_h \Rightarrow p_h = 0$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

A priori estimates: afraid of the inf–sup condition?

A priori error estimate for the fluxes

- subtracting the discrete and continuous formulations gives

$$(\underline{\mathbf{K}}^{-1}(\mathbf{u}_h - \mathbf{u}), \mathbf{v}_h) = (p_h - p, \nabla\cdot\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h$$

- take $\mathbf{v}_h := I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h$ and notice that $\nabla\cdot(I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h) = 0$

$$\Rightarrow (\underline{\mathbf{K}}^{-1}(\mathbf{u} - \mathbf{u}_h), I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h) = 0$$

- develop

$$\begin{aligned}\|\underline{\mathbf{K}}^{-1/2}(\mathbf{u} - \mathbf{u}_h)\|^2 &= (\underline{\mathbf{K}}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - \mathbf{u}_h) \\ &= (\underline{\mathbf{K}}^{-1}(\mathbf{u} - \mathbf{u}_h), \mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u})) \\ &\quad + \underbrace{(\underline{\mathbf{K}}^{-1}(\mathbf{u} - \mathbf{u}_h), I_{\mathbf{V}_h}(\mathbf{u}) - \mathbf{u}_h)}_{=0}\end{aligned}$$

- Cauchy–Schwarz inequality

$$\Rightarrow \underbrace{\|\underline{\mathbf{K}}^{-1/2}(\mathbf{u} - \mathbf{u}_h)\|}_{\text{approximation error}} \leq \underbrace{\|\underline{\mathbf{K}}^{-1/2}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}))\|}_{\text{interpolation error}}$$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla\tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- only valid in the lowest-order case on simplices or, when $\underline{\mathbf{K}}$ is diagonal, on rectangular parallelepipeds
- higher-order cases: $\mathbf{u}_h$ is a $P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}$ projection of $-\underline{\mathbf{K}}\nabla\tilde{p}_h$ onto $\mathbf{V}_h \Leftrightarrow$ weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla\tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- only valid in the lowest-order case on simplices or, when $\underline{\mathbf{K}}$ is diagonal, on rectangular parallelepipeds
- higher-order cases: $\mathbf{u}_h$ is a $P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}$ projection of $-\underline{\mathbf{K}}\nabla\tilde{p}_h$ onto $\mathbf{V}_h$ $\Leftrightarrow$ weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla\tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- only valid in the lowest-order case on simplices or, when $\underline{\mathbf{K}}$ is diagonal, on rectangular parallelepipeds
- higher-order cases: $\mathbf{u}_h$ is a $P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}$ projection of $-\underline{\mathbf{K}}\nabla\tilde{p}_h$ onto $\mathbf{V}_h \Leftrightarrow$ weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla\tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Proof:
$$\begin{aligned} 0 &= -(\nabla\tilde{p}_h, \mathbf{v}_{e_{K,K'}})_{K\cup K'} - (\tilde{p}_h, \nabla\cdot\mathbf{v}_{e_{K,K'}})_{K\cup K'} \\ &= -\langle\mathbf{v}_{e_{K,K'}}\cdot\mathbf{n}_K, \tilde{p}_h\rangle_{\partial K} - \langle\mathbf{v}_{e_{K,K'}}\cdot\mathbf{n}_{K'}, \tilde{p}_h\rangle_{\partial K'} \\ &= \langle\mathbf{v}_{e_{K,K'}}\cdot\mathbf{n}_K, \tilde{p}_h|_{K'} - \tilde{p}_h|_K\rangle_{e_{K,K'}} \end{aligned}$$

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- only valid in the lowest-order case on simplices or, when $\underline{\mathbf{K}}$ is diagonal, on rectangular parallelepipeds
- higher-order cases: $\mathbf{u}_h$ is a $P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}$ projection of $-\underline{\mathbf{K}}\nabla \tilde{p}_h$ onto $\mathbf{V}_h \Leftrightarrow$ weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$

Elementwise postprocessing

Postprocessed scalar variable $\tilde{p}_h$

- $-\underline{\mathbf{K}}\nabla\tilde{p}_h|_K = \mathbf{u}_h|_K$ for all $K \in \mathcal{T}_h$,
- $(\tilde{p}_h, 1)_K/|K| = p_K$ for all $K \in \mathcal{T}_h$

