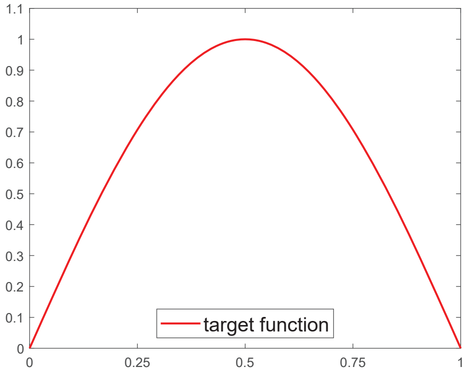
p-robust equivalence of global continuous (constrained) and local discontinuous (unconstrained) approximations, p-stable local (commuting) projectors, and optimal elementwise hp approximation estimates in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$

Théophile Chaumont-Frelet, Leszek Demkowicz, and **Martin Vohralík**

Inria Paris & Ecole des Ponts

Institute of Mathematics, Czech Academy of Sciences, 12 April 2024

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



Target function in $H_0^1(\Omega)$

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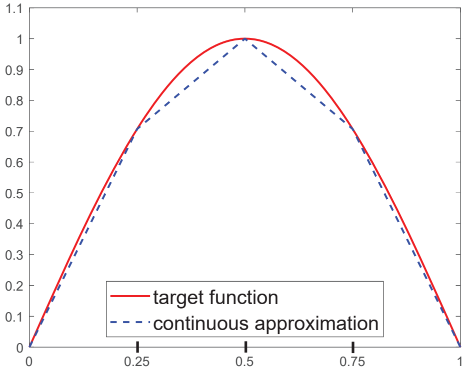

Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

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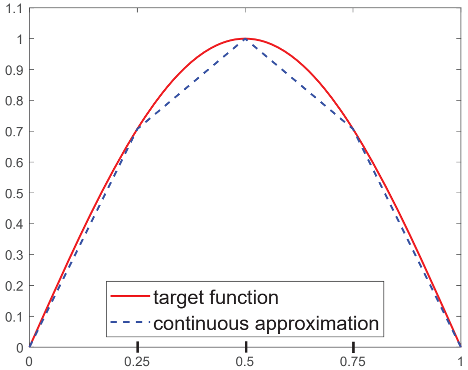

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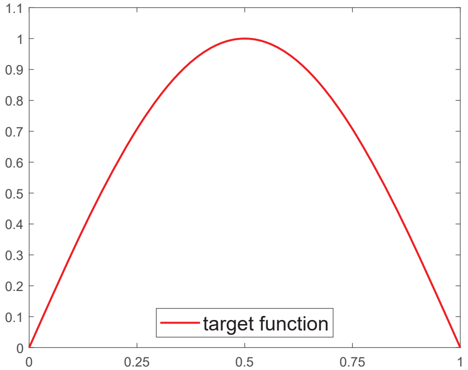

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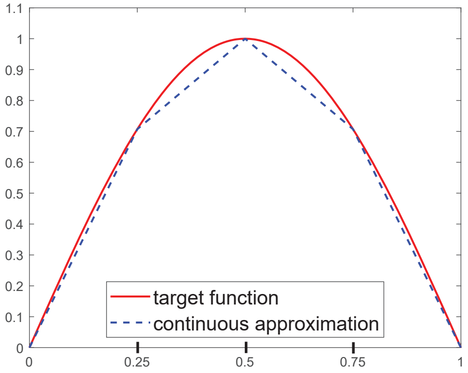

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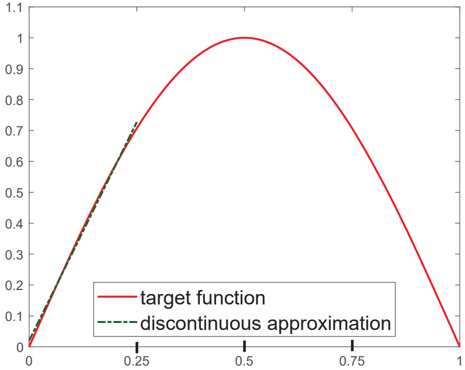

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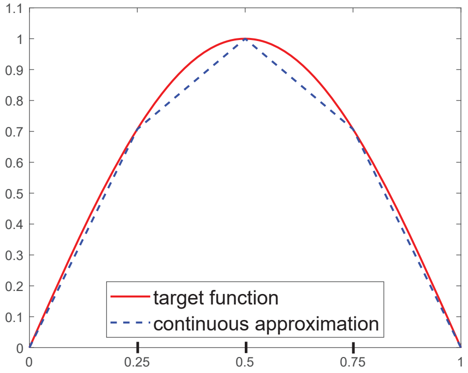

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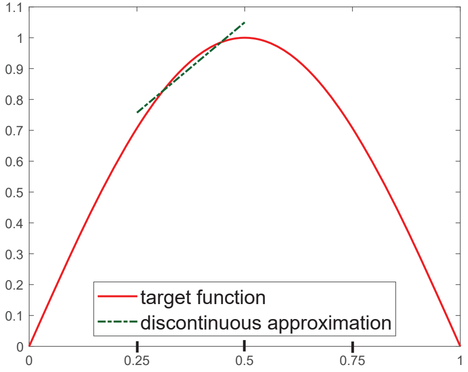

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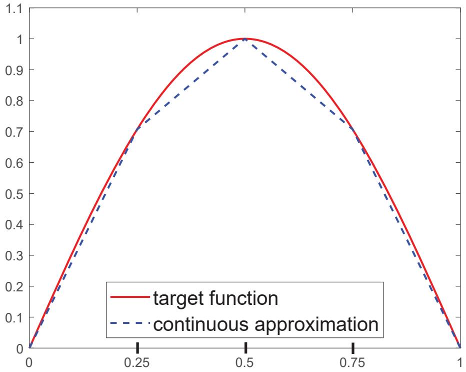

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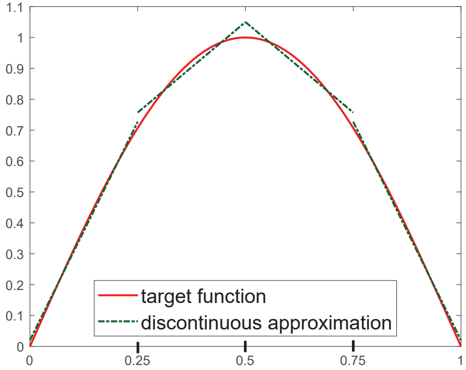

