p-robust multigrid and domain decomposition solvers in the H^1 and $\boldsymbol{H}(\mathrm{div})$ settings

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ÉCOLE NATIONALE DES PONTS ET CHAUSSÉES

INSTITUT POLYTECHNIQUE DE PARIS

Outline

Outline

1. Introduction
   - Basis dependency
   - Moments and orthogonal decomposition
   - Line search
   - Stable patchwise decomposition (p-robust)
   - Stable levelwise decomposition ($p = 1$)
   - A posteriori error localization and adaptivity
2. p-robust multigrid in the H^1 setting
   - Solver
   - Numerical results
   - Adaptive number of smoothing steps and adaptive local smoothing
3. p-robust multigrid and domain decomposition in the $\boldsymbol{H}(\mathrm{div})$ setting
   - Solvers
   - Numerical results
4. Conclusions and future directions

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Basis dependency

Finite elements

Find $u_J \in V_J := \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$ such that

$$(\nabla u_J, \nabla v) = (f, v) \qquad \forall v \in V_J.$$

Basis dependency

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Basis-indepenent

functional formulation

Basis dependency

Finite elements

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Basis-indepenent

functional formulation

Algebraic problem

Find $\mathrm{U}_J \in \mathbb{R}^{|V_J|}$ such that

$$\mathbb{A}_J \mathrm{U}_J = \mathrm{F}_J.$$

Basis dependency

Finite elements

Find $u_J \in V_J := \mathcal{P}_p(\mathcal{T}_J) \cap H^1_0(\Omega)$ such that

$$(\nabla u_J, \nabla v) = (f, v) \qquad \forall v \in V_J.$$

Basis-indepenent

functional formulation

Algebraic problem

Find $\mathrm{U}_J \in \mathbb{R}^{|V_J|}$ such that

$$\mathbb{A}_J \mathrm{U}_J = \mathrm{F}_J.$$

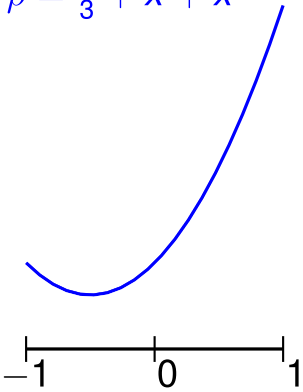
Basis-depenent

sparsity, condition number, ...

Outline

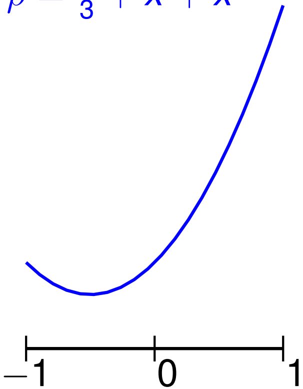
Moments

$$\rho = \frac{2}{3} + x + x^2$$



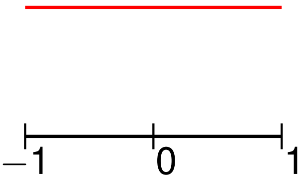
Moments

$\rho = \frac{2}{3} + x + x^2$



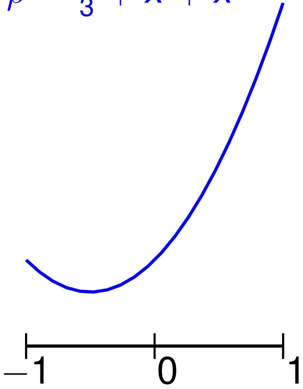
$\rho = \rho_0$

$\rho_0 = 1$

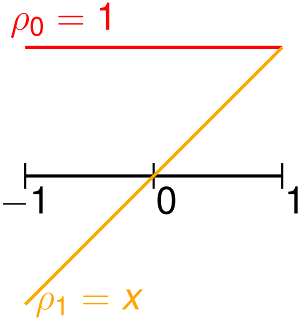


Moments

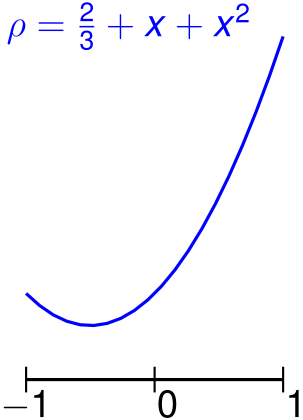
$$\rho = \frac{2}{3} + x + x^2$$

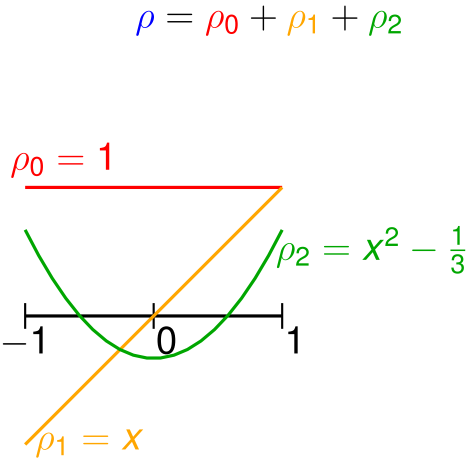


$$\rho = \rho_0 + \rho_1$$

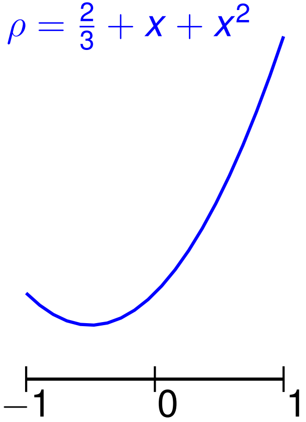


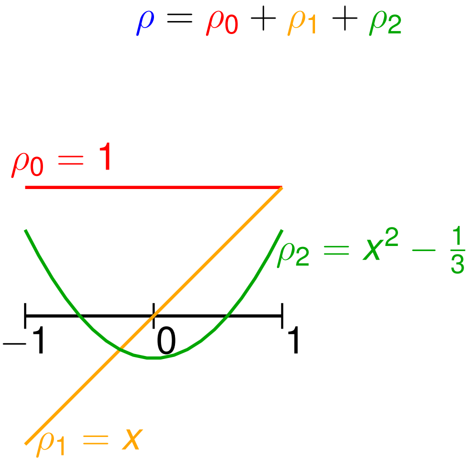
Moments


$\rho = \frac{2}{3} + x + x^2$
−1
0
1

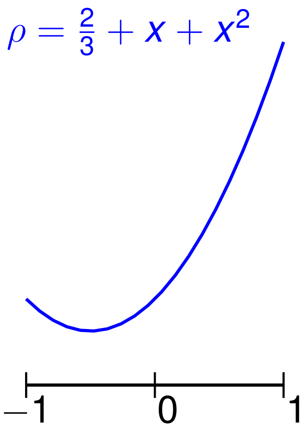

$\rho = \rho_0 + \rho_1 + \rho_2$
$\rho_0 = 1$
$\rho_2 = x^2 - \frac{1}{3}$
−1
0
1
$\rho_1 = x$

Moments, decomposition



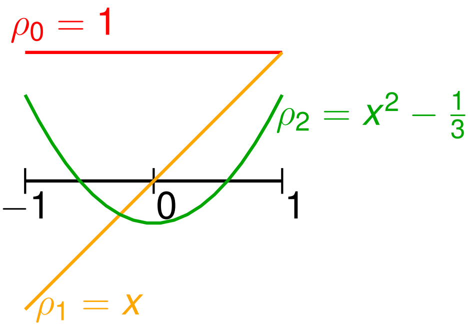


Moments, orthogonal decomposition



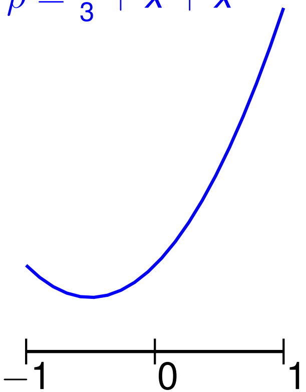
$$\rho = \rho_0 + \rho_1 + \rho_2$$

$$\|\rho\|^2 = \sum_{j=0}^{2} \|\rho_j\|^2$$



Moments, orthogonal decomposition, nonsymmetry

$\rho = \frac{2}{3} + x + x^2$



$\rho = \rho_0 + \rho_1 + \rho_2$

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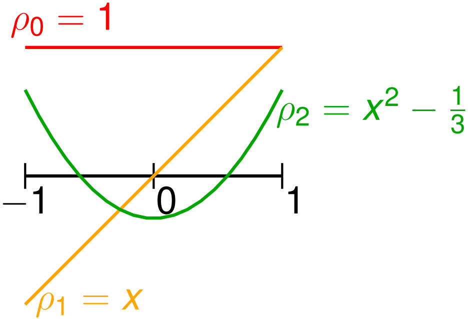


Nonsymmetric procedure

From coarsest (most important) to finest (details)

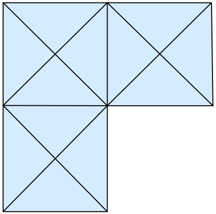
Multigrid setting

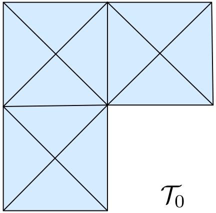
meshes

Multigrid setting

meshes

- $\mathcal{T}_0$



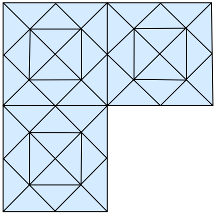


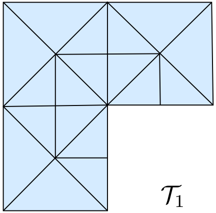
$\mathcal{T}_0$

Multigrid setting

meshes

- $\mathcal{T}_0$, uniformly or adaptively refined



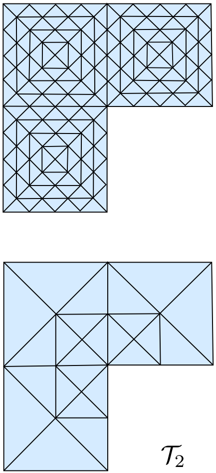


$\mathcal{T}_1$

Multigrid setting

meshes

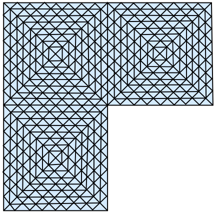
- $\mathcal{T}_0$, uniformly or adaptively refined

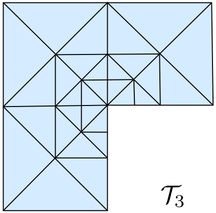


Multigrid setting

meshes

- $\mathcal{T}_0$, uniformly or adaptively refined



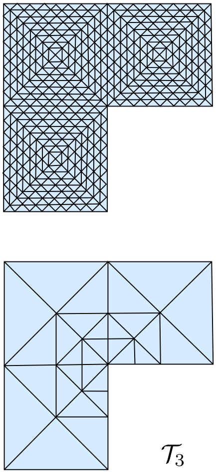


$\mathcal{T}_3$

Multigrid setting

Hierarchy of meshes $\{\mathcal{T}_j\}_{0\le j\le J}$

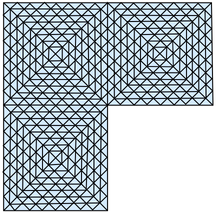
- $\mathcal{T}_0$, uniformly or adaptively refined

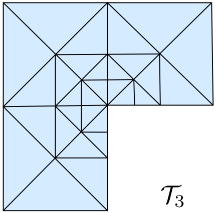


Multigrid setting

Hierarchy of meshes $\{\mathcal{T}_j\}_{0 \le j \le J}$

- $\mathcal{T}_0$, uniformly or adaptively refined
- $\mathcal{T}_0$ is quasi-uniform
- all $\mathcal{T}_j$ are shape-regular
- maximum strength of refinement between $\mathcal{T}_{j-1}$ and $\mathcal{T}_j$





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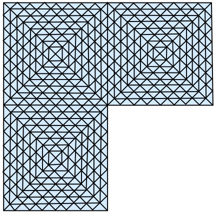
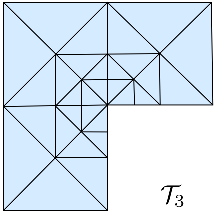


Hierarchy of increasing polynomial degrees

For given $p \ge 1$, let

$$1 = p_0 \le p_1 \le p_2 \le \ldots \le p_J = p$$



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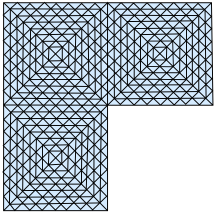

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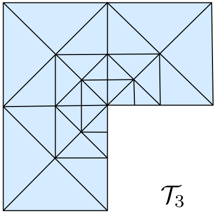
Economical choice

$$1 = p_0 = p_1 = \ldots p_{J-1},\ p_J = p$$



$\mathcal{T}_3$

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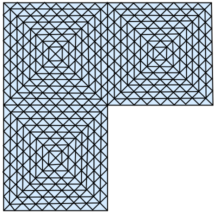

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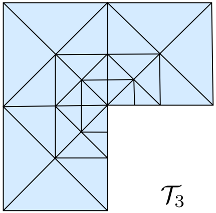

$\mathcal{T}_3$

Hierarchy of nested spaces

$$V_j := \mathcal{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\Omega).$$

Economical choice

$$1 = p_0 = p_1 = \ldots p_{J-1},\; p_J = p$$

$u_J^i \in V_J$: moments

$u_J^i \in V_J$: moments

Coarse-grid moment

$\rho_0^i \in V_0$ such that $\underbrace{(\nabla \rho_0^i, \nabla v_0)}_{\text{global lifting}} = \underbrace{(f, v_0) - (\nabla u_J^i, \nabla v_0)}_{\text{initial algebraic residual}} \quad \forall v_0 \in V_0$

$u_J^i \in V_J$: moments

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Level 1 moment

$\rho_1^i \in V_1$, the solution of

$$\underbrace{(\nabla \rho_1, \nabla v_1)}_{\text{global lifting}} = \underbrace{(f, v_1) - (\nabla(u_J^i + \rho_0^i), \nabla v_1)}_{\text{current algebraic residual}} \quad \forall v_1 \in V_1$$

$u_J^i \in V_J$: moments

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$\rho_0^i \in V_0$ such that

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Level j moments

$\rho_j^i \in V_j$, $1 \leq j \leq J$, the solutions of

$$\underbrace{(\nabla \rho_j^i, \nabla v_j)}_{\text{global lifting}} = \underbrace{(f, v_j) - \left(\nabla \left(u_J^i + \sum_{k=0}^{j-1} \rho_k^i\right), \nabla v_j\right)}_{\text{current algebraic residual}} \quad \forall v_j \in V_j$$

$u_J^i \in V_J$: moments

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Level j moments: impractical

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$u_J^i \in V_J$: moments

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$$u_J^{i+1} := u_J^i + \sum_{j=0}^{J} \rho_j^i,$$