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ is a piecewise second-order polynomial
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$
- means of traces of $\tilde{p}_h$ on the sides continuous
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- only valid in the lowest-order case on simplices or, when $\underline{\mathbf{K}}$ is diagonal, on rectangular parallelepipeds
- higher-order cases: $\mathbf{u}_h$ is a $P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}$ projection of $-\underline{\mathbf{K}}\nabla\tilde{p}_h$ onto $\mathbf{V}_h$ $\Leftrightarrow$ weak connection of $\tilde{p}_h$ and $\mathbf{u}_h$

A priori error estimates for the pressures

Lowest-order Raviart–Thomas case

- $\|\underline{\mathbf{K}}^{1/2}\nabla(p-\tilde{p}_h)\| = \|\underline{\mathbf{K}}^{-1/2}(\mathbf{u}-\mathbf{u}_h)\| \leq \|\underline{\mathbf{K}}^{-1/2}(\mathbf{u}-I_{\mathbf{V}_h}(\mathbf{u}))\| \leq Ch$
- means of traces of $\tilde{p}_h$ continuous $\Rightarrow$ broken Friedrichs inequality

$$\|p-\tilde{p}_h\| \leq C_{\mathrm{bF}}\|\nabla(p-\tilde{p}_h)\|$$

- $\Rightarrow \|p-\tilde{p}_h\| \leq Ch$
- superconvergence: $\|p-\tilde{p}_h\| \leq Ch^2$

General case

- a little bit more complicated since we only have $\mathbf{u}_h = -P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}(\underline{\mathbf{K}}\nabla\tilde{p}_h)$ instead of $\mathbf{u}_h = -\underline{\mathbf{K}}\nabla\tilde{p}_h$
- one still easily recovers all the known a priori error estimates for mixed finite elements

A priori error estimates for the pressures

Lowest-order Raviart–Thomas case

- $\|\underline{\mathbf{K}}^{1/2}\nabla(p - \tilde{p}_h)\| = \|\underline{\mathbf{K}}^{-1/2}(\mathbf{u} - \mathbf{u}_h)\| \leq \|\underline{\mathbf{K}}^{-1/2}(\mathbf{u} - I_{\mathbf{V}_h}(\mathbf{u}))\| \leq Ch$
- means of traces of $\tilde{p}_h$ continuous $\Rightarrow$ broken Friedrichs inequality

$$\|p - \tilde{p}_h\| \leq C_{\mathrm{bF}}\|\nabla(p - \tilde{p}_h)\|$$

- $\Rightarrow \|p - \tilde{p}_h\| \leq Ch$
- superconvergence: $\|p - \tilde{p}_h\| \leq Ch^2$

General case

- a little bit more complicated since we only have $\mathbf{u}_h = -P_{\mathbf{V}_h,\underline{\mathbf{K}}^{-1}}(\underline{\mathbf{K}}\nabla\tilde{p}_h)$ instead of $\mathbf{u}_h = -\underline{\mathbf{K}}\nabla\tilde{p}_h$
- one still easily recovers all the known a priori error estimates for mixed finite elements

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Goal

Eliminate equivalently the flux unknowns

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\Leftrightarrow$$

$$\mathbb{S}P = H$$

Local flux expression from the Lagrange multipliers

Nonconforming finite element method

find $\lambda_h \in \Psi_h$ such that

$$(\underline{\mathbf{K}}\nabla\lambda_h, \nabla\psi_h) = (f, \psi_h) \qquad \forall \psi_h \in \Psi_h$$

Local flux expression from the Lagrange multipliers

there holds Marini (1985)

$$\mathbf{u}_h|_K = -\underline{\mathbf{K}}_K \nabla\lambda_h|_K + \frac{f_K}{d}(\mathbf{x} - \mathbf{x}_K) \qquad \forall K \in \mathcal{T}_h$$

- $\mathbf{x}_K$: barycenter of K
- f_K: mean value of the source term f over K

Local flux expression from the Lagrange multipliers

Nonconforming finite element method

find $\lambda_h \in \Psi_h$ such that

$$(\underline{\mathbf{K}}\nabla\lambda_h, \nabla\psi_h) = (f, \psi_h) \qquad \forall \psi_h \in \Psi_h$$