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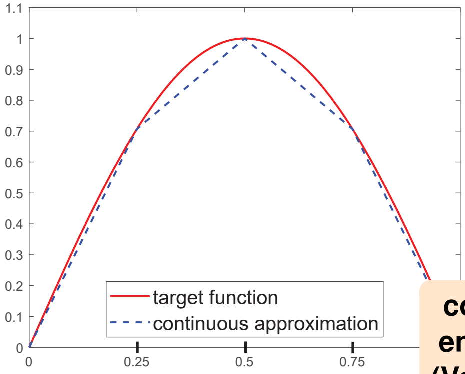
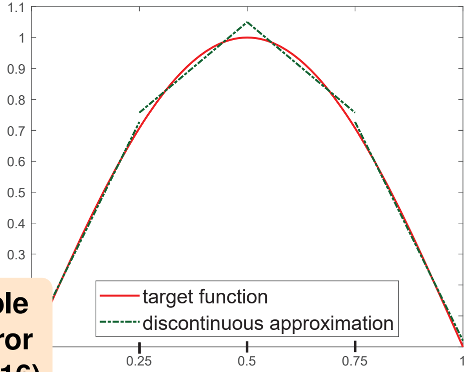


comparable energy error (Veeser 2016)



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1. Spaces $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$, finite element spaces, and the Laplace equation
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Two key Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega); \nabla v \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{div}, \Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak divergences in $L^2(\Omega)$,
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Two key Sobolev spaces with BCs

$H^1_{0,\mathrm{D}}(\Omega)$

$H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_\mathrm{D}\}$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N}$ in appropriate sense$\}$

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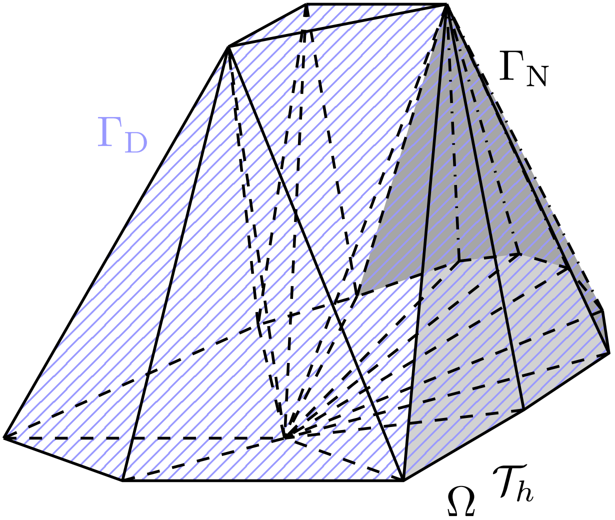
Meshes, elements, and patches



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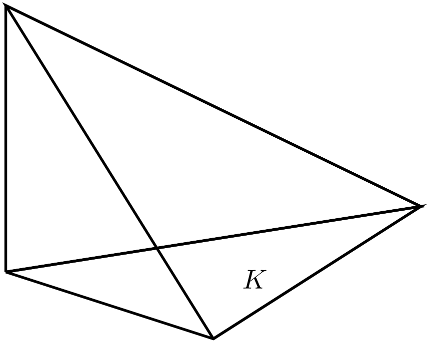

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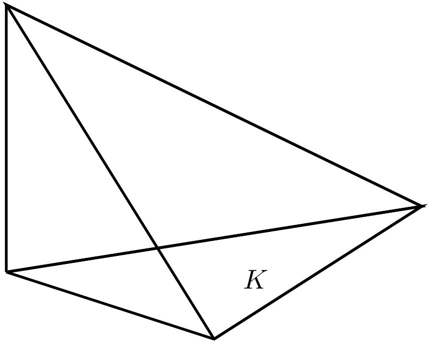

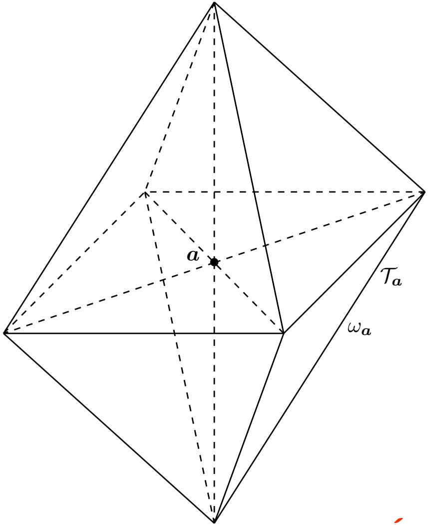

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Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

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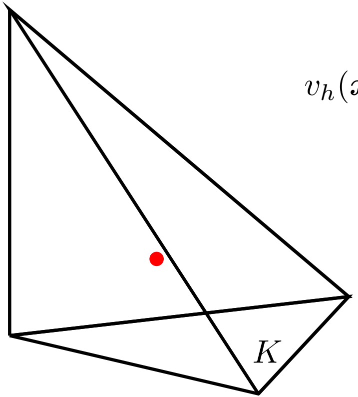

$v_h(\boldsymbol{x}) = a$

$p = 0$

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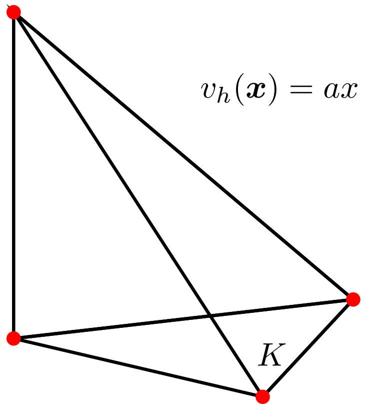
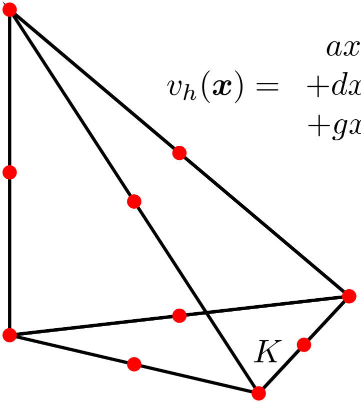


$$v_h(\boldsymbol{x}) = ax + by + cz + d$$

$p = 1$

Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$



$$v_h(\boldsymbol{x}) = \begin{array}{l} ax^2 + by^2 + cz^2 \\ +dxy + eyz + fxz \\ +gx + hy + iz + j \end{array}$$