$u_J^i \in V_J$: moments, decomposition

Coarse-grid moment

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$$\underbrace{(\nabla \rho_0^i, \nabla v_0)}_{\text{global lifting}} = \underbrace{(f, v_0) - (\nabla u_J^i, \nabla v_0)}_{\text{initial algebraic residual}} \quad \forall v_0 \in V_0$$

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decomposition

$$u_J = u_J^{i+1} := u_J^i + \sum_{j=0}^{J} \rho_j^i,$$

$u_J^i \in V_J$: moments, orthogonal decomposition

Coarse-grid moment

$\rho_0^i \in V_0$ such that

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Orthogonal decomposition

$$u_J = u_J^{i+1} := u_J^i + \sum_{j=0}^{J} \rho_j^i, \qquad 0 = \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^i)\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \|\nabla \rho_j^i\|^2}_{\text{error decrease}}.$$

$u_J^i \in V_J$: moments, orthogonal decomposition, nonsymmetry

Coarse-grid moment

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Nonsymmetric procedure

From coarsest (most important) to finest (details)

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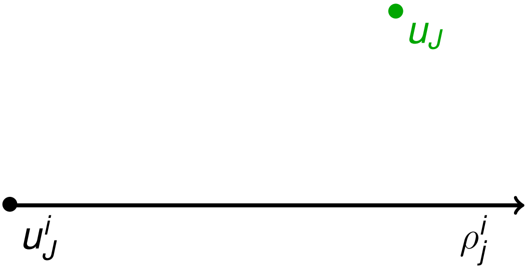
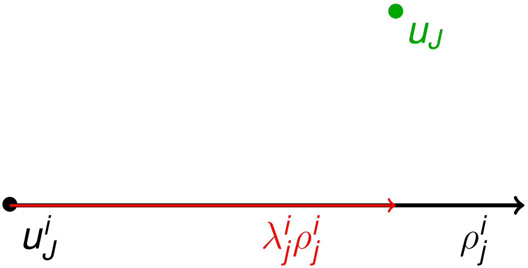
Line search

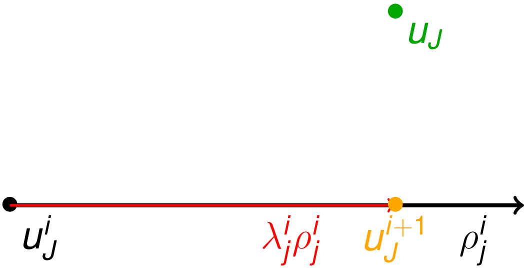
Line search



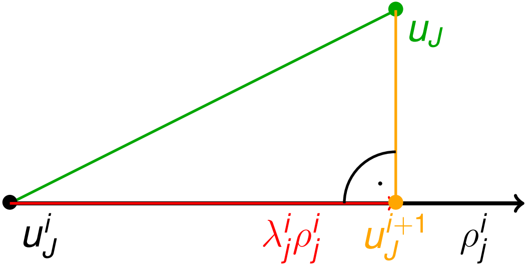
Line search



Line search



Line search



$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^i)\|^2}_{\text{old error}} - \underbrace{(\lambda_j^i \|\nabla \rho_j^i\|)^2}_{\text{computable decrease}}$$

Line search in multigrid

- current approximation $u^i_{J,j-1} := u^i_J + \sum_{k=0}^{j-1} \lambda^i_k \rho^i_k$
- j-level update (descent direction): ρ^i_j

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Lemma (Line search)

$$\textit{error}\ \|\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\|^2$$

Line search in multigrid

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- j-level update (descent direction): ρ^i_j

Lemma (Line search)

minimize the error $\|\nabla(u_J - (u^i_{J,j-1} + \lambda\rho^i_j))\|^2$ *over all possible* $\lambda \in \mathbb{R}$

Line search in multigrid

- current approximation $u^i_{J,j-1} := u^i_J + \sum_{k=0}^{j-1} \lambda^i_k \rho^i_k$
- j-level update (descent direction): ρ^i_j

Lemma (Line search)

The choice

$$\lambda^i_j := \frac{(f, \rho^i_j) - (\nabla u^i_{J,j-1}, \nabla \rho^i_j)}{\|\nabla \rho^i_j\|^2}$$

minimizes the error $\|\nabla(u_J - (u^i_{J,j-1} + \lambda \rho^i_j))\|^2$ *over all possible* $\lambda \in \mathbb{R}$

Line search in multigrid

- current approximation $u^i_{J,j-1} := u^i_J + \sum_{k=0}^{j-1} \lambda^i_k \rho^i_k$
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Proof. (Minimization of a quadratic function $\mathbb{R} \to \mathbb{R}$).

$$\|\nabla(u_J - (u^i_{J,j-1} + \lambda \rho^i_j))\|^2 = \|\nabla(u_J - u^i_{J,j-1})\|^2 - 2\lambda \underbrace{(\nabla(u_J - u^i_{J,j-1}), \nabla \rho^i_j)}_{(f,\rho^i_j)-(\nabla u^i_{J,j-1},\nabla \rho^i_j)} + \lambda^2 \|\nabla \rho^i_j\|^2$$

Line search in multigrid

- current approximation $u^i_{J,j-1} := u^i_J + \sum_{k=0}^{j-1} \lambda^i_k \rho^i_k$
- j-level update (descent direction): ρ^i_j

Lemma (Line search: error decrease Pythagoras formula)

The choice

$$\lambda^i_j := \frac{(f, \rho^i_j) - (\nabla u^i_{J,j-1}, \nabla \rho^i_j)}{\|\nabla \rho^i_j\|^2}$$

minimizes the error $\|\nabla(u_J - (u^i_{J,j-1} + \lambda \rho^i_j))\|^2$ *over all possible* $\lambda \in \mathbb{R}$ *and gives*

$$\underbrace{\|\nabla(u_J - (u^i_{J,j-1} + \lambda^i_j \rho^i_j))\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u^i_{J,j-1})\|^2}_{\text{old error}} - \underbrace{(\lambda^i_j)^2 \|\nabla \rho^i_j\|^2}_{\text{computable decrease}}.$$

Proof. (Minimization of a quadratic function $\mathbb{R} \to \mathbb{R}$).

$$\|\nabla(u_J - (u^i_{J,j-1} + \lambda \rho^i_j))\|^2 = \|\nabla(u_J - u^i_{J,j-1})\|^2 - 2\lambda \underbrace{(\nabla(u_J - u^i_{J,j-1}), \nabla \rho^i_j)}_{(f,\rho^i_j) - (\nabla u^i_{J,j-1}, \nabla \rho^i_j)} + \lambda^2 \|\nabla \rho^i_j\|^2$$

Moments and line search in multigrid

Impractical global lifting

$$0 = \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \|\nabla \rho_j^i\|^2}_{\text{error decrease}}.$$

Moments and line search in multigrid

Impractical global lifting

$$0 = \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \|\nabla \rho_j^i\|^2}_{\text{error decrease}}.$$

Line search

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}.$$

Moments and line search in multigrid

Impractical global lifting

$$0 = \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \|\nabla \rho_j^i\|^2}_{\text{error decrease}} .$$

Line search: error decrease Pythagoras formula

$$0 \neq \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2} .$$

Moments and line search in multigrid

Impractical global lifting

$$0 = \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \|\nabla \rho_j^i\|^2}_{\text{error decrease}}.$$

Line search: error decrease Pythagoras formula

$$0 \neq \underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}.$$

Guaranteed a posteriori error estimate on the algebraic error

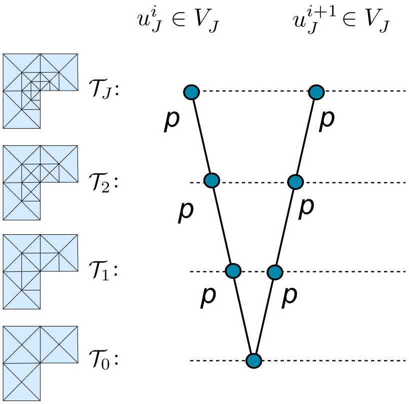
$$\eta_{\mathrm{alg}}^i \leq \|\nabla(u_J - u_J^i)\|$$

Moments and line search in multigrid

MG(p,p)-J: MG with p pre-smoothing and p post-smoothing by Jacobi (diagonal, symmetric)

Moments and line search in multigrid

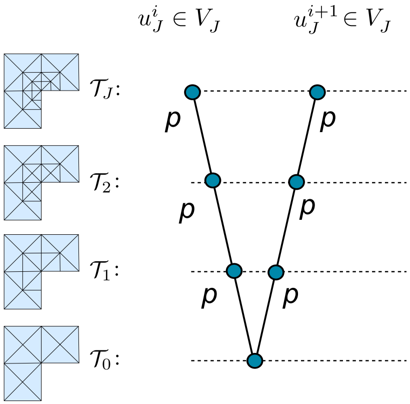
MG(p,p)-J: MG with p pre-smoothing and p post-smoothing by Jacobi (diagonal, symmetric)



Moments and line search in multigrid

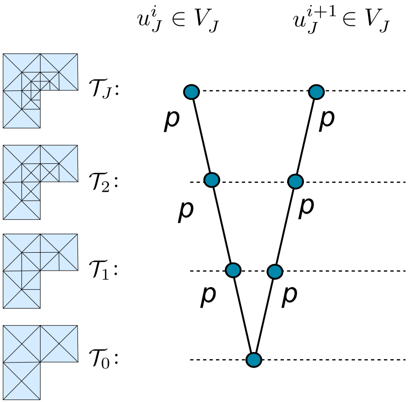
MG(p,p)-J: MG with p pre-smoothing and p post-smoothing by Jacobi (diagonal, symmetric)

MG(0,1)$^{\mathrm{ls}}$-J: MG with 0 pre-smoothing and 1 post-smoothing (**moments**) by Jacobi (diagonal, nonsymmetric) and **line search** on each level

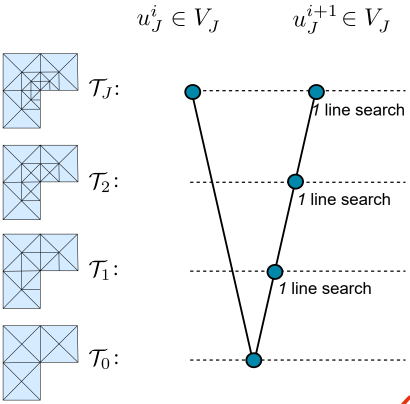


Moments and line search in multigrid

MG(p,p)-J: MG with p pre-smoothing and p post-smoothing by Jacobi (diagonal, symmetric)



MG(0,1)$^{\mathrm{ls}}$-J: MG with 0 pre-smoothing and 1 post-smoothing (**moments**) by Jacobi (diagonal, nonsymmetric) and **line search** on each level



Moments and line search

		Sine		Peak		L-shape	
J	p	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J
4	1	19	11	10	8	33	20
	2	32	19	21	24	63	102
	4	101	160	47	-	145	-
	6	893	-	393	-	2484	-
5	1	20	20	10	9	33	25
	2	34	-	20	26	64	104
	4	77	-	66	-	56	-
	6	856	-	403	-	2485	-

Moments and line search

		Sine			Peak			L-shape		
J	p	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J
4	1	19	19	11	10	10	8	33	33	20
	2	19	32	19	13	21	24	34	63	102
	4	20	101	160	14	47	-	36	145	-
	6	20	893	-	14	393	-	36	2484	-
5	1	20	20	20	10	10	9	33	33	25
	2	20	34	-	13	20	26	33	64	104
	4	20	77	-	14	66	-	33	56	-
	6	20	856	-	14	403	-	33	2485	-

- MG(0,1)$^{\mathrm{ls}}$-bJ: MG with 0 pre-smoothing and 1 post-smoothing (**moments**) by **block** Jacobi and **line search** on each level (the economical choice $p_0 = p_1 = \ldots p_{J-1} = 1$)

Moments and line search

J	p	Sine MG(0,1)$^{\mathrm{ls}}$-bJ	Sine MG(0,1)$^{\mathrm{ls}}$-J	Sine MG(p,p)-J	Peak MG(0,1)$^{\mathrm{ls}}$-bJ	Peak MG(0,1)$^{\mathrm{ls}}$-J	Peak MG(p,p)-J	L-shape MG(0,1)$^{\mathrm{ls}}$-bJ	L-shape MG(0,1)$^{\mathrm{ls}}$-J	L-shape MG(p,p)-J
4	1	19	19	11	10	10	8	33	33	20
	2	19	32	19	13	21	24	34	63	102
	4	20	101	160	14	47	-	36	145	-
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	2	20	34	-	13	20	26	33	64	104
	4	20	77	-	14	66	-	33	56	-
	6	20	856	-	14	403	-	33	2485	-

- MG(0,1)$^{\mathrm{ls}}$-bJ: MG with 0 pre-smoothing and 1 post-smoothing (**moments**) by **block** Jacobi and **line search** on each level (the economical choice $p_0 = p_1 = \ldots p_{J-1} = 1$)
- the usual requirement of "**sufficient number of smoothing steps**" (unknown parameter in practice) is **avoided**

Moments and line search & block Jacobi: 1 post-smoothing sufficient

		Sine			Peak			L-shape		
J	p	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J	MG(0,1)$^{\mathrm{ls}}$-bJ	MG(0,1)$^{\mathrm{ls}}$-J	MG(p,p)-J
4	1	19	19	11	10	10	8	33	33	20
	2	19	32	19	13	21	24	34	63	102
	4	20	101	160	14	47	-	36	145	-
	6	20	893	-	14	393	-	36	2484	-
5	1	20	20	20	10	10	9	33	33	25
	2	20	34	-	13	20	26	33	64	104
	4	20	77	-	14	66	-	33	56	-
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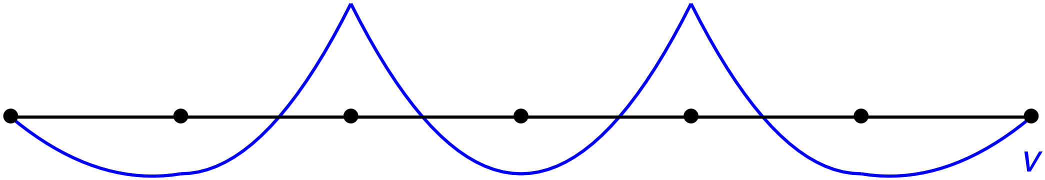
- MG(0,1)$^{\mathrm{ls}}$-bJ: MG with 0 pre-smoothing and 1 post-smoothing (**moments**) by **block** Jacobi and **line search** on each level (the economical choice $p_0 = p_1 = \dots p_{J-1} = 1$)
- the usual requirement of "**sufficient number of smoothing steps**" (unknown parameter in practice) is **avoided**