Local flux expression from the Lagrange multipliers

there holds Marini (1985)

$$\mathbf{u}_h|_K = -\underline{\mathbf{K}}_K \nabla\lambda_h|_K + \frac{f_K}{d}(\mathbf{x} - \mathbf{x}_K) \qquad \forall K \in \mathcal{T}_h$$

- $\mathbf{x}_K$: barycenter of K
- f_K: mean value of the source term f over K

Outline

1 Introduction

2 Primal-formulation-based a priori analysis
- Discrete existence and uniqueness
- A priori estimates for the fluxes
- A priori estimates for the pressures

3 Implementation
- One unknown per element and a link to the MPFA method
- General polygonal meshes

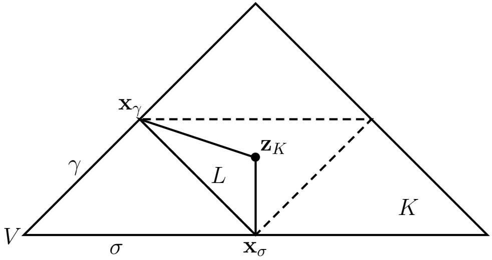
4 Use in a posteriori analysis
- Equilibration by local Neumann MFE problems
- Stability of MFEs for a posteriori efficiency
- Numerical experiment

5 Conclusions and future directions

A new element value

A new element value in $K \in \mathcal{T}_h$

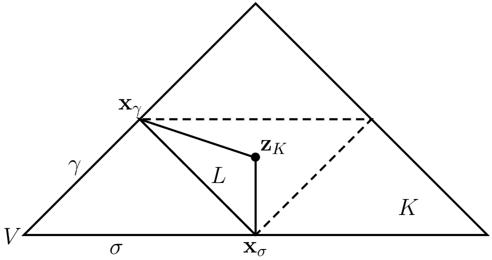
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K



A new element value

A new element value in $K \in \mathcal{T}_h$

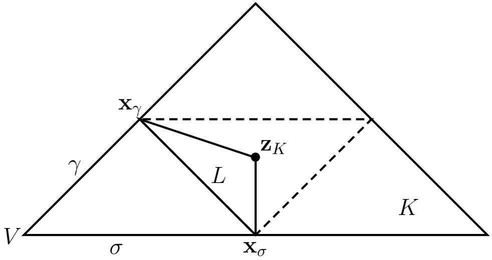
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K



A new element value

A new element value in $K \in \mathcal{T}_h$

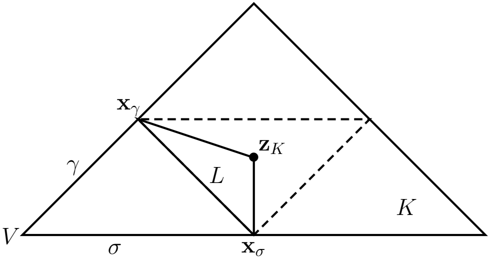
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K



A new element value

A new element value in $K \in \mathcal{T}_h$

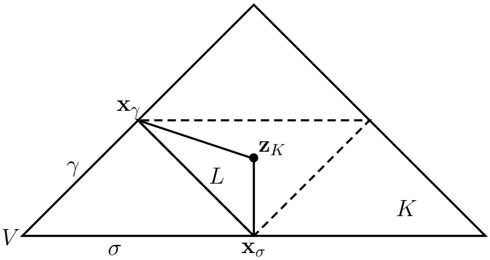
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K



A new element value

A new element value in $K \in \mathcal{T}_h$

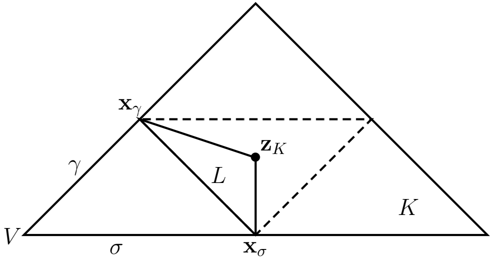
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K
- $\mathbf{u}_h|_K = -\underline{\mathbf{K}}_K \nabla \lambda_h|_K + \frac{f_K}{d}(\mathbf{x} - \mathbf{x}_K)$ $\Rightarrow$