$p = 2$

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

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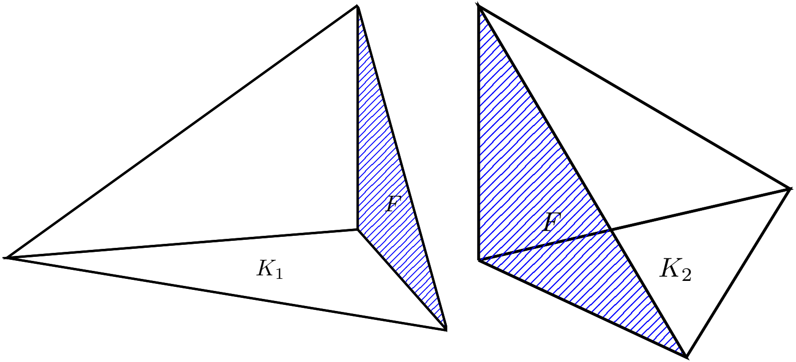

- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

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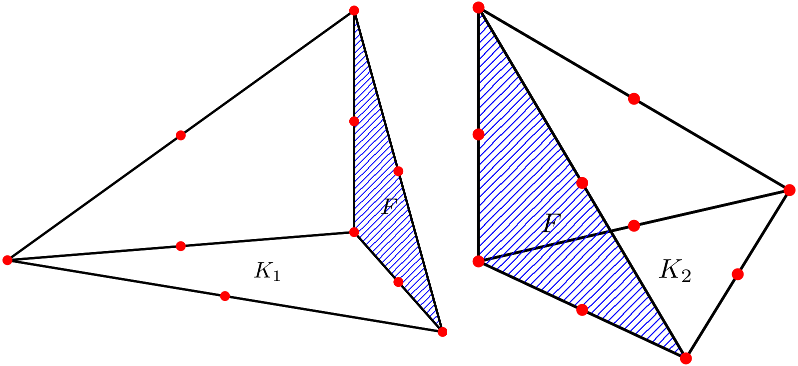

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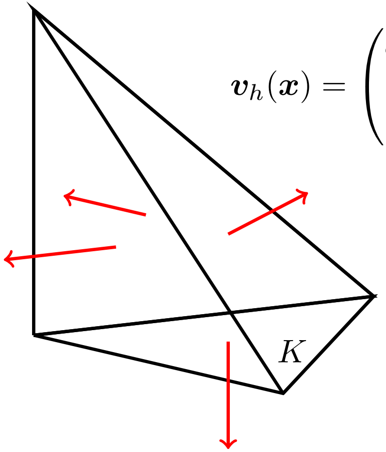
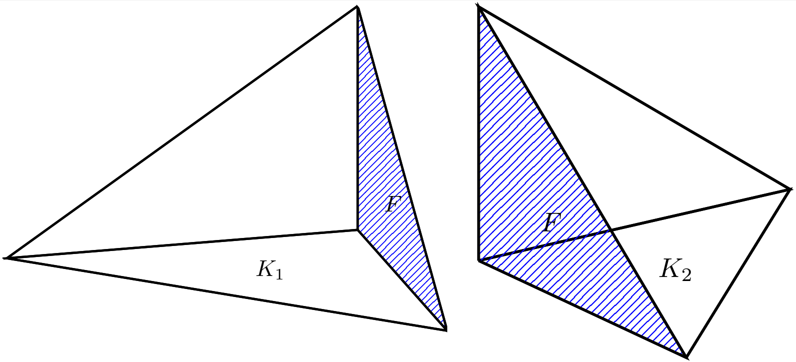


$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$p = 0$

Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$



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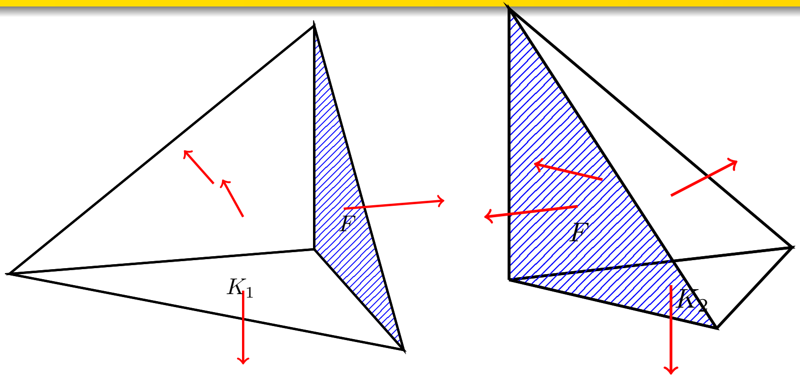

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The Laplace equation (source term $f \in L^2(\Omega)$)

The Laplace equation

Find $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{D}}, \\ -\nabla u \cdot \boldsymbol{n}_\Omega &= 0 && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H^1_{0,\mathrm{D}}(\Omega)$ *such that*

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega).$$

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Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, $p \geq 1$, *s.t.*

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Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

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Dual weak formulation

$\boldsymbol{\sigma} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with* $\nabla\cdot\boldsymbol{\sigma} = f$ *such that*

$$(\boldsymbol{\sigma}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \text{ with } \nabla\cdot\boldsymbol{v} = 0.$$

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$$(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) = 0 \quad \forall \boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \text{ with } \nabla\cdot\boldsymbol{v}_h = 0.$$

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

Error characterisation

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(u - v_h)\|$$

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Outline

1. Spaces $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$, finite element spaces, and the Laplace equation
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   - Finite element spaces
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   - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
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Approximation error estimates in the $H^1(\Omega)$ context

h approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. *Then*

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005)($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
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Approximation error estimates in the $H^1(\Omega)$ context

hp approximation estimate

Let $v \in H^s(\not{\Omega})$, $s \geq 1$. *Then*

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, \not{p}) \frac{h^{\min\{p, s-1\}}}{p^{s-1}} \|v\|_{H^s(\not{\Omega})}.$$

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Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ *with*

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- $P_h^p : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$: a locally defined projector
- $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

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Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ *with*

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$. *Then*

$$\|\nabla(v - P_h^p v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s) \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

- $P_h^p : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$: a locally defined projector
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Approximation error estimates in the $\boldsymbol{H}(\mathrm{div},\Omega)$ context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \le C(\kappa_{\mathcal{T}_h}, d, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
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Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

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Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, ~~$s > 1/2$~~. *Then*

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Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^{s_K}(K), \quad (\nabla \cdot \boldsymbol{v})|_K \in [\mathcal{P}_p(K)]^3 \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$.