Outline

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

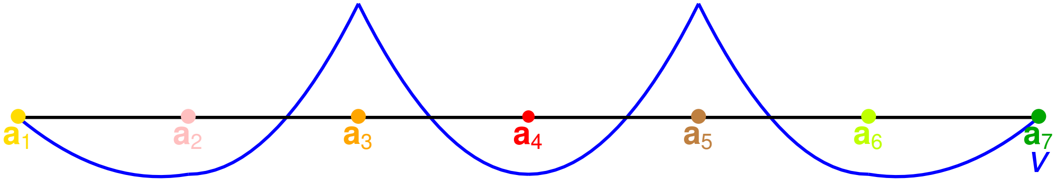
Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H^1_0(\Omega)$

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

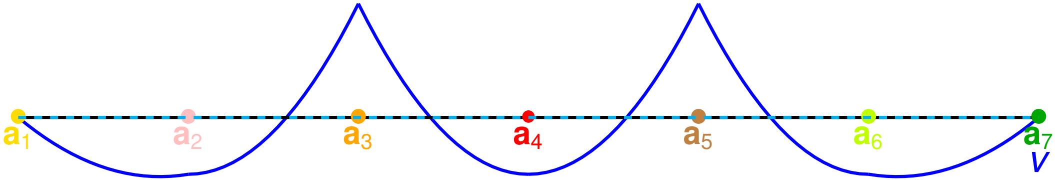
Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$
- vertices $\mathbf{a} \in \mathcal{V}_J$

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

Schöberl, Melenk, Pechstein, & Zaglmayr (2008)

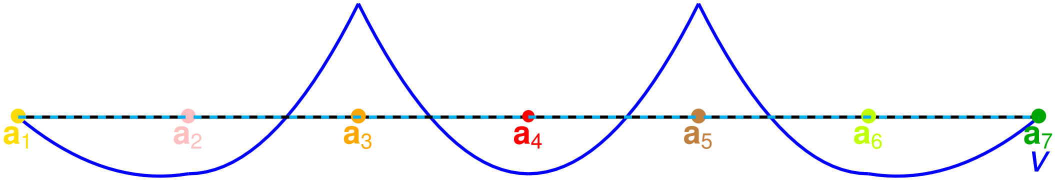


- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$
- vertices $\mathbf{a} \in \mathcal{V}_J$
- **decomposition** $v = v_0 +$,

$v_0 \in \mathcal{P}_1(\mathcal{T}_J) \cap H_0^1(\Omega)$

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

Schöberl, Melenk, Pechstein, & Zaglmayr (2008)

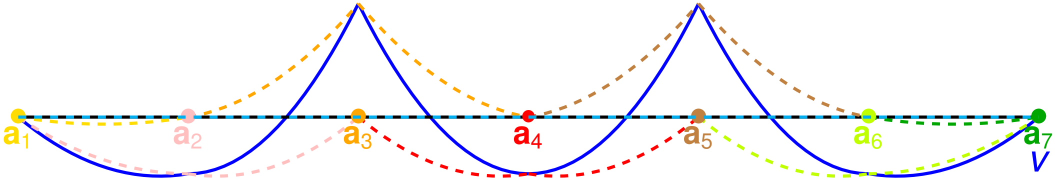


- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$
- vertices $\mathbf{a} \in \mathcal{V}_J$
- **decomposition** $v = v_0 + \sum_{\mathbf{a} \in \mathcal{V}_J} v_{\mathbf{a}}$,

$$v_0 \in \mathcal{P}_1(\mathcal{T}_J) \cap H_0^1(\Omega), \; v_{\mathbf{a}} \in \mathcal{P}_p(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_J^{\mathbf{a}})$$

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

Schöberl, Melenk, Pechstein, & Zaglmayr (2008)

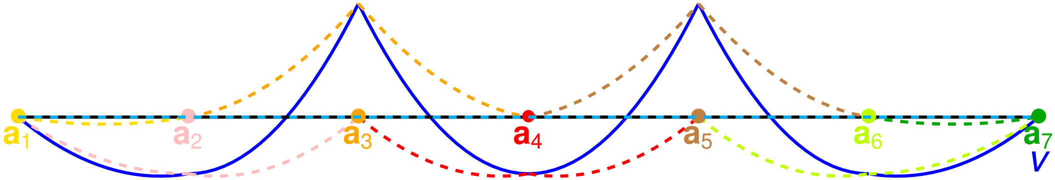


- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$
- vertices $\mathbf{a} \in \mathcal{V}_J$
- **decomposition** $v = v_0 + \sum_{\mathbf{a}\in\mathcal{V}_J} v_{\mathbf{a}} = v_0 + v_{\mathbf{a}_1} + v_{\mathbf{a}_2} + v_{\mathbf{a}_3} + v_{\mathbf{a}_4} + v_{\mathbf{a}_5} + v_{\mathbf{a}_6} + v_{\mathbf{a}_7}$,

  $v_0 \in \mathcal{P}_1(\mathcal{T}_J) \cap H_0^1(\Omega)$, $v_{\mathbf{a}} \in \mathcal{P}_p(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_J^{\mathbf{a}})$

p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



- $v \in \mathcal{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega)$
- vertices $\mathbf{a} \in \mathcal{V}_J$
- **decomposition** $v = v_0 + \sum_{\mathbf{a}\in\mathcal{V}_J} v_{\mathbf{a}} = v_0 + v_{\mathbf{a}_1} + v_{\mathbf{a}_2} + v_{\mathbf{a}_3} + v_{\mathbf{a}_4} + v_{\mathbf{a}_5} + v_{\mathbf{a}_6} + v_{\mathbf{a}_7}$,

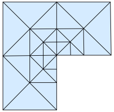
  $v_0 \in \mathcal{P}_1(\mathcal{T}_J) \cap H_0^1(\Omega)$, $v_{\mathbf{a}} \in \mathcal{P}_p(\mathcal{T}_{\mathbf{a}}) \cap H_0^1(\omega_J^{\mathbf{a}})$
- p-**stable**:

$$\|\nabla v_0\|^2 + \sum_{\mathbf{a}\in\mathcal{V}_J} \|\nabla v_{\mathbf{a}}\|^2_{\omega_J^{\mathbf{a}}} \lesssim \|\nabla v\|^2$$

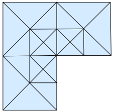
Outline

1. Introduction
   - Basis dependency
   - Moments and orthogonal decomposition
   - Line search
   - Stable patchwise decomposition (p-robust)
   - Stable levelwise decomposition ($p = 1$)
   - A posteriori error localization and adaptivity
2. p-robust multigrid in the H^1 setting
   - Solver
   - Numerical results
   - Adaptive number of smoothing steps and adaptive local smoothing
3. p-robust multigrid and domain decomposition in the $\boldsymbol{H}(\mathrm{div})$ setting
   - Solvers
   - Numerical results
4. Conclusions and future directions

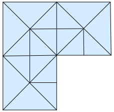
$p = 1$ stable decomposition by levels Xu, Chen, & Nochetto (2009)



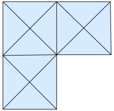
$\mathcal{T}_J$:



$\mathcal{T}_2$:



$\mathcal{T}_1$:



$\mathcal{T}_0$:

- $v \in \mathcal{P}_1(\mathcal{T}_J) \cap H_0^1(\Omega)$
- **decomposition** $v = v_0 + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} v_{j,\mathbf{a}}$, $v_0 \in \mathcal{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$,

  $v_{j,\mathbf{a}} \in \mathcal{P}_1(\mathcal{T}_j^{\mathbf{a}}) \cap H_0^1(\omega_j^{\mathbf{a}})$
- J-**stable**:

$$\|\nabla v_0\|^2 + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla v_{j,\mathbf{a}}\|^2_{\omega_j^{\mathbf{a}}} \lesssim \|\nabla v\|^2$$

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A posteriori error localization and adaptivity

Moments, line search

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease} \left(\eta_{\mathrm{alg}}^i\right)^2}$$

A posteriori error localization and adaptivity

Moments, line search

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

$$\lambda_j^i \|\nabla \rho_j^i\|^2 = (f, \rho_j^i) - (\nabla u_{J,j-1}^i, \nabla \rho_j^i) = \sum_{\mathbf{a} \in \mathcal{V}_j} \left\{ (f, \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} - (\nabla u_{J,j-1}^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} \right\}$$

A posteriori error localization and adaptivity

Moments, line search, and block Jacobi

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\|\right)^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

$$\lambda_j^i \|\nabla \rho_j^i\|^2 = (f, \rho_j^i) - (\nabla u_{J,j-1}^i, \nabla \rho_j^i) = \sum_{\mathbf{a} \in \mathcal{V}_j} \left\{ (f, \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} - (\nabla u_{J,j-1}^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} \right\} = \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$$

A posteriori error localization and adaptivity

Moments, line search, and block Jacobi

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\|\nabla \rho_0^i\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

$$\lambda_j^i \|\nabla \rho_j^i\|^2 = (f, \rho_j^i) - (\nabla u_{J,j-1}^i, \nabla \rho_j^i) = \sum_{\mathbf{a}\in\mathcal{V}_j} \left\{(f, \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} - (\nabla u_{J,j-1}^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}}\right\} = \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$$

A posteriori error localization and adaptivity

Moments, line search, and block Jacobi: local error decrease Pythagoras

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\|\nabla \rho_0^i\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

$$\lambda_j^i \|\nabla \rho_j^i\|^2 = (f, \rho_j^i) - (\nabla u_{J,j-1}^i, \nabla \rho_j^i) = \sum_{\mathbf{a}\in\mathcal{V}_j} \left\{(f, \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}} - (\nabla u_{J,j-1}^i, \nabla \rho_{j,\mathbf{a}}^i)_{\omega_j^{\mathbf{a}}}\right\} = \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$$

A posteriori error localization and adaptivity

Moments, line search, and block Jacobi: local error decrease Pythagoras

$$\underbrace{\|\nabla(u_J - u_J^{i+1})\|^2}_{\text{new error}} = \underbrace{\|\nabla(u_J - u_J^{i})\|^2}_{\text{old error}} - \underbrace{\|\nabla \rho_0^i\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

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- **error localization** over **mesh levels** and **vertex patches**

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- **error localization** over **mesh levels** and **vertex patches**
- bulk-chasing criterion: θ-fraction of the error
- solver adaptivity (possibly on top of mesh adaptivity): **adaptive number of smoothing steps** and **local adaptive smoothing**

Outline

Multigrid for high-order finite elements

Line search

- Heinrichs (1988)

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J-robust solver on graded simplicial meshes
Xu, Chen, & Nochetto: $p = 1$-stable decomposition by levels

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V -cycle multigrid

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V(0,1)-cycle multigrid

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- **zero** pre- and a **single** post-smoothing step

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$u_J^i \in V_J \qquad u_J^{i+1} = u_J^i + \sum_{j=0}^{J} \lambda_j^i \rho_j^i \in V_J \qquad \eta_{\mathrm{alg}}^i = \left(\sum_{j=0}^{J} \left(\lambda_j^i \|\nabla \rho_j^i\| \right)^2 \right)^{\frac{1}{2}}$

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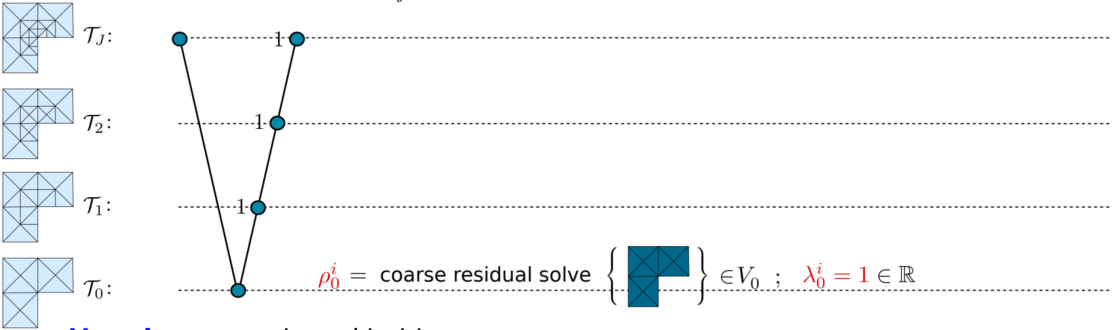

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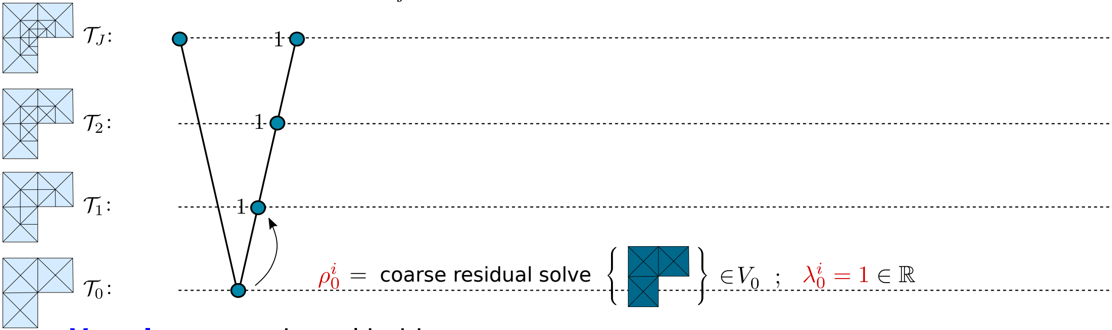

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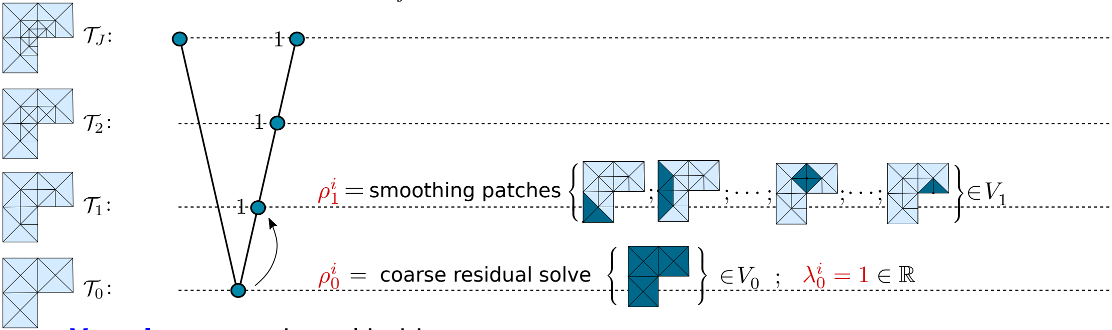