A new element value

A new element value in $K \in \mathcal{T}_h$

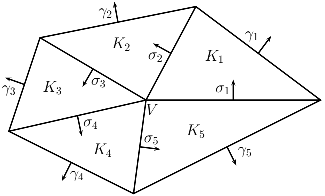
- $\mathbf{z}_K$: a new point related to K (not necessarily inside K)
- new element value: $\bar{p}_K = \lambda_h(\mathbf{z}_K)$
- λ_h expressed in the three points $\mathbf{x}_\sigma$, $\mathbf{x}_\gamma$, and $\mathbf{z}_K$ ($d = 2$)
- Lagrange basis functions φ_σ, φ_γ, and φ_K
- $\mathbf{u}_h|_K = -\underline{\mathbf{K}}_K \nabla \left(\sum_{\sigma \in \mathcal{E}_{V,K}} \lambda_\sigma \varphi_\sigma + \bar{p}_K \varphi_K \right) + \frac{f_K}{d}(\mathbf{x} - \mathbf{x}_K)$



Definition of a local problem

Definition of a local problem

- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\text{int}}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
- impose the continuity of $\mathbf{u}_h$ on the interior sides ($\mathcal{E}_V^{\text{int}}$) of the patch
$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\text{int}}$$
- **local problem**: given $\bar{P}_V = \{\bar{p}_K\}_{K \in \mathcal{T}_V}$, find $\Lambda_V^{\text{int}} = \{\lambda_\gamma\}_{\gamma \in \mathcal{E}_V^{\text{int}}}$ s.t.
$$\mathbb{M}_V \Lambda_V^{\text{int}} = \widetilde{G}_V - \mathbb{J}_V \bar{P}_V$$
- the **same building principle** as that of **MPFA methods**

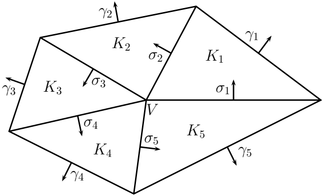


$$\begin{aligned} \mathcal{T}_V &= \{K_i\}_{i=1}^5 \\ \mathcal{E}_V^{\text{int}} &= \{\sigma_i\}_{i=1}^5 \\ \mathcal{E}_V^{\text{ext}} &= \{\gamma_i\}_{i=1}^5 \\ \mathcal{E}_V &= \mathcal{E}_V^{\text{int}} \cup \mathcal{E}_V^{\text{ext}} \end{aligned}$$

Definition of a local problem

Definition of a local problem

- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\rm int}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
- impose the continuity of $\mathbf{u}_h$ on the interior sides ($\mathcal{E}_V^{\rm int}$) of the patch
$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\rm int}$$
- **local problem**: given $\bar{P}_V = \{\bar{p}_K\}_{K \in \mathcal{T}_V}$, find $\Lambda_V^{\rm int} = \{\lambda_\gamma\}_{\gamma \in \mathcal{E}_V^{\rm int}}$ s.t.
$$\mathbb{M}_V \Lambda_V^{\rm int} = \widetilde{G}_V - \mathbb{J}_V \bar{P}_V$$
- the **same building principle** as that of **MPFA methods**

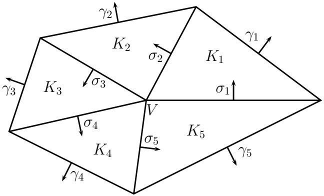


$$\begin{aligned}
\mathcal{T}_V &= \{K_i\}_{i=1}^5 \\
\mathcal{E}_V^{\rm int} &= \{\sigma_i\}_{i=1}^5 \\
\mathcal{E}_V^{\rm ext} &= \{\gamma_i\}_{i=1}^5 \\
\mathcal{E}_V &= \mathcal{E}_V^{\rm int} \cup \mathcal{E}_V^{\rm ext}
\end{aligned}$$

Definition of a local problem

Definition of a local problem

- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\mathrm{int}}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
- impose the continuity of $\mathbf{u}_h$ on the interior sides ($\mathcal{E}_V^{\mathrm{int}}$) of the patch
$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\mathrm{int}}$$
- **local problem**: given $\bar{P}_V = \{\bar{p}_K\}_{K \in \mathcal{T}_V}$, find $\Lambda_V^{\mathrm{int}} = \{\lambda_\gamma\}_{\gamma \in \mathcal{E}_V^{\mathrm{int}}}$ s.t.
$$\mathbb{M}_V \Lambda_V^{\mathrm{int}} = \widetilde{G}_V - \mathbb{J}_V \bar{P}_V$$
- the **same building principle** as that of **MPFA methods**



$$\begin{aligned} \mathcal{T}_V &= \{K_i\}_{i=1}^5 \\ \mathcal{E}_V^{\mathrm{int}} &= \{\sigma_i\}_{i=1}^5 \\ \mathcal{E}_V^{\mathrm{ext}} &= \{\gamma_i\}_{i=1}^5 \\ \mathcal{E}_V &= \mathcal{E}_V^{\mathrm{int}} \cup \mathcal{E}_V^{\mathrm{ext}} \end{aligned}$$