- also for varying polynomial degree

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$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \nabla\cdot\boldsymbol{v}}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s) \sum_{K\in\mathcal{T}_h} \left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}} \|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2.$$

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for $s_K \geq 0$ *and* $t_K \geq 0$.

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$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)\sum_{K\in\mathcal{T}_h}\left[\left(\frac{h_K^{\min\{p+1,s_K\}}}{(p+1)^{s_K}}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s_K}(K)}\right)^2 + \left(\frac{h_K}{p+1}\frac{h_K^{\min\{p+1,t_K\}}}{(p+1)^{t_K}}\|\nabla\cdot\boldsymbol{v}\|_{H^{t_K}(K)}\right)^2\right].$$

- also for varying polynomial degree

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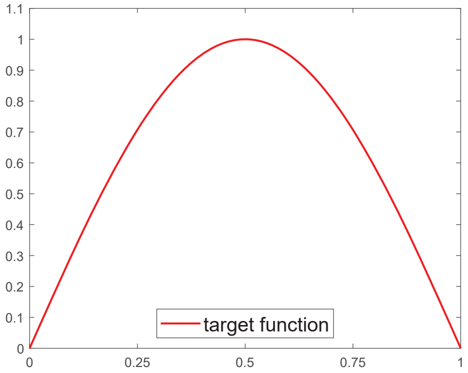
Global-best approximation ≈ local-best approximation

Previous contributions

- Carstensen, Peterseim, and Schedensack (2012): H^1 (lowest-order case $p = 1$)
- Aurada, Feischl, Kemetmüller, Page, and Praetorius (2013): H^1 (boundary approximation context)
- Veeser (2016): H^1 (any p, p-dependent constant)
- Canuto, Nochetto, Stevenson, and Verani (2017): H^1 (improvement of the p dependence of the equivalence constant in 2D)
- Ern, Gudi, Smears, and Vohralík (2022): $\boldsymbol{H}(\mathrm{div})$ (any p, p-dependent constant)
- Chaumont-Frelet and Vohralík (2021, 2022): $\boldsymbol{H}(\mathrm{curl})$ (any p, p-dependent constant)
- Gawlik, Holst, and Licht (2021): finite element exterior calculus context (any p, p-dependent constant)

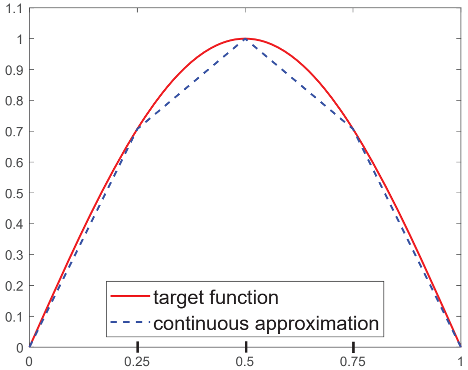
Outline

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



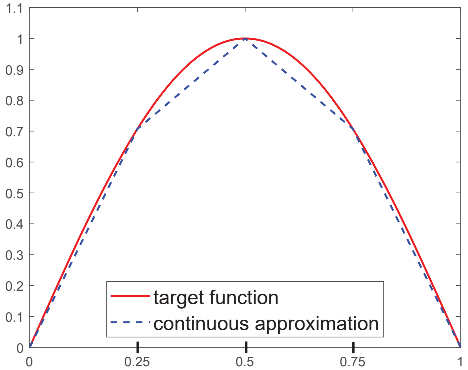
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



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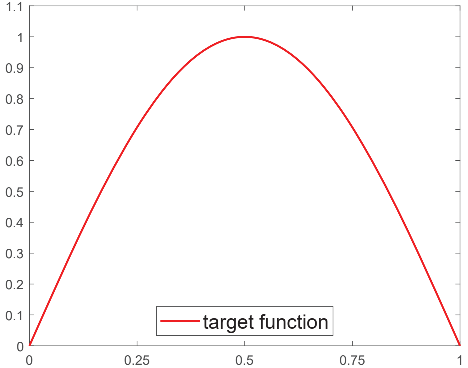

Target function in $H_0^1(\Omega)$

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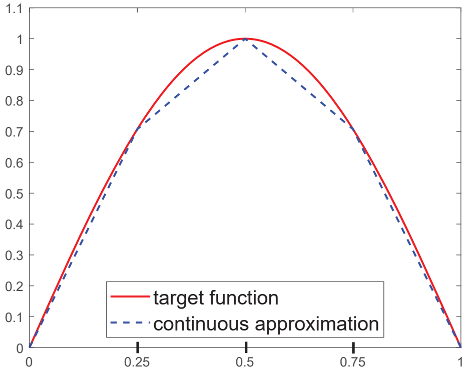