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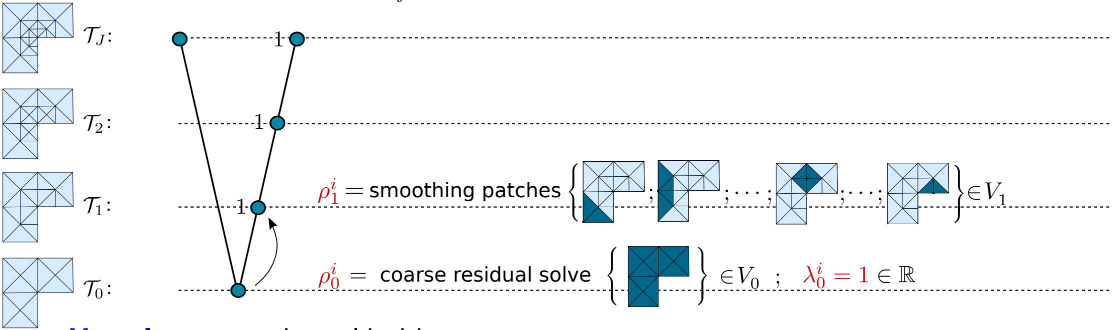

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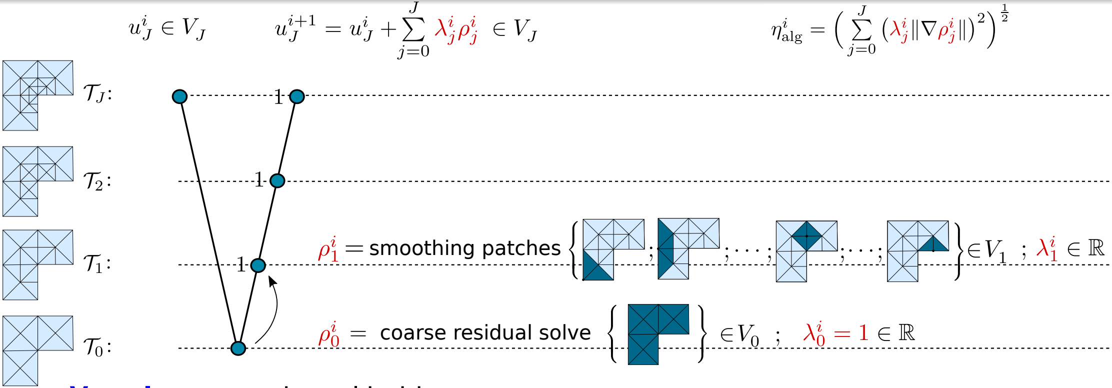

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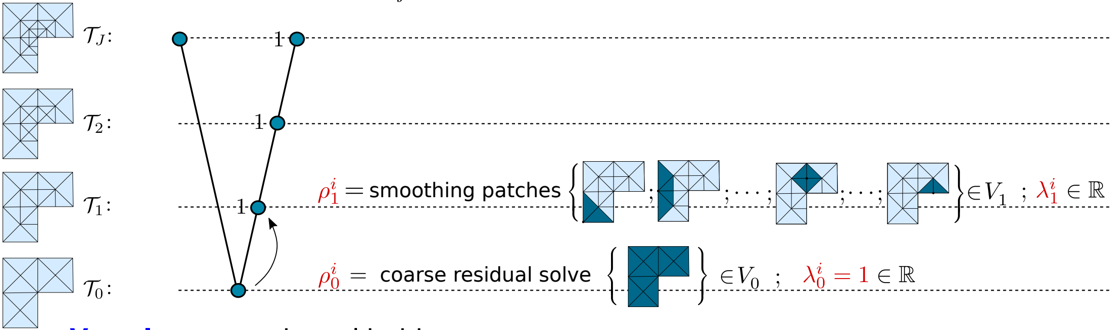

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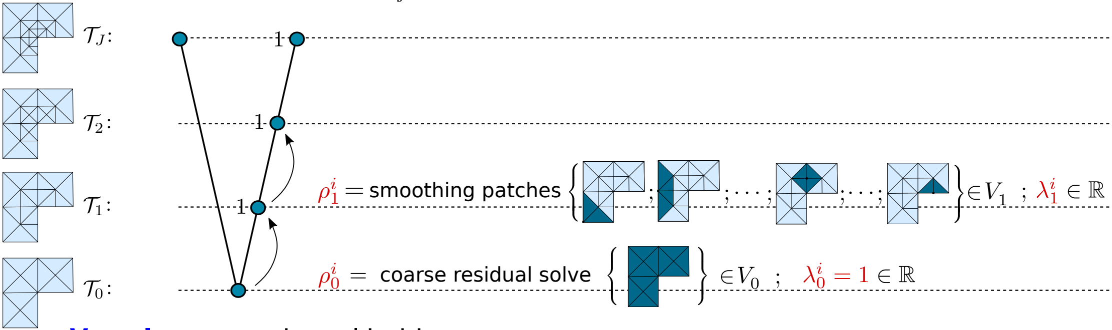

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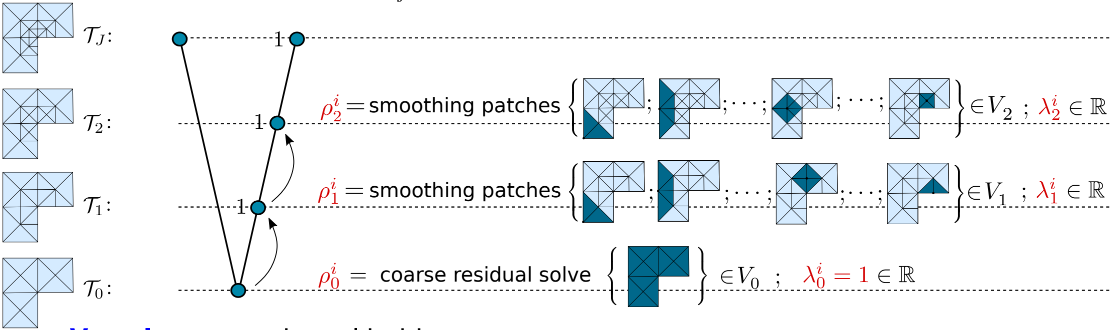

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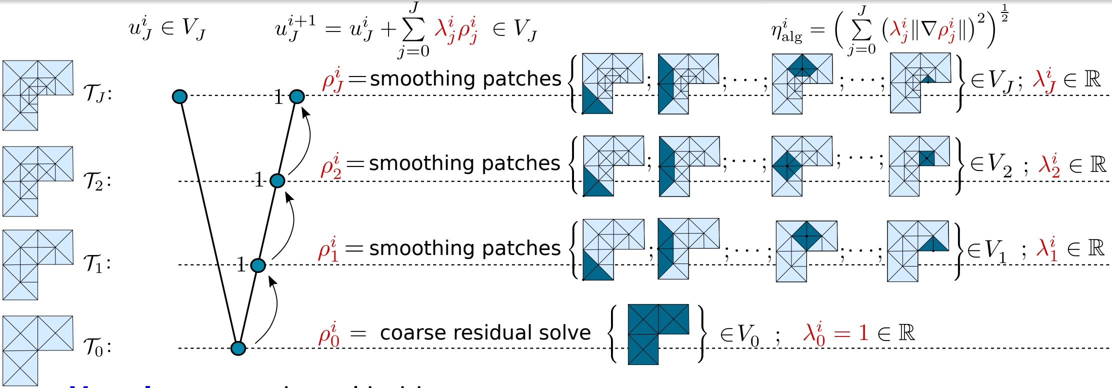

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Functional writing

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Coarse solve: Define $\rho_0^i \in V_0$ by $\underbrace{(\nabla \rho_0^i, \nabla v_0)}_{\text{global lifting}} = \underbrace{(f, v_0) - (\nabla u_J^i, \nabla v_0)}_{\text{initial algebraic residual}} \quad \forall v_0 \in V_0.$

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There holds

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There holds $\eta_{\mathrm{alg}}^{i} \leq \|\nabla(u_J - u_J^{i})\|$ *and*

$$\eta_{\mathrm{alg}}^{i} \geq \beta \|\nabla(u_J - u_J^{i})\|, \qquad \beta = \sqrt{1-\alpha^2}.$$

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- complete *independence* of J is obtained under H^2-regularity
- recent extension (Miraçi, Praetorius, Schimanko, & Streitberger): *smoothing near refined elements only* and *independence* of J in H^1

Outline

1. Introduction
   - Basis dependency
   - Moments and orthogonal decomposition
   - Line search
   - Stable patchwise decomposition (p-robust)
   - Stable levelwise decomposition ($p = 1$)
   - A posteriori error localization and adaptivity
2. p-robust multigrid in the H^1 setting
   - Solver
   - Numerical results
   - Adaptive number of smoothing steps and adaptive local smoothing
3. p-robust multigrid and domain decomposition in the $\boldsymbol{H}(\mathrm{div})$ setting
   - Solvers
   - Numerical results
4. Conclusions and future directions

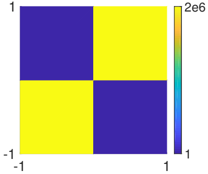
5 test cases

Sine: $u(x,y) = \sin(2\pi x)\sin(2\pi y),\ \Omega := (-1,1)^2$

Peak: $u(x,y) = x(x-1)y(y-1)\mathrm{e}^{-100((x-0.5)^2-(y-0.117)^2)},\ \Omega := (0,1)^2$

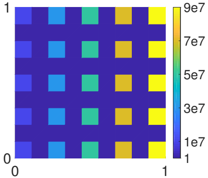
L-shape: $u(r,\theta) = r^{2/3}\sin(2\theta/3),\quad \Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$

Checkerboard: $u(r,\varphi) = r^\gamma \mu(\varphi),\quad \Omega := (-1,1)^2$



with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^6)$ or no jump

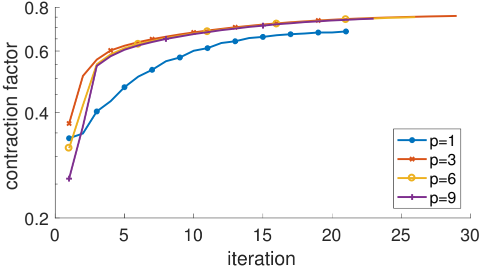
Skyscraper: unknown analytic solution, $\Omega := (0,1)^2$



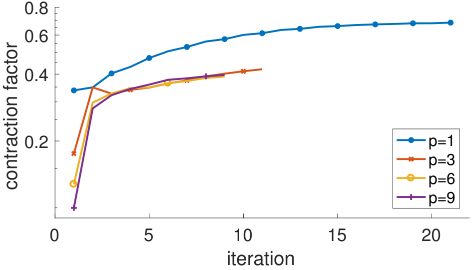
with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^7)$ or $\mathcal{J}(\mathbf{K}) = O(1)$

Confirmation of p-robustness: contraction factors

L-shape problem, $J = 3$, $p_j = 1$ (left) and $p_j = p$ (right), $1 \le j \le J$



$1 \to 1, p$



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Confirmation of p-robustness: iteration numbers

Stopping criterion:
$$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \leq 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	2e^4	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	1e^5	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	6e^5	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	1e^6	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	6e^4	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	6e^5	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	2e^6	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	5e^6	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Confirmation of p-robustness: iteration numbers

Stopping criterion: $$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \le 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular													
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Confirmation of p-robustness: iteration numbers

Stopping criterion: $$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \le 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$	$1\to 1, p$	$1, p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Confirmation of p-robustness: iteration numbers

Stopping criterion:
$$\frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{i_s}\|}{\|\mathrm{F}_J\|} \leq 10^{-5} \frac{\|\mathrm{F}_J - \mathbb{A}_J \mathrm{U}_J^{0}\|}{\|\mathrm{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Numerical **K**- and J-robustness observed even in low-regularity cases.

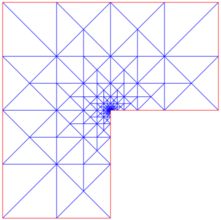
Tests for graded meshes and H^1-regular solutions

L-shape, $\mathbf{K} = I$, 1, $p \to p$

J	p	i_s
5	1	16
	3	7
	6	6
	9	5

J	p	i_s
10	1	15
	3	6
	6	5
	9	5

J	p	i_s
15	1	17
	3	11
	6	5
	9	4



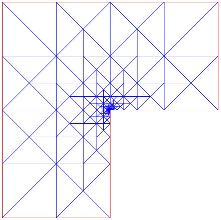
Tests for graded meshes and H^1-regular solutions

L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s
5	1	16
	3	7
	6	6
	9	5

J	p	i_s
10	1	15
	3	6
	6	5
	9	5

J	p	i_s
15	1	17
	3	11
	6	5
	9	4

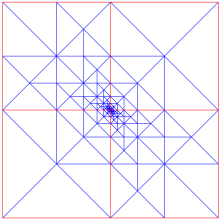


Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

J	p	i_s
5	1	33
	3	15
	6	12
	9	11

J	p	i_s
10	1	57
	3	23
	6	15
	9	12

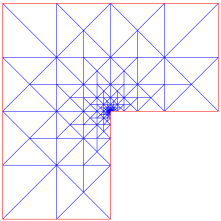
J	p	i_s
15	1	97
	3	32
	6	20
	9	15



Tests for graded meshes and H^1-regular solutions

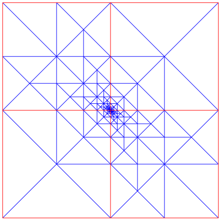
L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



These H^1-regular test cases indicate the possibility of *linear J-dependence*, in accordance with the theoretical results.

Three space dimensions

Test cases: uniform mesh refinement, $1 \to 1, p$, and $J = 4$.

Cube: $\Omega := (0,1)^3$,

$$u(x,y,z) = x(x-1)y(y-1)z(z-1),$$

$$\mathbf{K} = I.$$

Nested cubes: $\Omega := (-1,1)^3$,

unknown analytic solution,

$\mathbf{K} = I$ and $10^5 * I$ in $(-0.5, 0.5)^3$.