Definition of a local problem

Definition of a local problem

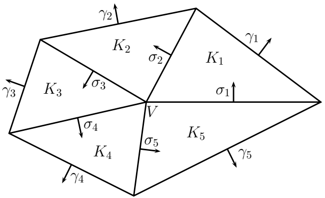
- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\mathrm{int}}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
- impose the continuity of $\mathbf{u}_h$ on the interior sides ($\mathcal{E}_V^{\mathrm{int}}$) of the patch

$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\mathrm{int}}$$

- **local problem**: given $\bar{P}_V = \{\bar{p}_K\}_{K \in \mathcal{T}_V}$, find $\Lambda_V^{\mathrm{int}} = \{\lambda_\gamma\}_{\gamma \in \mathcal{E}_V^{\mathrm{int}}}$ s.t.

$$\mathbb{M}_V \Lambda_V^{\mathrm{int}} = \widetilde{G}_V - \mathbb{J}_V \bar{P}_V$$

- the **same building principle** as that of **MPFA methods**



$$\mathcal{T}_V = \{K_i\}_{i=1}^5$$
$$\mathcal{E}_V^{\mathrm{int}} = \{\sigma_i\}_{i=1}^5$$
$$\mathcal{E}_V^{\mathrm{ext}} = \{\gamma_i\}_{i=1}^5$$
$$\mathcal{E}_V = \mathcal{E}_V^{\mathrm{int}} \cup \mathcal{E}_V^{\mathrm{ext}}$$

Definition of a local problem

Definition of a local problem

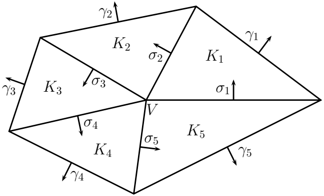
- consider a patch $\mathcal{T}_V$ of the elements around a vertex V
- given the new element values $\bar{p}_K$ and λ_σ, $\sigma \in \mathcal{E}_V^{\mathrm{int}}$, in the patch, express the fluxes $\mathbf{u}_h$ in the patch
- impose the continuity of $\mathbf{u}_h$ on the interior sides ($\mathcal{E}_V^{\mathrm{int}}$) of the patch

$$\sum_{K \in \mathcal{T}_V;\, \sigma \in \mathcal{E}_K} \langle \mathbf{u}_h \cdot \mathbf{n}_K, 1 \rangle_\sigma = 0 \qquad \forall \sigma \in \mathcal{E}_V^{\mathrm{int}}$$

- **local problem**: given $\bar{P}_V = \{\bar{p}_K\}_{K \in \mathcal{T}_V}$, find $\Lambda_V^{\mathrm{int}} = \{\lambda_\gamma\}_{\gamma \in \mathcal{E}_V^{\mathrm{int}}}$ s.t.

$$\mathbb{M}_V \Lambda_V^{\mathrm{int}} = \widetilde{G}_V - \mathbb{J}_V \bar{P}_V$$

- the **same building principle** as that of **MPFA methods**



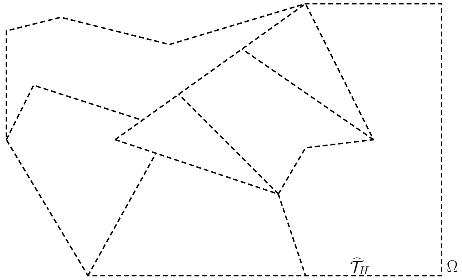
$$\mathcal{T}_V = \{K_i\}_{i=1}^5$$
$$\mathcal{E}_V^{\mathrm{int}} = \{\sigma_i\}_{i=1}^5$$
$$\mathcal{E}_V^{\mathrm{ext}} = \{\gamma_i\}_{i=1}^5$$
$$\mathcal{E}_V = \mathcal{E}_V^{\mathrm{int}} \cup \mathcal{E}_V^{\mathrm{ext}}$$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