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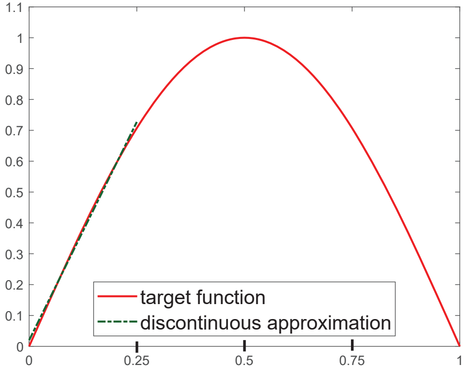
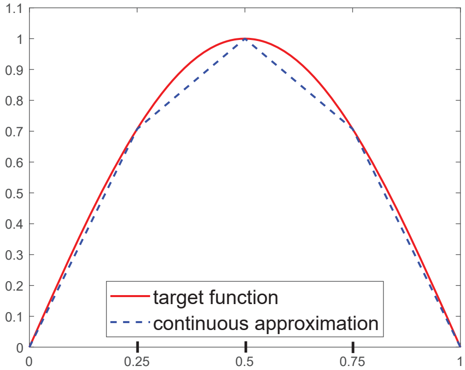
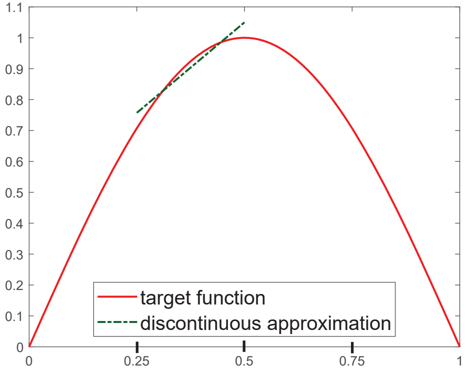


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

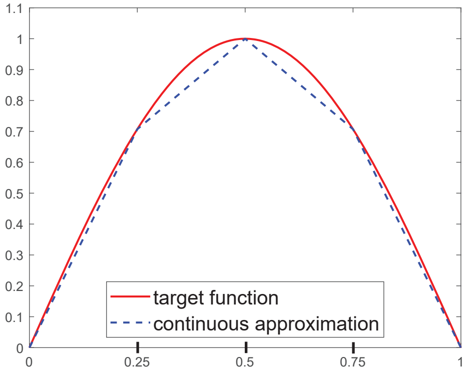


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

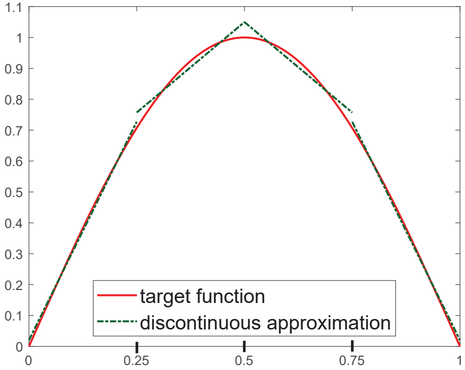


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

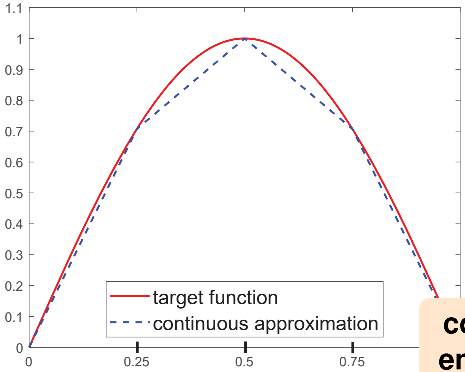


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

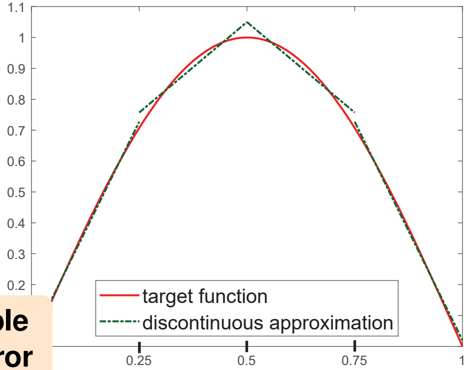


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



comparable energy error



Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, Verani (2017))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, Verani (2017))

$$\min_{\textit{smaller space}} \approx_p \min_{\textit{bigger space}}$$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, Verani (2017))

$$\min_{\textit{CG space}} \approx_p \min_{\textit{DG space}}$$

Equivalence of global- and local-best approximations in $H^1_0(\Omega)$

Theorem (Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, Verani (2017))

Let $v \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h)\cap H^1_0(\Omega)} \|\nabla(v-v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\min_{v_h\in\mathcal{P}_p(K)} \|\nabla(v-v_h)\|_K^2}_{\substack{\text{local-best on each } K\in\mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, Verani (2017))

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- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p

Main result

Theorem (Equivalence in $H^1_{0,\mathrm{D}}(\Omega)$)

Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

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Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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Let $v \in H^1_{0,\mathrm{D}}(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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- $\approx$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$
- also for varying polynomial degree

Outline

1. Spaces $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$, finite element spaces, and the Laplace equation
   - Sobolev spaces
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5. Conclusions

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022)**)**

$$\min_{\textit{smaller space with constraints}} \approx_p \min_{\textit{bigger space without constraints}}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

$$\min_{\text{MFE space with constraints}} \approx_p \min_{\text{broken MFE space without constraints}}$$

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RTN}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot \boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx_p \underbrace{\sum_{K\in\mathcal{T}_h} \left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

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$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div},\Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(\mathcal{T}_h)\cap \boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on }\Omega\\ \text{normal trace-continuity constraint}\\ \text{divergence constraint}\\ \text{MFE space (much smaller)}}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h}\underbrace{\left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RTN}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v}-\Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each }K\\ \text{no normal trace-continuity constraint}\\ \text{no divergence constraint}\\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Equivalence of global- and local-best approximations in $\boldsymbol{H}(\mathrm{div}, \Omega)$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

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$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_h^p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Main result

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RTN}}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

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Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary. Then,

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global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RTN}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p\nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\text{local-best on each } K \text{; no normal trace-continuity constraint; no divergence constraint; broken MFE space (much bigger)}}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx$: only depends on d and shape-regularity of $\mathcal{T}_h$

Outline

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Commuting de Rham diagram with operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Commuting de Rham diagram with operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

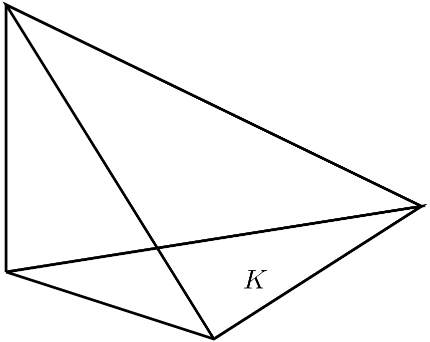
Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (**minimal regularity**)
2. is defined **locally** (in neighborhood of mesh elements)
3. is defined simply (starting from the **elementwise L^2 orthogonal projection**)
4. has optimal *hp* approximation properties, that of **elementwise div-unconstrained L^2-orthogonal projector** (global–local equivalence)
5. is **stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting properties** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