Checkers cubes: $\Omega := (0,1)^3$,

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$\mathbf{K} = I$ and $10^6 * I$ in $(0,0.5)^3 \cup (0.5,1)^3$

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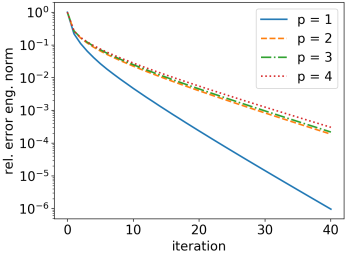

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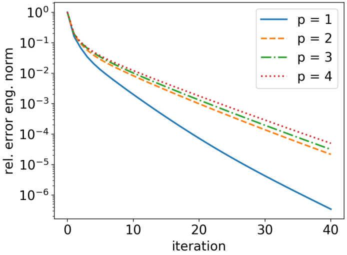

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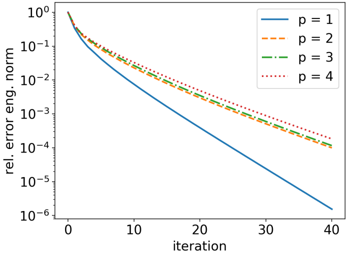

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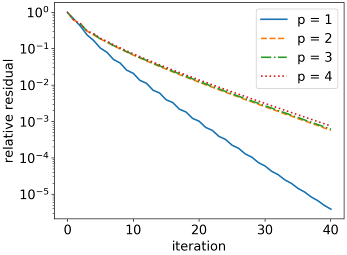

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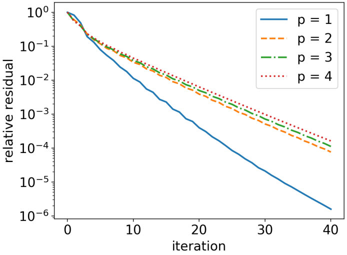

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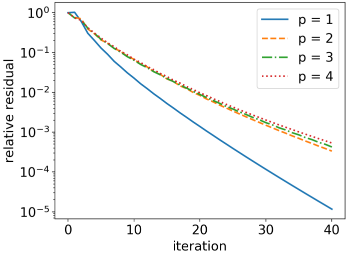
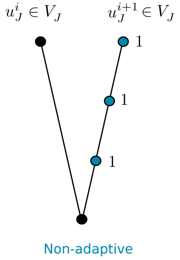


Outline

Adaptive number of smoothing steps

Recall: $\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - \sum_{0=1}^{J}(\lambda_j^i \|\nabla \rho_j^i\|)^2$



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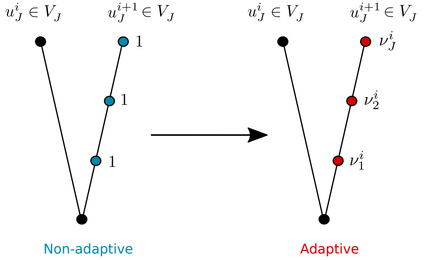

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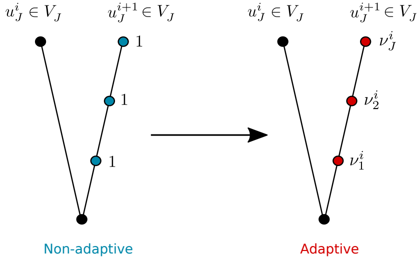

Variable number of smoothing steps

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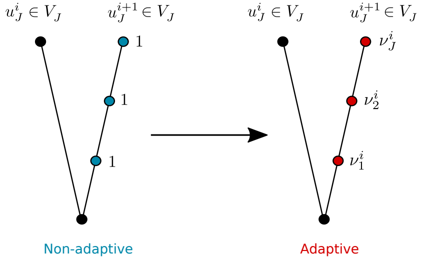
Recall: $\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - \sum_{0=1}^{J}(\lambda_j^i \|\nabla \rho_j^i\|)^2$



Variable number of smoothing steps/multigrid cycles:

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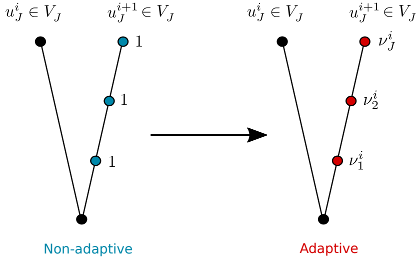

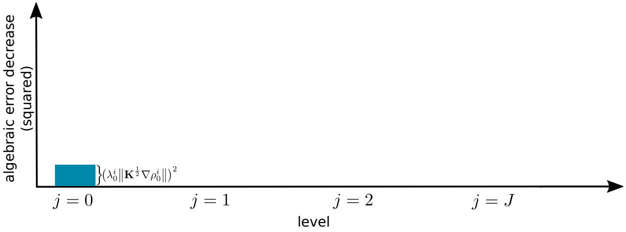

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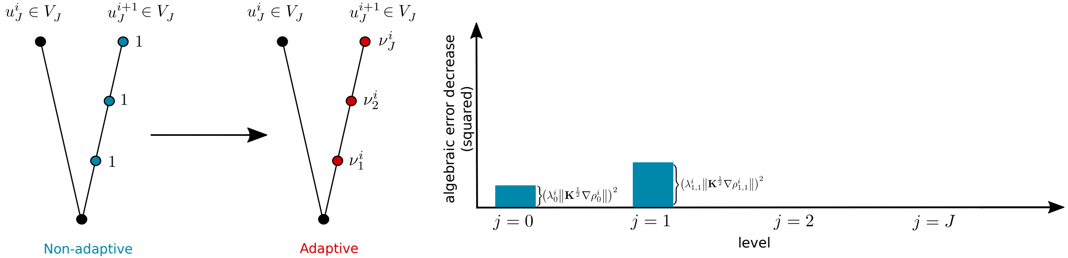

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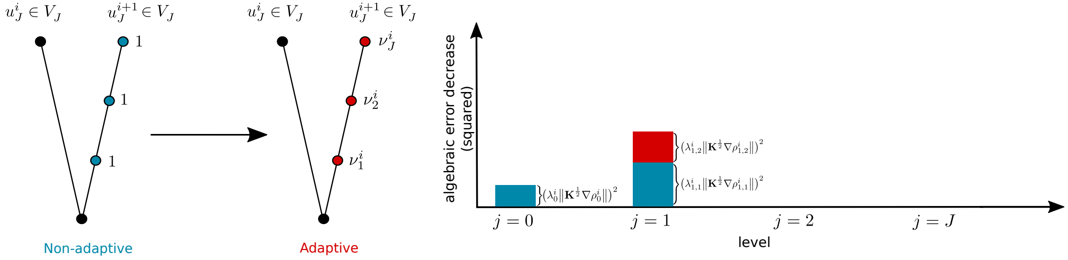

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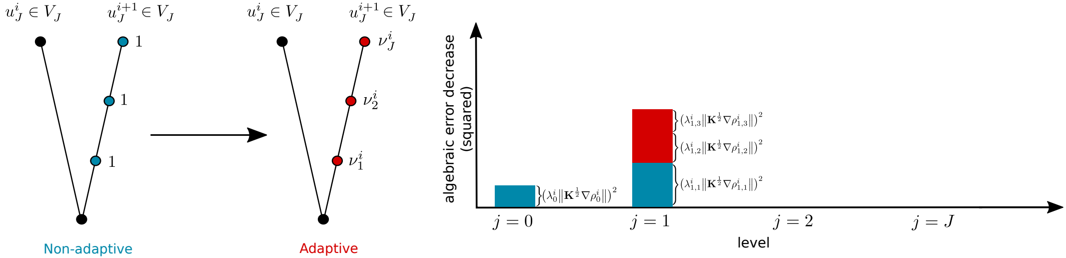

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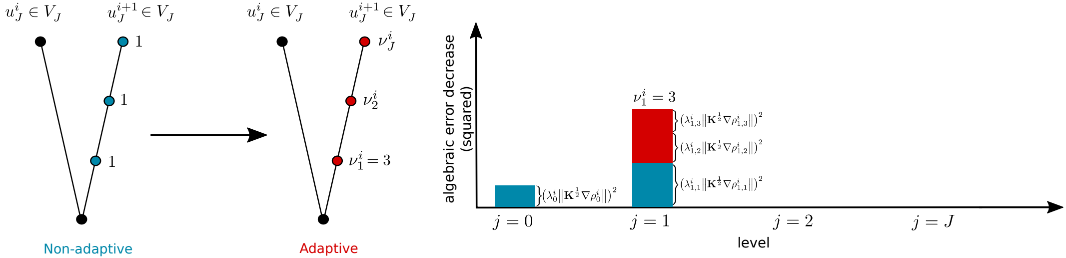

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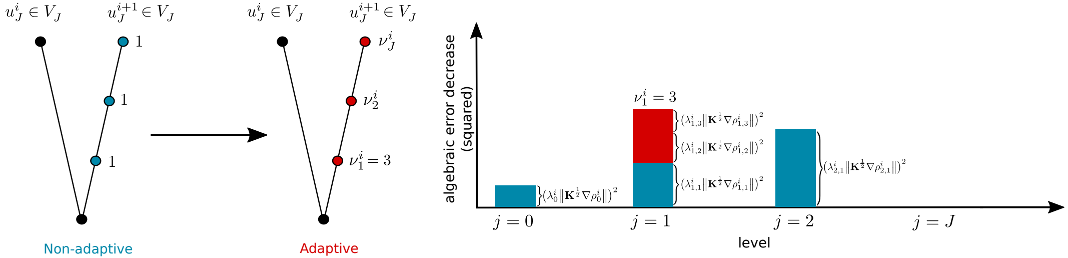

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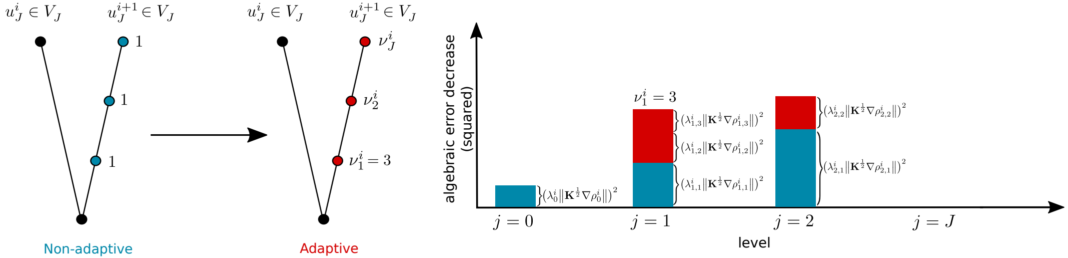

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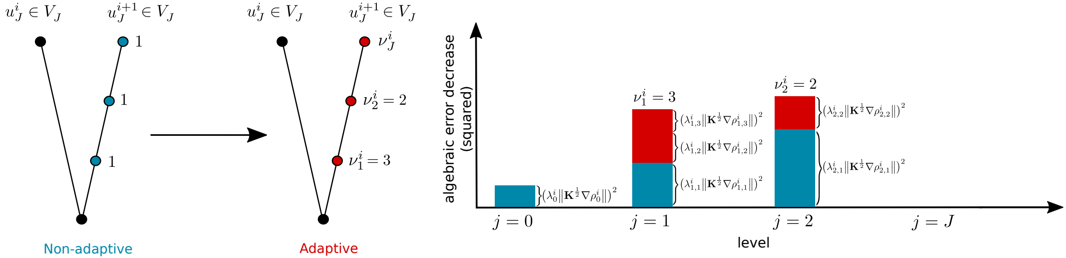
Adaptive number of smoothing steps

Recall: $\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - \sum_{0=1}^{J} (\lambda_j^i \|\nabla \rho_j^i\|)^2$



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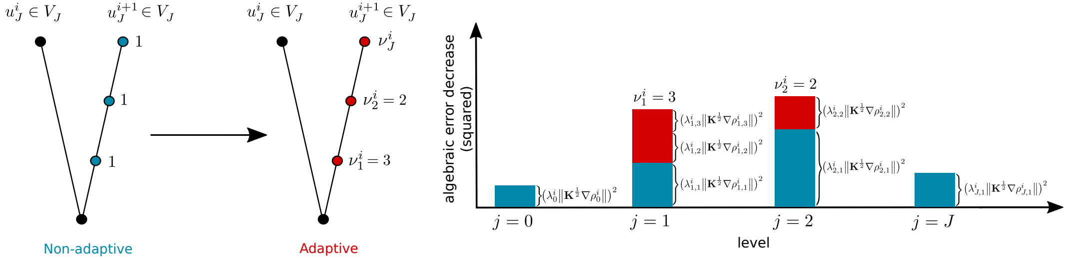

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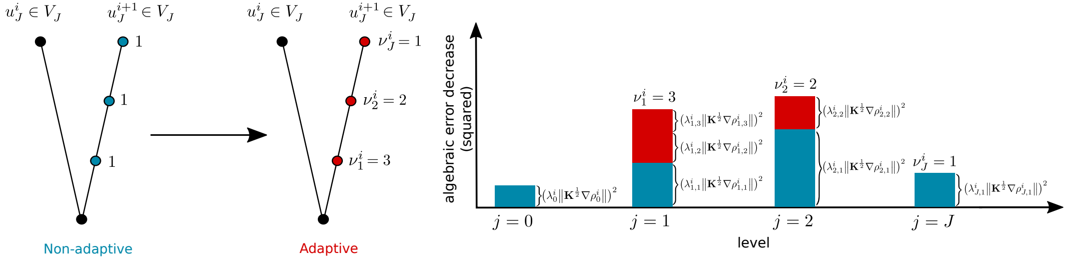

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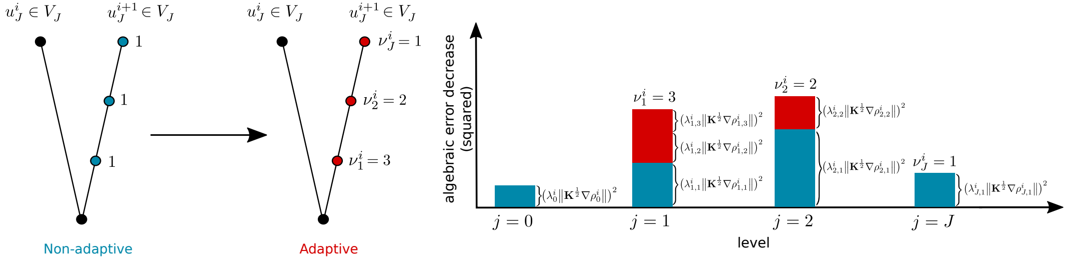

$$\left(\lambda_{j,\nu}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2 \geq \theta^2\left(\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda_{k,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2 + \sum_{\ell=1}^{\nu-1}\left(\lambda_{j,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2\right)$$

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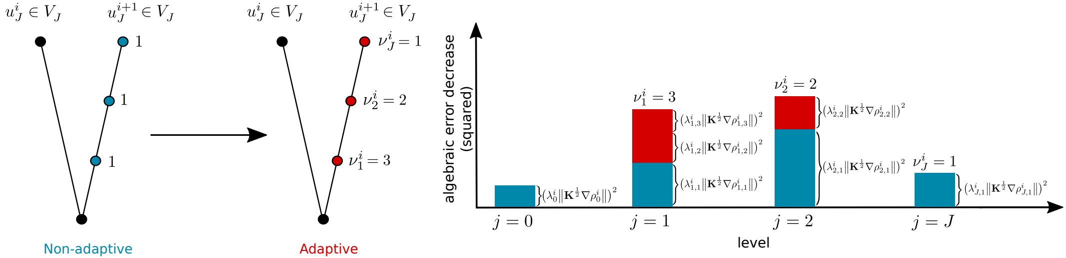

$$\underbrace{\left(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac12}\nabla\rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2 \left(\sum_{k=0}^{j-1} \sum_{\ell=1}^{\nu_k^i} \left(\lambda_{k,\ell}^i \|\mathbf{K}^{\frac12}\nabla\rho_{k,\ell}^i\|\right)^2 + \sum_{\ell=1}^{\nu-1} \left(\lambda_{j,\ell}^i \|\mathbf{K}^{\frac12}\nabla\rho_{j,\ell}^i\|\right)^2 \right)$$