General polygonal meshes

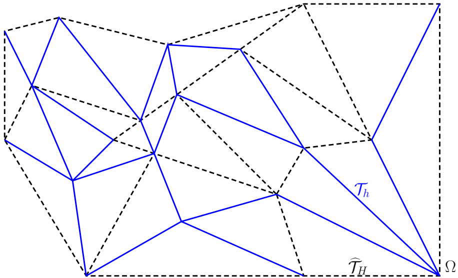
A general polygonal mesh $\widehat{\mathcal{T}}_H$



- nonconvex and non star-shaped elements in $\widehat{\mathcal{T}}_H$
- $\widehat{\mathcal{T}}_H$ can be nonmatching
- maximal number of sides of $K \in \widehat{\mathcal{T}}_H$ is not limited
- $\widehat{\mathcal{T}}_H$ is not necessarily shape-regular
- only assumption: existence of a simplicial submesh $\mathcal{T}_h$

General polygonal meshes

A general polygonal mesh $\widehat{\mathcal{T}}_H$



- nonconvex and non star-shaped elements in $\widehat{\mathcal{T}}_H$
- $\widehat{\mathcal{T}}_H$ can be nonmatching
- maximal number of sides of $K \in \widehat{\mathcal{T}}_H$ is not limited
- $\widehat{\mathcal{T}}_H$ is not necessarily shape-regular
- only assumption: existence of a simplicial submesh $\mathcal{T}_h$

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle, ...)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle, ...)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle, ...)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to the **formulation on** $\mathcal{T}_h$ (a priori and a posteriori error estimates, discrete maximum principle, ...)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\begin{pmatrix} \widehat{\mathbb{A}} & \widehat{\mathbb{B}}^t \\ \widehat{\mathbb{B}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{U} \\ \widehat{P} \end{pmatrix} = \begin{pmatrix} \widehat{F} \\ \widehat{G} \end{pmatrix}$$

- $\widehat{U}$: flux unknowns related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{P}$: potential unknowns related to the elements of $\widehat{\mathcal{T}}_H$ only
- indefinite, saddle point system, well-posed
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of loc. matrices corresponding to local Neumann problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle, ...)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to the **formulation on** $\mathcal{T}_h$ (a priori and a posteriori error estimates, discrete maximum principle . . .)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to the **formulation on** $\mathcal{T}_h$ (a priori and a posteriori error estimates, discrete maximum principle . . .)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle . . .)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle . . .)

MFEs on general polygonal meshes

MFEs on $\mathcal{T}_h$

$$\mathbb{Z}\Lambda = E$$

MFEs on $\widehat{\mathcal{T}}_H$

$$\widehat{\mathbb{Z}}\widehat{\Lambda} = \widehat{E}$$

- $\widehat{\Lambda}$: Lagrange multipliers related to the sides of $\widehat{\mathcal{T}}_H$ only
- $\widehat{\mathbb{Z}}$ is symmetric and positive definite
- derived by static condensation from MFEs on $\mathcal{T}_h$ (inverses of local matrices corresponding to local Dirichlet problems)
- works for arbitrary order
- **equivalent** to the **formulation on $\mathcal{T}_h$** (a priori and a posteriori error estimates, discrete maximum principle . . .)

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

A posteriori estimation context

Model problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

A posteriori error estimation

- $\mathcal{T}_h$ a simplicial mesh
- $u_h \in V_h \subset H_0^1(\Omega)$ an approximate solution (FEs)
- $-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
  - normal trace of $-\nabla u_h$ is not continuous
  - $\nabla\cdot(-\nabla u_h)$ is far from the source term f

Equilibrated flux Prager and Synge (1947)

- $\boldsymbol{\sigma}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$
- $\nabla\cdot\boldsymbol{\sigma}_h = \Pi_{Q_h} f$

$$\Rightarrow \|\nabla(u - u_h)\| \le \|\nabla u_h + \boldsymbol{\sigma}_h\| + h/\pi\|f - \Pi_{Q_h} f\|$$

A posteriori estimation context

Model problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

A posteriori error estimation

- $\mathcal{T}_h$ a simplicial mesh
- $u_h \in V_h \subset H_0^1(\Omega)$ an approximate solution (FEs)
- $-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
  - normal trace of $-\nabla u_h$ is not continuous
  - $\nabla\cdot(-\nabla u_h)$ is far from the source term f