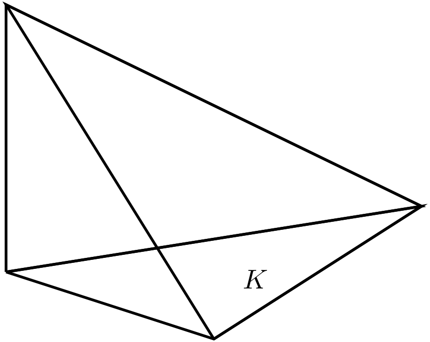
- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable
- Ern, Gudi, Smears, and Vohralík (2022): all the above properties in $\boldsymbol{H}(\mathrm{div})$
- none is p-robust

Classical elementwise interpolation



- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$

Classical elementwise interpolation



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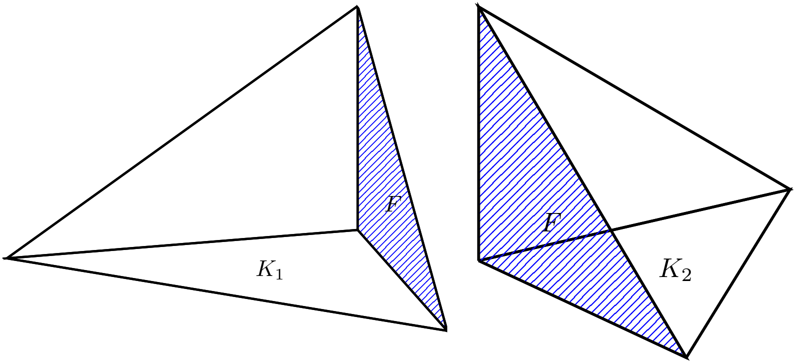
Classical elementwise interpolation: conformity enforcement



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K_2)$, and $(\boldsymbol{v}|_{K_1} \cdot \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \cdot \boldsymbol{n}_F)|_F$ in appropriate sense ($p = 0$)

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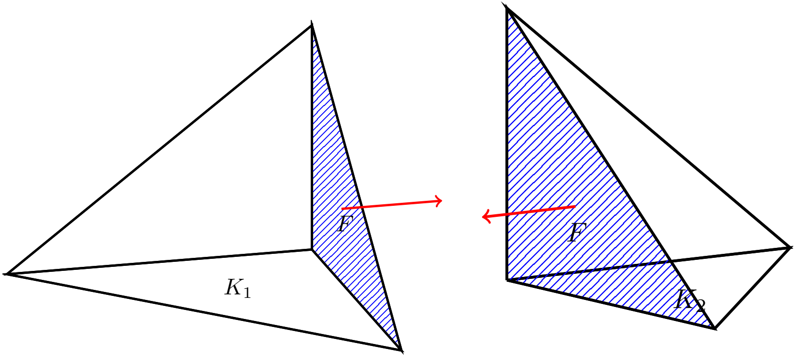

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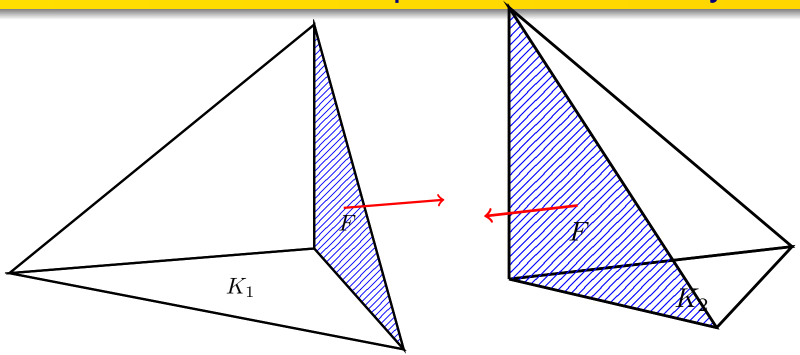

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Clash

Face normal trace integrals not available in $\boldsymbol{H}(\mathrm{div})$.

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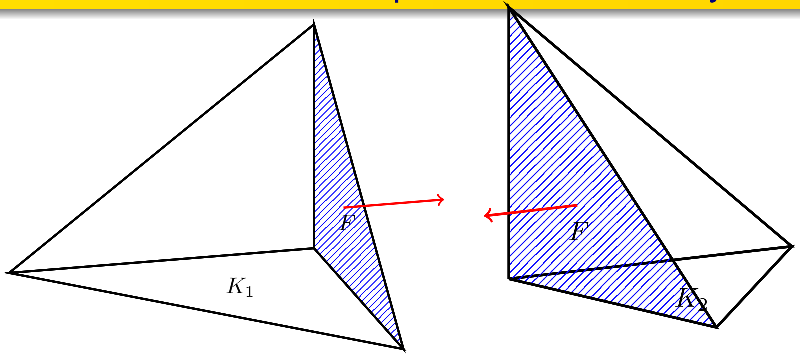

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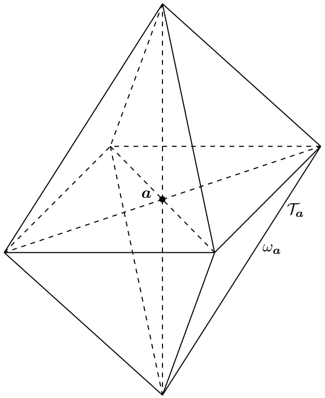
Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ requested.

Classical patchwise interpolation (Clément)



- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

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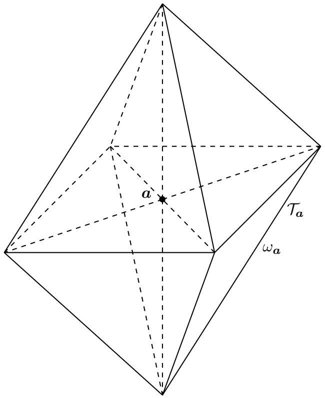

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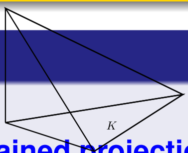
Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$



Definition (A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given (minimal regularity).