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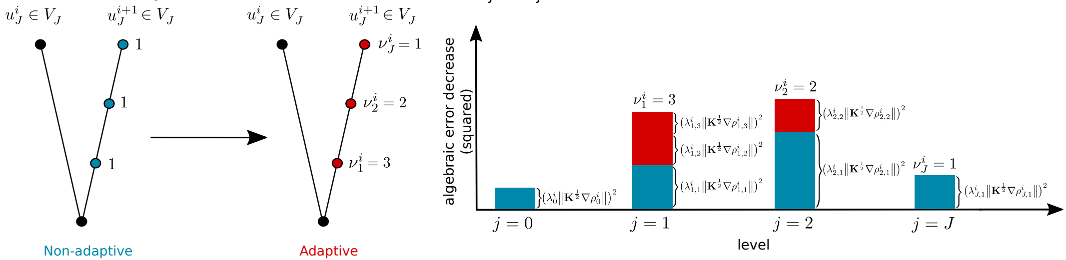

$$\underbrace{\left(\lambda_{j,\nu}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2\left(\underbrace{\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda_{k,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2}_{\text{all smoothings on previous levels}} + \sum_{\ell=1}^{\nu-1}\left(\lambda_{j,\ell}^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2\right)$$

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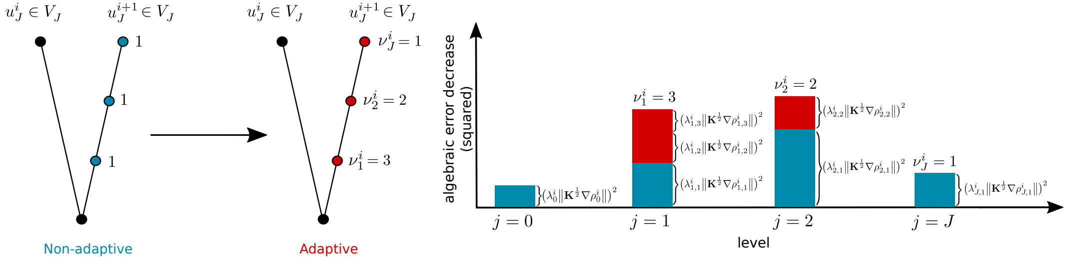

$$\underbrace{\left(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2 \Bigg(\underbrace{\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda_{k,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2}_{\text{all smoothings on previous levels}} + \underbrace{\sum_{\ell=1}^{\nu-1}\left(\lambda_{j,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2}_{\text{previous smoothings on current level}} \Bigg)$$

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Adaptive vs. fixed number of smoothing steps

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and spaces hierarchy $1, p \to p$.

	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

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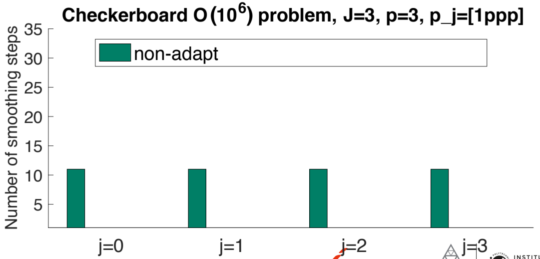
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1



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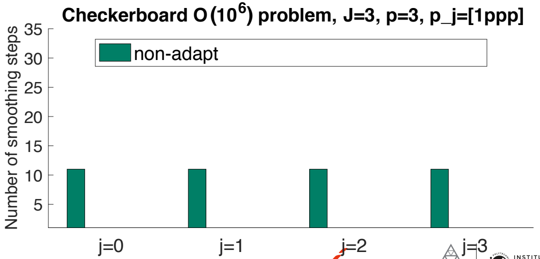
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

$p_j = p$, $\theta = 0.2$

	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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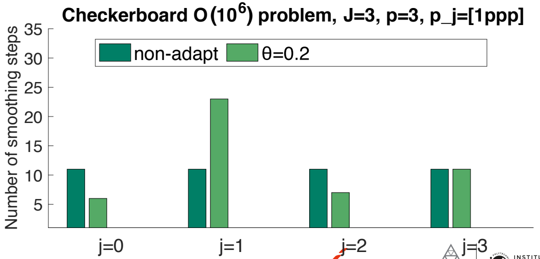
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

$p_j = p$, $\theta = 0.2$

	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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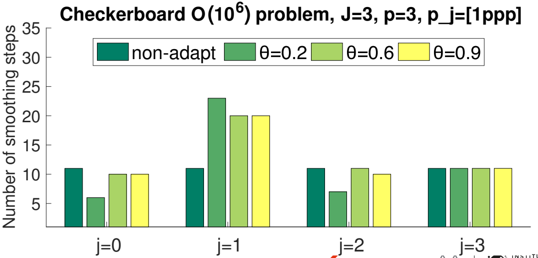
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



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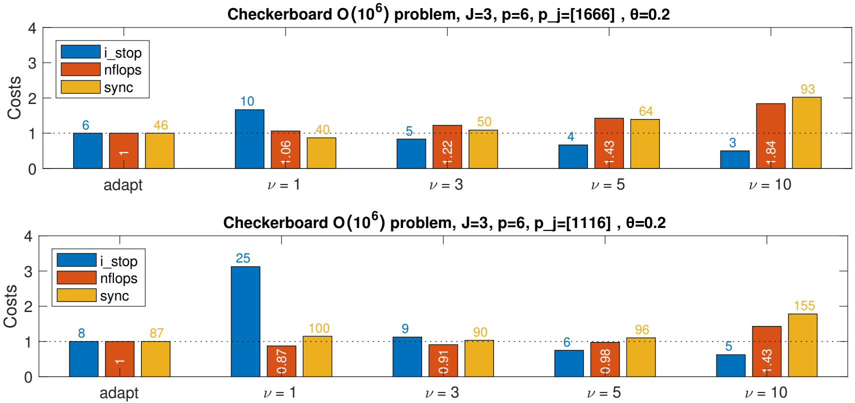

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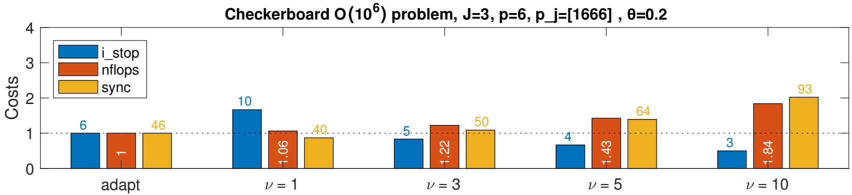

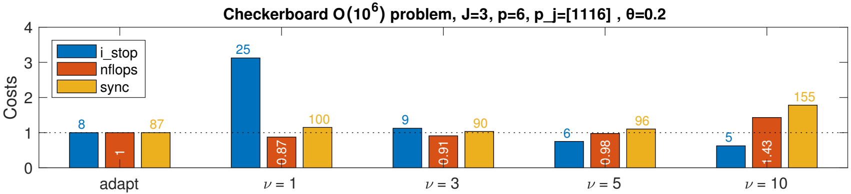

$$\textbf{nflops} := \frac{|\mathcal{V}_0|^3}{3} + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \frac{\mathrm{ndof}(V_j^{\mathbf{a}})^3}{3} + \sum_{i=1}^{i_{\mathrm{s}}} \left[2|\mathcal{V}_0|^2 + \sum_{j=1}^{J} \nu_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} 2\mathrm{ndof}(V_j^{\mathbf{a}})^2 \right] + \sum_{i=1}^{i_{\mathrm{s}}} \sum_{j=1}^{J} \left[2\,\mathrm{nnz}(\mathcal{I}_{j-1}^j) + 2\,\mathrm{nnz}(\mathcal{I}_j^{j-1}) + 2\nu_j^i\,\mathrm{nnz}(\mathbb{A}_j) + 3\nu_j^i(2\,\mathrm{size}(\mathbb{A}_j)) \right];$$

$$\textbf{sync} := i_{\mathrm{s}} + \sum_{i=1}^{i_{\mathrm{s}}} \sum_{j=1}^{J} \nu_j^i.$$

Adaptive local smoothing

Recall: $\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - \|\nabla \rho_0^i\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2$

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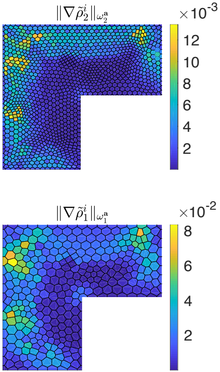

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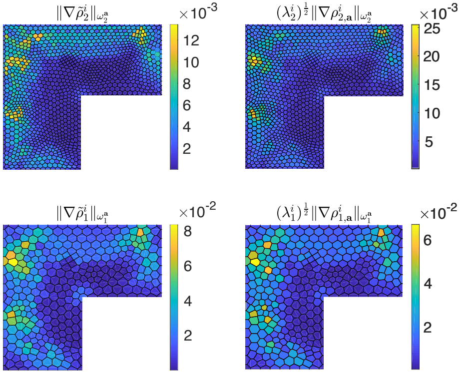

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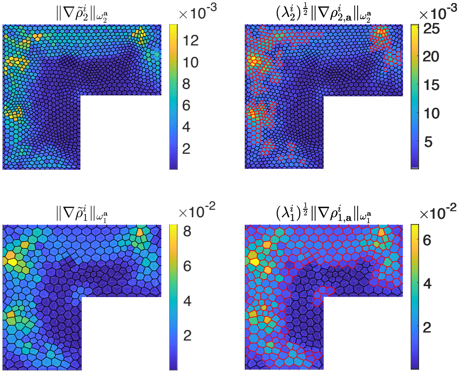

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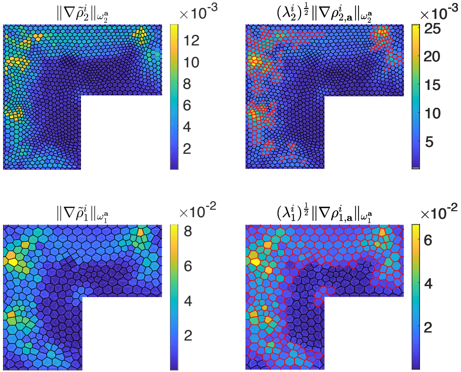

Local smoothing in adaptively-refined meshes

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- Rüde. “Mathematical and computational techniques for multilevel adaptive methods.” SIAM 1993.
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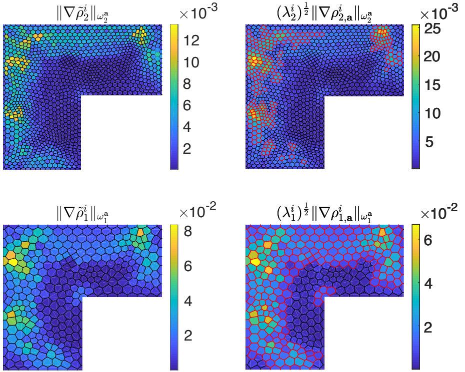

Local smoothing in adaptively-refined meshes vs adaptive local smoothing on a given mesh hierarchy

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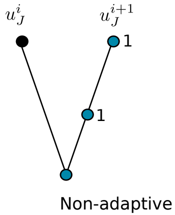

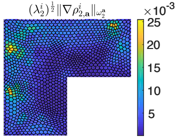

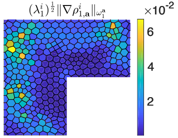

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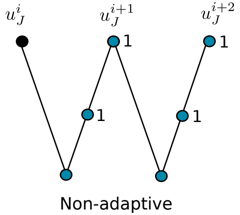

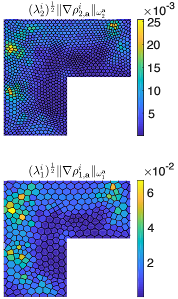

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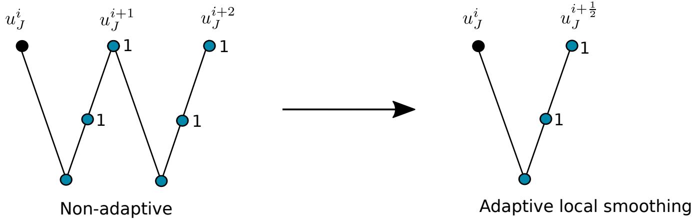

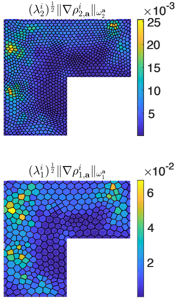

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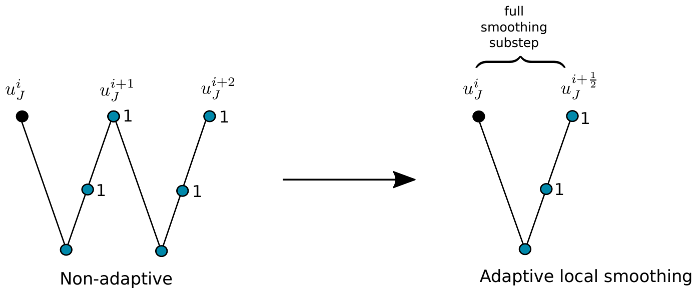

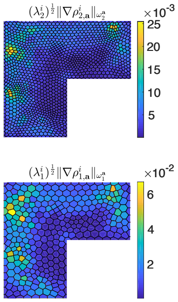

Local smoothing in adaptively-refined meshes vs adaptive local smoothing on a given mesh hierarchy

- Bai and Brandt. "Local mesh refinement multilevel techniques." *SIAM J. Sci. Statist. Comput.* 1987.
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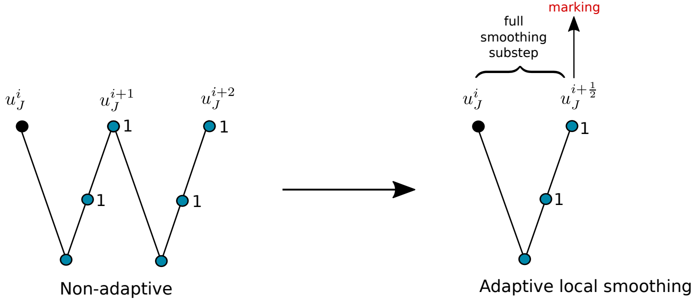
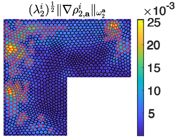
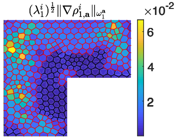
[11] Dörfler. "A convergent adaptive algorithm for Poisson's equation". *SIAM J. Numer. Anal.* 1996.