Equilibrated flux Prager and Synge (1947)

- $\sigma_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$
- $\nabla\cdot\sigma_h = \Pi_{Q_h} f$

$$\Rightarrow \|\nabla(u - u_h)\| \leq \|\nabla u_h + \sigma_h\| + h/\pi \|f - \Pi_{Q_h} f\|$$

A posteriori estimation context

Model problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

A posteriori error estimation

- $\mathcal{T}_h$ a simplicial mesh
- $u_h \in V_h \subset H_0^1(\Omega)$ an approximate solution (FEs)
- $-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$
  - normal trace of $-\nabla u_h$ is not continuous
  - $\nabla\cdot(-\nabla u_h)$ is far from the source term f

Equilibrated flux Prager and Synge (1947)

- $\boldsymbol{\sigma}_h \in \mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$
- $\nabla\cdot\boldsymbol{\sigma}_h = \Pi_{Q_h} f$

$$\Rightarrow \|\nabla(u - u_h)\| \leq \|\nabla u_h + \boldsymbol{\sigma}_h\| + h/\pi\|f - \Pi_{Q_h} f\|$$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Local flux equilibration

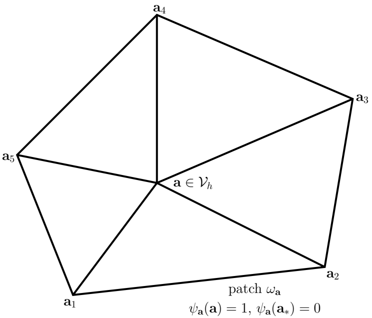
Patchwise Neumann problems by mixed finite elements

Destuynder and Métivet (1999), Braess and Schöberl (2008)

- partition of unity idea

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}$
- **local minimization**



Equilibrated flux reconstruction by MFEs

Assumption A (Galerkin orthogonality)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

Definition (Construction of σ_h)

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving **the local MFE problem**

$$\begin{aligned}(\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla{\cdot}\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\\ (\nabla{\cdot}\sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}{\cdot}\nabla u_h, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}},\end{aligned}$$

with $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ mixed finite element spaces (homogeneous Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and on $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$). Set

$$\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}}.$$

Equilibrated flux reconstruction by MFEs

Assumption A (Galerkin orthogonality)

There holds $u_h \in H^1(\mathcal{T}_h)$ and

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

Definition (Construction of $\boldsymbol{\sigma}_h$)

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving **the local MFE problem**

$$\begin{aligned}
(\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\\
(\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}},
\end{aligned}$$

with $\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ mixed finite element spaces (homogeneous Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and on $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$). Set

$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}.$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Continuous efficiency

Theorem (Cont. efficiency Carstensen & Funken (1999), Braess, Pillwein, & Schöberl (2009)**)**

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $\boldsymbol{\sigma}^{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ and $\bar{r}^{\mathbf{a}} \in L^2_*(\omega_{\mathbf{a}})$ be given by

$$
\begin{aligned}
(\boldsymbol{\sigma}^{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} - (\bar{r}^{\mathbf{a}}, \nabla\cdot\mathbf{v})_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v})_{\omega_{\mathbf{a}}} && \forall \mathbf{v} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}),\\
(\nabla\cdot\boldsymbol{\sigma}^{\mathbf{a}}, q)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q)_{\omega_{\mathbf{a}}} && \forall q \in L^2_*(\omega_{\mathbf{a}}),
\end{aligned}
$$

with

- $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$: $L^2_*(\omega_{\mathbf{a}}) := L^2(\omega_{\mathbf{a}})$ *(zero mean value);* $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) := \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with zero normal trace on* $\partial\omega_{\mathbf{a}}$*;*
- $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$: $L^2_*(\omega_{\mathbf{a}}) := L^2(\omega_{\mathbf{a}})$; $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) := \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with zero normal trace on* $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$.