1. For each $K \in \mathcal{T}_h$, define $\iota_h|_K$ by the **elementwise unconstrained projection**

$$\iota_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

(discrete but nonconforming (normal-trace discontinuous)).

2. Obtain $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ by applying the **flux equilibration procedure** to ι_h; in particular, $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$, where $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ are obtained by local energy minimizations on the patch subdomains $\omega_{\boldsymbol{a}}$.
3. Apply a p-**stable decomposition** to conforming local projections of the reminder $\iota_h - \boldsymbol{\sigma}_h$, yielding a correction $\boldsymbol{\zeta}_h$. Obtain $\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) := \boldsymbol{\sigma}_h + \boldsymbol{\zeta}_h$.

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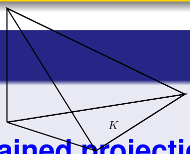

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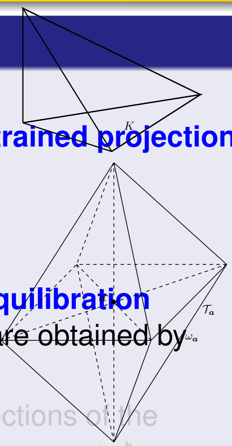

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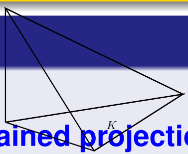

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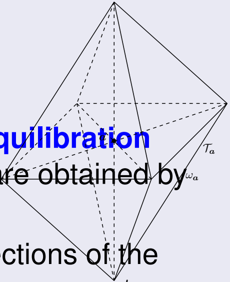

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Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}}: \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is a* **commuting projector** *since*

$$\nabla\cdot \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$
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Moreover, it has **local-best approximation properties** *and is* **L^2 stable** *up to data oscillation, since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $K \in \mathcal{T}_h$,*

$$\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\cdot(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}))\|_K\right)^2$$
$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\},$$

$$\|\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K^2 \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left\{ \|\boldsymbol{v}\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2 \right\}.$$

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Outline

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Piecewise polynomial approximation in $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$:

- a *p*-stable **local (commuting) projector**
- *p*-robust **global**-best–**local**-best **equivalence**
- elementwise *hp* **approximation** error **estimates** under **minimal Sobolev regularity**

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Chaumont-Frelet T., Vohralík M. p-robust equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{curl})$ based on local minimizations. Application to a posteriori analysis of the curl–curl problem. *SIAM Journal on Numerical Analysis* **61** (2023), 1783–1818.

Demkowicz L., Vohralík M. p-robust equivalence of global continuous constrained and local discontinuous unconstrained approximation, a p-stable local commuting projector, and optimal elementwise *hp* approximation estimates in $\boldsymbol{H}(\mathrm{div})$. HAL Preprint 04503603, 2024.

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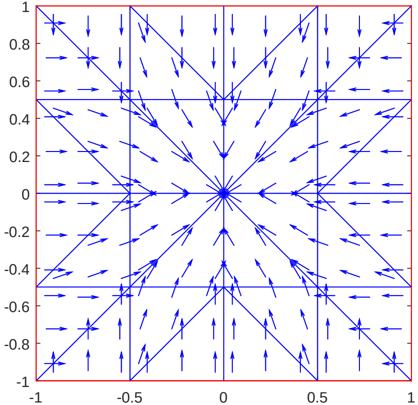
Thank you for your attention!

Outline

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



Flux $\boldsymbol{\iota}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\iota}_h \neq f$

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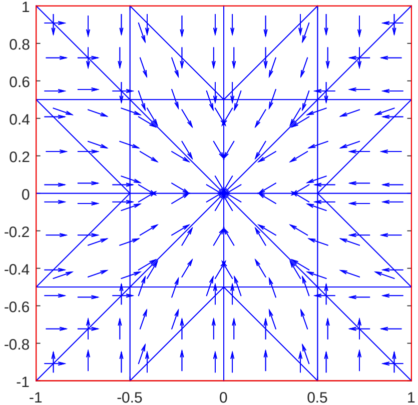

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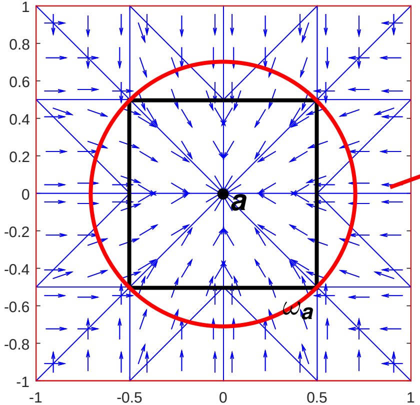

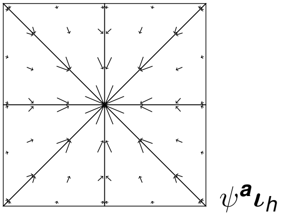

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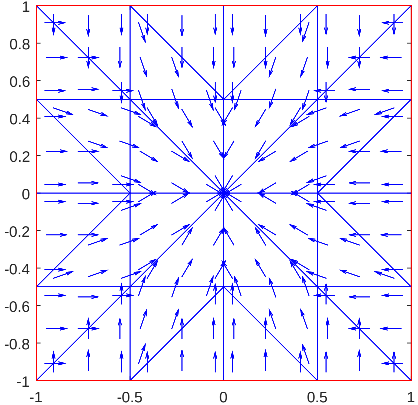

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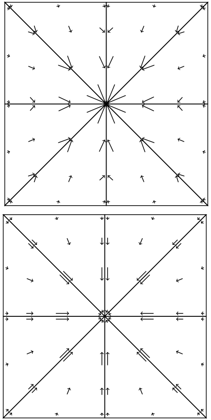
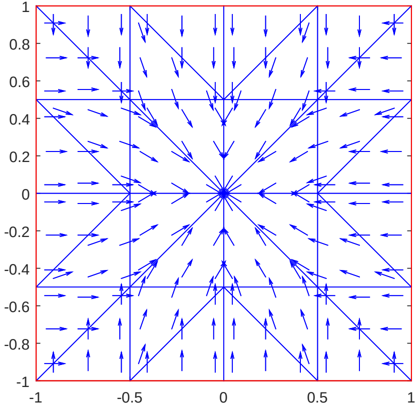