Adaptive local smoothing

Recall: $\|\nabla(u_J - u_J^{i+1})\|^2 = \|\nabla(u_J - u_J^i)\|^2 - \|\nabla \rho_0^i\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$







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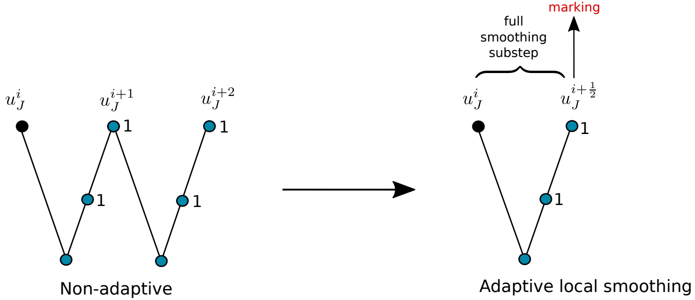

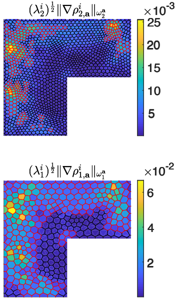

bulk-chasing criterion[11]: $\theta^2 \left(\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}} \right) \le \sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$

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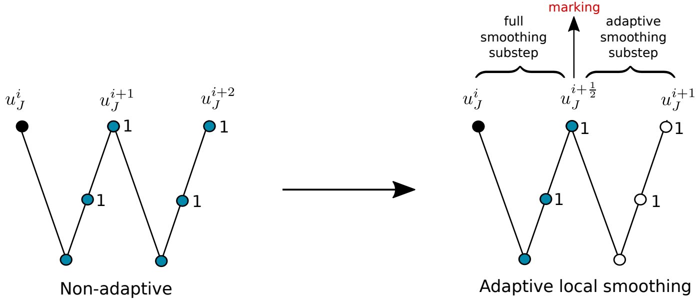

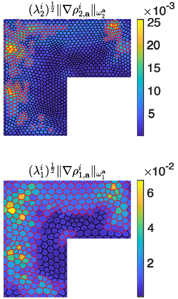

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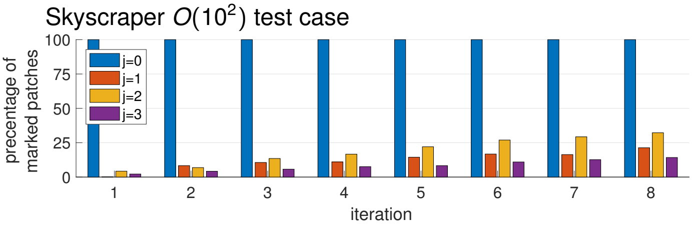
Does the adaptivity pay off?



Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

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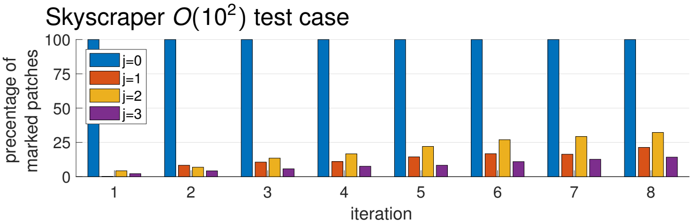

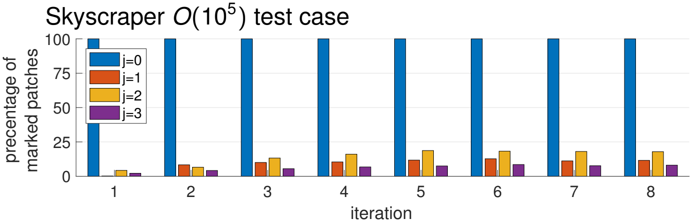

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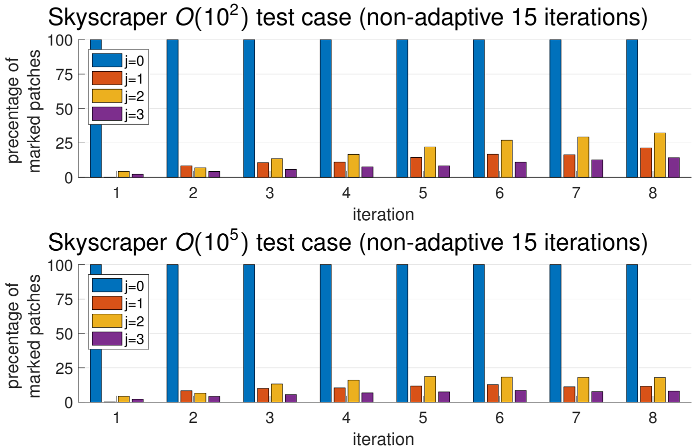

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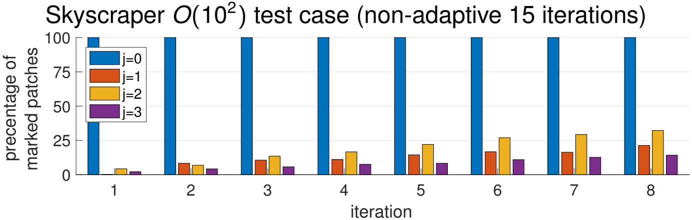

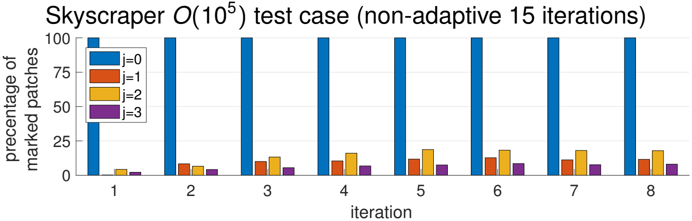

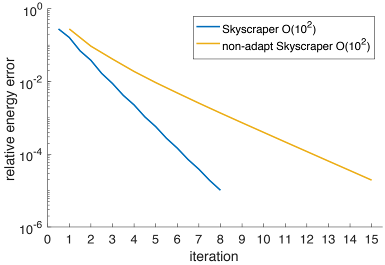

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Outline

Mixed finite element approximation

Mixed finite element approximation

Find $\boldsymbol{u}_J \in \boldsymbol{V}_J$ and $\gamma_J \in W_J$ such that

$$
\begin{aligned}
(\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) - (\gamma_J, \nabla\cdot\boldsymbol{v}_J) &= 0 \qquad && \forall \boldsymbol{v}_J \in \boldsymbol{V}_J, \\
(\nabla\cdot\boldsymbol{u}_J, w_J) &= (f, w_J) \qquad && \forall w_J \in W_J.
\end{aligned}
$$

Mixed finite element approximation

Mixed finite element approximation

Find $\boldsymbol{u}_J \in \boldsymbol{V}_J$ *and* $\gamma_J \in W_J$ *such that*

$$\begin{aligned} (\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) - (\gamma_J, \nabla\cdot\boldsymbol{v}_J) &= 0 && \forall \boldsymbol{v}_J \in \boldsymbol{V}_J, \\ (\nabla\cdot\boldsymbol{u}_J, w_J) &= (f, w_J) && \forall w_J \in W_J. \end{aligned}$$

- $\boldsymbol{V}_J := \{\boldsymbol{v}_J \in \boldsymbol{H}_0(\mathrm{div}, \Omega),\ \boldsymbol{v}_J|_K \in \boldsymbol{RT}_p(K)\ \forall K \in \mathcal{T}_J\}$: Raviart–Thomas, degree p
- W_J: piecewise polynomials on $\mathcal{T}_J$ of degree p and mean value 0 on Ω

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Dual formulation

Find $\boldsymbol{u}_J \in \boldsymbol{V}_J^f$ such that

$$(\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) = 0 \qquad \forall \boldsymbol{v}_J \in \boldsymbol{V}_J^0.$$

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$$(\boldsymbol{K}^{-1}\boldsymbol{u}_J, \boldsymbol{v}_J) - (\gamma_J, \nabla\cdot\boldsymbol{v}_J) = 0 \qquad \forall \boldsymbol{v}_J \in \boldsymbol{V}_J,$$

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Hierarchy of nested spaces ($0 = p_0 \le p_1 \le p_2 \le \ldots \le p_J = p$)

$$\boldsymbol{V}_j^0 := \{\boldsymbol{v}_j \in \boldsymbol{H}_0(\mathrm{div}, \Omega),\ \boldsymbol{v}_j|_K \in \boldsymbol{RT}_{p_j}(K)\ \forall K \in \mathcal{T}_j,\ \nabla\cdot\boldsymbol{v}_j = 0\}$$

Multigrid and domain decomposition for mixed finite elements

Multigrid for saddle-point formulations

- Arnold, Falk, Winther (2000), Schöberl, Zulehner (2003), Xu, Chen, Nochetto (2009), Brenner (2009, 2018), . . .

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Multigrid for the nonconforming reformulation (non-nested spaces)

- Brenner (1992), Chen (1996), Wheeler, Yotov (2000), ...

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Domain decomposition

- Glowinski, Wheeler (1988), Cowsar, Mandel, Wheeler (1995), . . .
- Ewing, Wang (1992), . . .

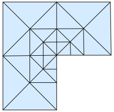
p-robust stable decomposition on $\mathcal{T}_J$ (additive Schwarz/block Jacobi)

Falk & Winther (2025)

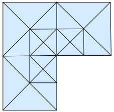
- $\boldsymbol{v} \in \boldsymbol{V}_J^0$ (p-degree div-free Raviart–Thomas on $\mathcal{T}_J$)
- **decomposition** $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{\mathbf{a} \in \mathcal{V}_J} \boldsymbol{v}_{\mathbf{a}}$, $\boldsymbol{v}_0 \in \boldsymbol{V}_0^J$ lowest-order, $\boldsymbol{v}_{\mathbf{a}} \in \boldsymbol{V}_J^{\mathbf{a},0}$
- ***p*-stable**:

$$\|\boldsymbol{v}_0\|^2 + \sum_{\mathbf{a} \in \mathcal{V}_J} \|\boldsymbol{v}_{\mathbf{a}}\|^2_{\omega_J^{\mathbf{a}}} \lesssim \|\boldsymbol{v}\|^2$$

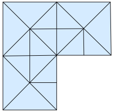
$p = 0$ stable decomposition by levels Hiptmair, Wu, and Zheng (2012)



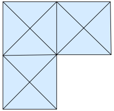
$\mathcal{T}_J$:



$\mathcal{T}_2$:



$\mathcal{T}_1$:



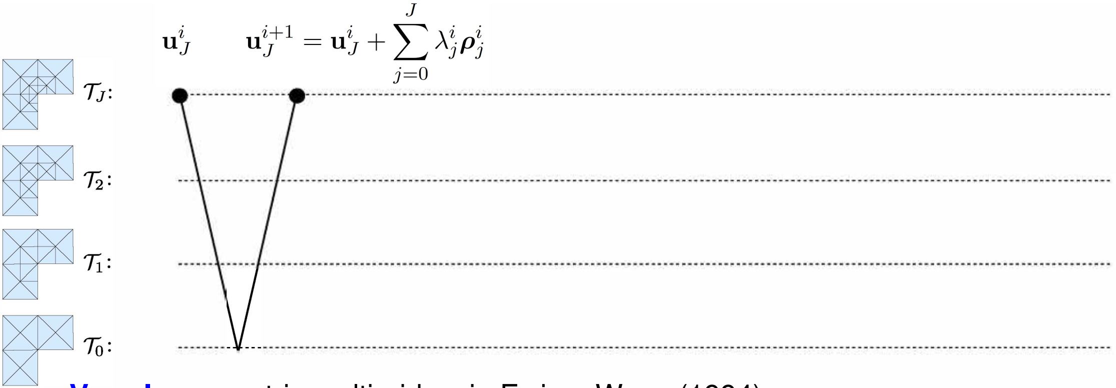
$\mathcal{T}_0$:

- $\boldsymbol{v} \in \boldsymbol{V}_J^0$ (0-degree div-free Raviart–Thomas on $\mathcal{T}_J$)
- **decomposition** $\boldsymbol{v} = \boldsymbol{v}_0 + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \boldsymbol{v}_{j,\mathbf{a}}$, $\boldsymbol{v}_0 \in \boldsymbol{V}_0^0$, $\boldsymbol{v}_{j,\mathbf{a}} \in \boldsymbol{V}_j^{\mathbf{a},0}$
- ***J*-stable**:

$$\|\boldsymbol{v}_0\|^2 + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \|\boldsymbol{v}_{j,\mathbf{a}}\|^2_{\omega_j^{\mathbf{a}}} \lesssim \|\boldsymbol{v}\|^2$$

Outline

V-cycle multigrid

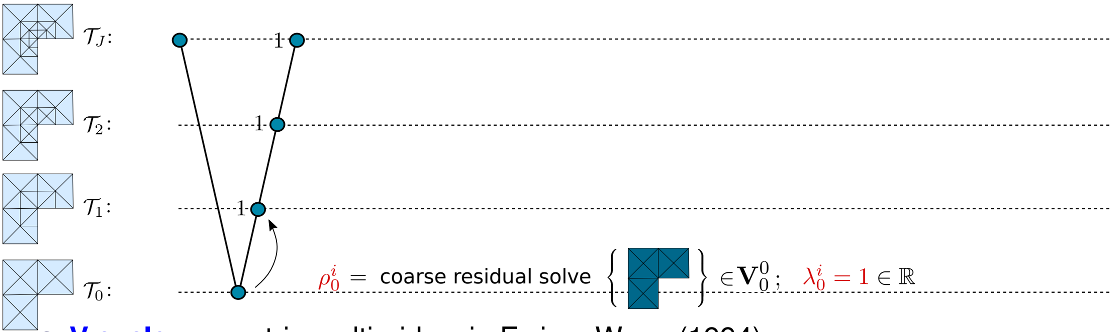


- **V-cycle** geometric multigrid as in Ewing, Wang (1994)

V(0,1)-cycle multigrid

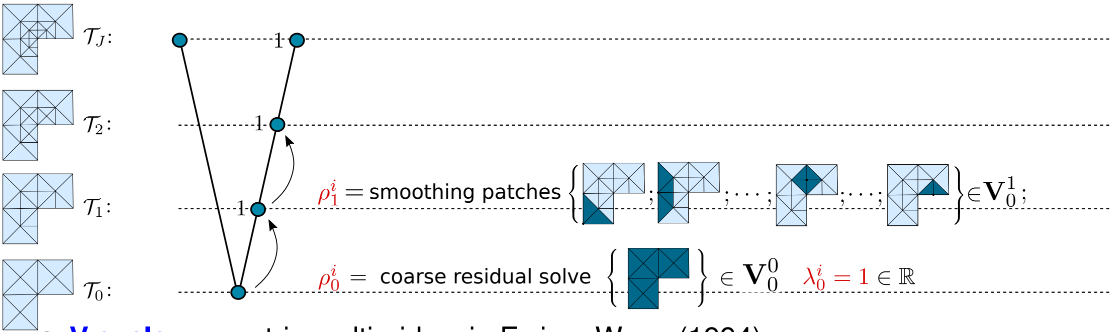
- **V-cycle** geometric multigrid as in Ewing, Wang (1994)
- **zero** pre- and a **single** post-smoothing step