Then there exists a constant $C_{\mathrm{cont,PF}} > 0$ only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$
\|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.
$$

Continuous efficiency

Theorem (Cont. efficiency Carstensen & Funken (1999), Braess, Pillwein, & Schöberl (2009)**)**

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $\boldsymbol{\sigma}^{\mathbf{a}} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}})$ and $\bar{r}^{\mathbf{a}} \in L^2_*(\omega_{\mathbf{a}})$ be given by

$$(\boldsymbol{\sigma}^{\mathbf{a}}, \mathbf{v})_{\omega_{\mathbf{a}}} - (\bar{r}^{\mathbf{a}}, \nabla \cdot \mathbf{v})_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v} \in \mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}),$$

$$(\nabla \cdot \boldsymbol{\sigma}^{\mathbf{a}}, q)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, q)_{\omega_{\mathbf{a}}} \qquad \forall q \in L^2_*(\omega_{\mathbf{a}}),$$

with

- $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$: $L^2_*(\omega_{\mathbf{a}}) := L^2(\omega_{\mathbf{a}})$ *(zero mean value);* $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) := \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with zero normal trace on* $\partial\omega_{\mathbf{a}}$*;*
- $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$: $L^2_*(\omega_{\mathbf{a}}) := L^2(\omega_{\mathbf{a}})$; $\mathbf{H}_*(\mathrm{div}, \omega_{\mathbf{a}}) := \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with zero normal trace on* $\partial\omega_{\mathbf{a}} \setminus \partial\Omega$.

Then there exists a constant $C_{\mathrm{cont,PF}} > 0$ only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{cont,PF}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Mixed finite elements stability ($d = 2$)

Theorem (MFE stability & continuous right inverse of the divergence operator Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010)**)**

Let u be the weak solution and let u_h and f be piecewise polynomial. Consider corresponding polynomial degree MFE reconstructions. Then there exists a constant $C_{\rm st} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}}.$$

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Numerics: discontinuous Galerkin

Model problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega :=]0,1[\times]0,1[,\\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$
\begin{aligned}
&u(\mathbf{x}) = (c_1 + c_2(1-x_1) + e^{-\alpha x_1})(c_1 + c_2(1-x_2) + e^{-\alpha x_2})\\
&c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10
\end{aligned}
$$

Discretization

incomplete interior penalty discontinuous Galerkin method

Numerics: discontinuous Galerkin

Model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega :=]0,1[\times]0,1[, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

incomplete interior penalty discontinuous Galerkin method

Numerics: discontinuous Galerkin

Model problem

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega :=]0,1[\times]0,1[,\\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$
\begin{aligned}
&u(\mathbf{x}) = (c_1 + c_2(1-x_1) + e^{-\alpha x_1})(c_1 + c_2(1-x_2) + e^{-\alpha x_2})\\
&c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10
\end{aligned}
$$

Discretization

incomplete interior penalty discontinuous Galerkin method

Estimates, errors, effectivity indices (calc. V. Dolejší)

h	p	$\|\nabla(u-u_h)\|$	$\|u-u_h\|_{\mathrm{DG}}$	$\|\nabla u_h+\sigma_h\|$	$\|\nabla(u_h-s_h)\|$	η_{osc}	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

Outline

1. Introduction
2. Primal-formulation-based a priori analysis
   - Discrete existence and uniqueness
   - A priori estimates for the fluxes
   - A priori estimates for the pressures
3. Implementation
   - One unknown per element and a link to the MPFA method
   - General polygonal meshes
4. Use in a posteriori analysis
   - Equilibration by local Neumann MFE problems
   - Stability of MFEs for a posteriori efficiency
   - Numerical experiment
5. Conclusions and future directions

Bibliography

Bibliography

- VOHRALÍK M., Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods, *Math. Comp.* **79** (2010), 2001–2032.
- VOHRALÍK M., WOHLMUTH B., Mixed finite element methods: implementation with one unknown per element, local flux expressions, positivity, polygonal meshes, and relations to other methods, *Math. Models Methods Appl. Sci.* **23** (2013), 803–838.
- ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, HAL Preprint 00921583, submitted for publication.

Conclusions and future directions

Conclusion

- Rocquencourt is a great place to do research

Future directions

- Austin, Bangalore, Cairo, Erlangen, Heildeberg, Kenitra, Paris, Prague, Rennes, Stuttgart, Washington ...

Thanks to Jean & Jérôme!

Conclusions and future directions

Conclusion

- Rocquencourt is a great place to do research

Future directions

- Austin, Bangalore, Cairo, Erlangen, Heildeberg, Kenitra, Paris, Prague, Rennes, Stuttgart, Washington . . .

Thanks to Jean & Jérôme!