$\psi^{\boldsymbol{a}} \boldsymbol{\iota}_h$

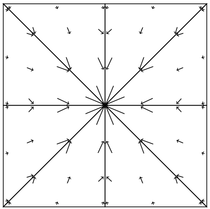
$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\iota}_h, \nabla \psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}}$$

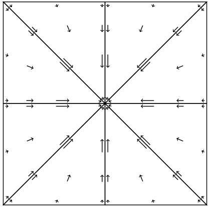
Equilibration in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



Flux $\boldsymbol{\iota}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\iota}_h \neq f$



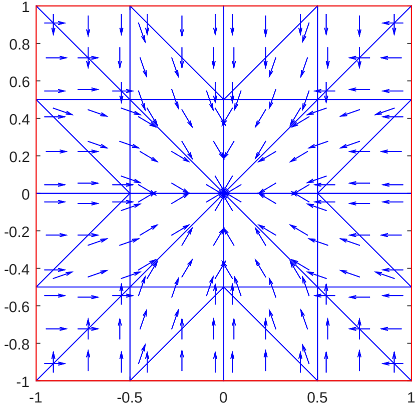
$\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h$



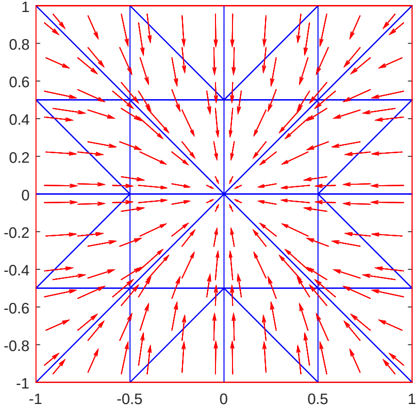
$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=f\psi^{\boldsymbol{a}}+\boldsymbol{\iota}_h\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h-\boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



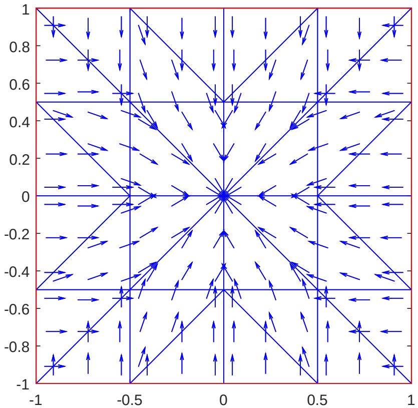
Flux $\boldsymbol{\iota}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla\cdot\boldsymbol{\iota}_h \neq f$



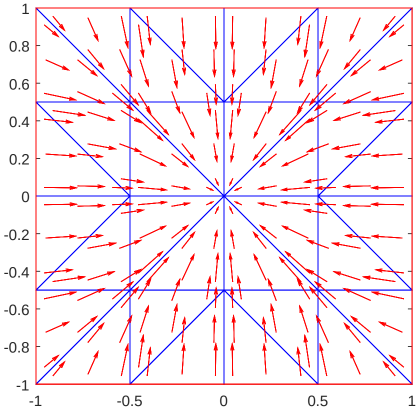
Equilibrated flux rec. $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}),\ \nabla\cdot\boldsymbol{\sigma}_h = f$$

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



Flux $\boldsymbol{\iota}_h \notin \boldsymbol{H}(\mathrm{div})$, $\nabla{\cdot}\boldsymbol{\iota}_h \neq f$



Equilibrated flux rec. $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div})$, $\nabla{\cdot}\boldsymbol{\sigma}_h = f$

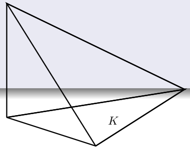
Outline

$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_{\mathcal{F}} \in \mathcal{P}_p(\Gamma_{\mathcal{F}})$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{v}_p = r_K \\ \boldsymbol{v}_p\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \nabla\cdot\boldsymbol{v} = r_K \\ \boldsymbol{v}\cdot\boldsymbol{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\boldsymbol{v}\|_K .$$



Comments

- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data r_K, $r_{\mathcal{F}}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{div}, K)$
- extension to a vertex patch: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

$\boldsymbol{H}$(div) polynomial extensions on a tetrahedron

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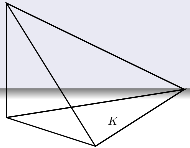

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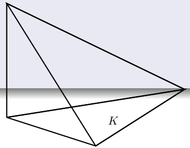

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$\boldsymbol{H}(\mathrm{div})$ polynomial extensions on a tetrahedron and on patches

Theorem ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)**)**

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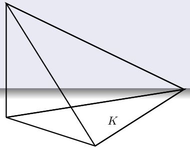

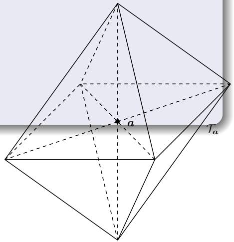

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Outline

6 Equilibration in $\boldsymbol{H}(\mathrm{div})$

7 1st key p-robustness tool: stable (broken) polynomial extensions

8 2nd key p-robustness tool: stable decompositions

$\boldsymbol{H}$(div) stable decomposition

Theorem ($\boldsymbol{H}$(div) stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008)**)**

Let $d = 2$ and let $\overline{\Omega}$ be contractible. Let

$$\boldsymbol{\delta}_p \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p = 0, \qquad \text{div-free}$$

$$(\boldsymbol{\delta}_p, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\ \forall K \in \mathcal{T}_h. \qquad \text{vanishing means}$$

Then there exists a decomposition of $\boldsymbol{\delta}_p$ as

$$\boldsymbol{\delta}_p = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\delta}_p^{\boldsymbol{a}}, \qquad \text{decomposition}$$

where

$\boldsymbol{\delta}_p^{\boldsymbol{a}}$ are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}}$, linearly depend on $\boldsymbol{\delta}_p$ on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$,

and satisfy

$$\boldsymbol{\delta}_p^{\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p^{\boldsymbol{a}} = 0, \qquad \text{local}$$

$$\|\boldsymbol{\delta}_p^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\delta}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \qquad p\text{-stable}$$