V(0,1)-cycle multigrid



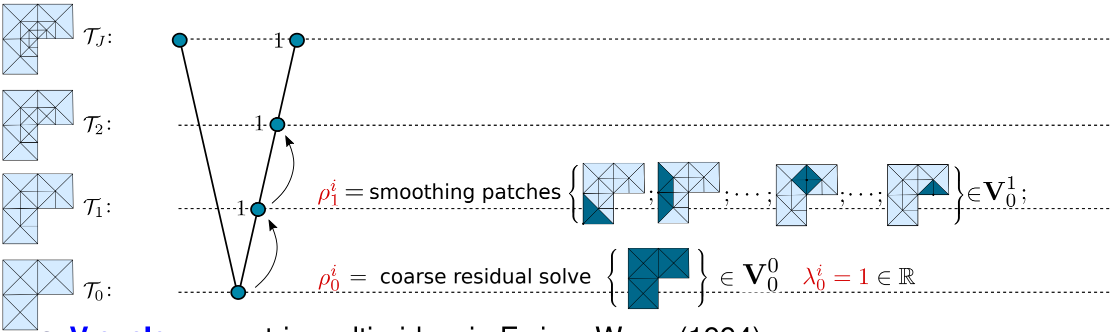
- **V-cycle** geometric multigrid as in Ewing, Wang (1994)
- **zero** pre- and a **single** post-smoothing step
- cheapest ***RT*$_0$ coarse solve**

V(0,1)-cycle multigrid with block-Jacobi smoothing



- **V-cycle** geometric multigrid as in Ewing, Wang (1994)
- **zero** pre- and a **single** post-smoothing step
- cheapest $\boldsymbol{RT}_0$ **coarse solve**
- **additive Schwarz/block-Jacobi** smoothing ρ_j^i

V(0,1)-cycle multigrid with block-Jacobi smoothing



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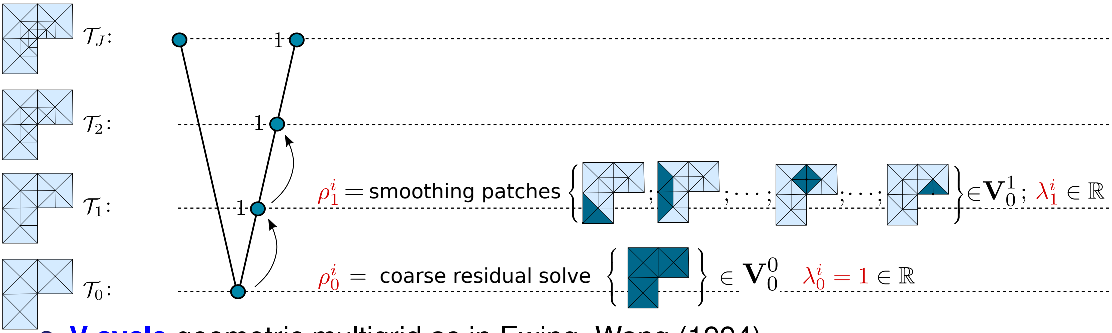

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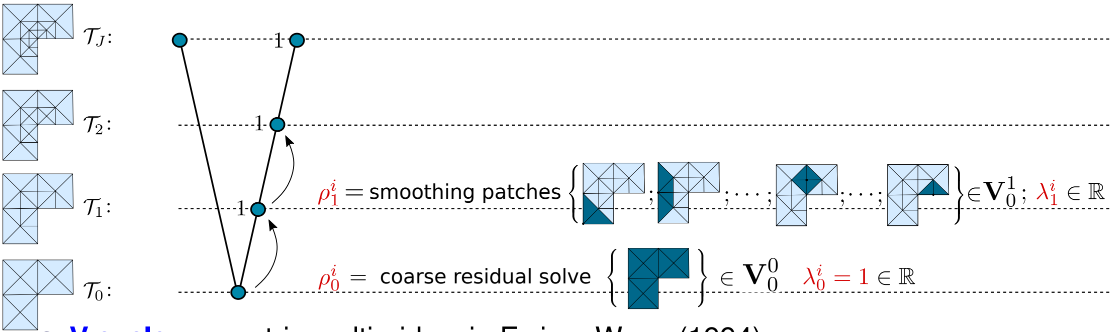
- level-wise *step-sizes* λ_j^i in correction stage

V(0,1)-cycle multigrid with block-Jacobi smoothing and **line search**



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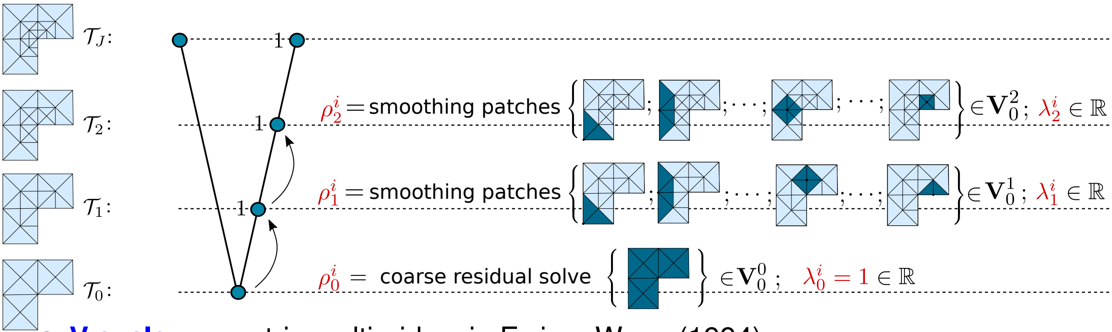
V(0,1)-cycle multigrid with block-Jacobi smoothing and **line search**



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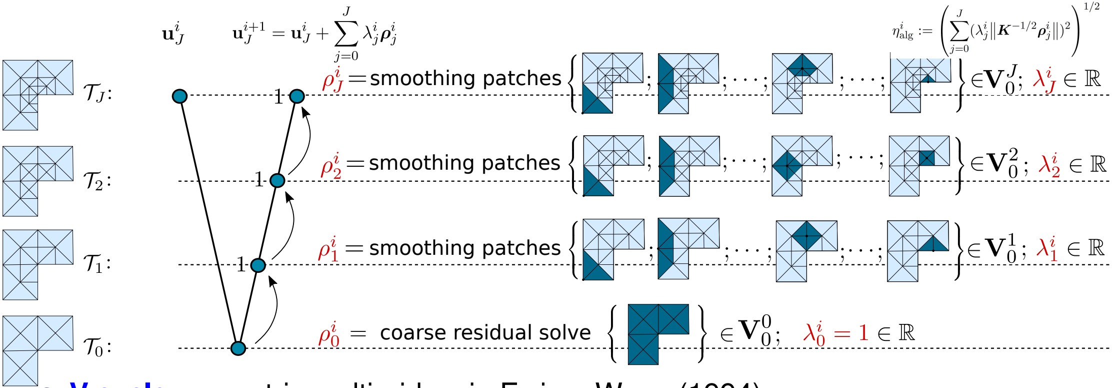
V(0,1)-cycle multigrid with block-Jacobi smoothing and line search



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- level-wise *step-sizes* λ_j^i in correction stage: optimally chosen by **line search**

Functional writing: for $\boldsymbol{u}_J^i \in \boldsymbol{V}_J^f$ arbitrary, construct ρ_j^i, $\lambda_j^i \Longrightarrow \boldsymbol{u}_J^{i+1}$

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Coarse solve: $\rho_0^i \in \boldsymbol{V}_0^0$ s.t. $\underbrace{(\boldsymbol{K}^{-1}\rho_0^i, \boldsymbol{v}_0)}_{\text{global lifting}} = \underbrace{-(\boldsymbol{K}^{-1}\boldsymbol{u}_J^i, \boldsymbol{v}_0)}_{\text{global algebraic residual}} \quad \forall \boldsymbol{v}_0 \in \boldsymbol{V}_0^0;$

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Main results

Theorem (p-robust error contraction of the multilevel solver)

There holds

$$\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i+1})\| \leq \alpha \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, d, \mathbf{K}, J) < 1.$$

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There holds $\eta_{\mathrm{alg}}^i \leq \left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\right\|$ *and*

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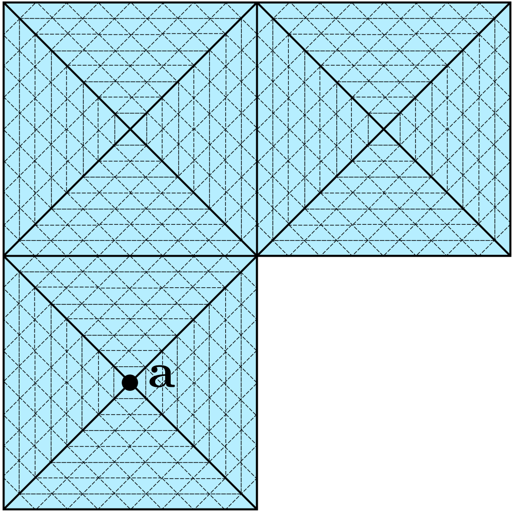
Moments, line search, and block Jacobi: local error decrease Pythagoras

$$\underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i+1})\right\|^2}_{\text{new error}} = \underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\right\|^2}_{\text{old error}} - \underbrace{\left\|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_0^i\right\|^2 - \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \left\|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_j^i\right\|_{\omega_j^{\mathbf{a}}}^2}_{\text{error decrease } \left(\eta_{\mathrm{alg}}^i\right)^2}$$

Domain decomposition for high-order mixed finite elements



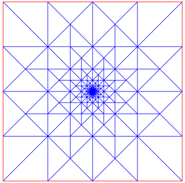
Coarse grid $\mathcal{T}_H$ (solid line), fine grid $\mathcal{T}_J$ (dashed line), patch domain $\omega^{\mathbf{a}}$

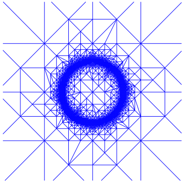
Outline

3 test cases

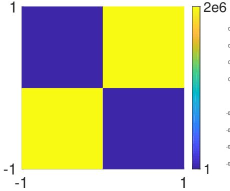
Smooth: $\gamma(x,y) = \cos(\pi x)\cos(\pi y), \ \Omega = (0,1)^2$

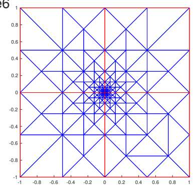
Well wavefront: $\gamma(r) = \tan^{-1}(\alpha(r - r_0)), \ \Omega = (0,1)^2$





Checkerboard: $u(r,\varphi) = r^{\gamma}\mu(\varphi); \quad \Omega := (-1,1)^2$





jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^6)$

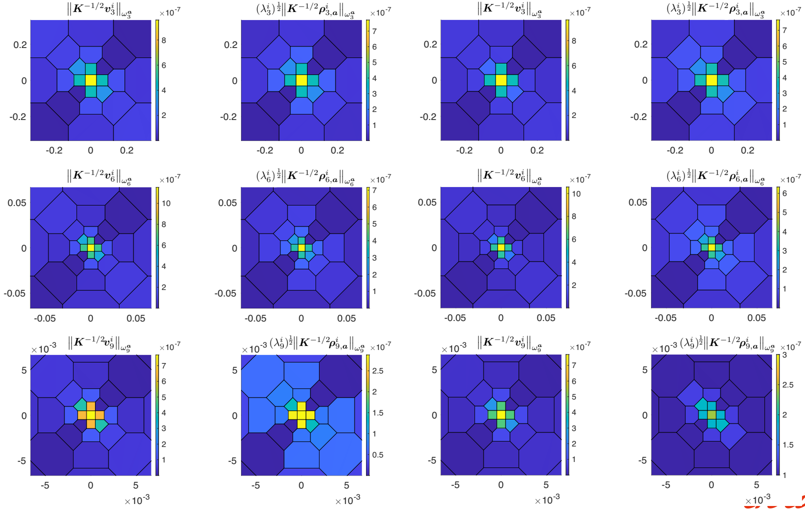
Setting

		Smooth $J=5$ $p=1$	Smooth $J=5$ $p=6$	Wellwavefront $J=12$ $p=1$	Wellwavefront $J=12$ $p=6$	Checkerboard $J=28$ $p=1$	Checkerboard $J=28$ $p=6$
	#DoF (mixed)	130 816	1 318 016	191 486	1 927 051	33 195	334 215
	#DoF (div-free)	82 176	861 056	119 734	1 256 969	20 794	218 134
	$\#\mathcal{T}_J$	16 384	16 384	23 940	23 940	4153	4153
	$\#\mathcal{T}_0$	16	16	16	16	24	24
MG (per patch)	min/max #elements	2/6	2/6	2/10	2/10	2/9	2/9
	min/max #DoF (mixed)	11/41	147/462	11/69	147/770	11/62	147/693
	min/max #DoF (div-free)	6/24	91/294	6/40	91/490	6/36	91/441
DD (per subdomain)	min/max #elements	2048/6144	2048/6144	97/22 785	97/22 785	20/3898	20/3898
	min/max #DoF (mixed)	16 256/48 960	164 416/493 920	749/182 203	7714/1 833 923	146/31 156	1561/313 691
	min/max #DoF (div-free)	10 112/30 528	107 072/321 888	458/113 848	4998/1 195 943	86/19 462	1001/204 547

Results

p	Smooth						Wellwavefront						Checkerboard					
	$J=3$		$J=4$		$J=5$		$J=4$		$J=8$		$J=12$		$J=7$		$J=14$		$J=28$	
	MG	DD	MG	DD	MG	DD	MG	DD	MG	DD	MG	DD	MG	DD	MG	DD	MG	DD
1	8	9	8	9	7	9	12	9	10	5	10	5	19	10	37	13	85	13
2	9	9	8	9	7	9	12	7	10	5	8	5	16	11	30	12	64	12
3	8	8	7	8	6	8	10	7	9	5	8	5	15	11	26	11	56	12
6	5	7	4	7	4	7	9	4	8	4	6	4	11	10	18	10	37	11

Error and algebraic a posteriori error estimator, $p = 1$ (left), $p = 3$ (right), levels $j = 3, 6, 9$ (top to bottom)



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Our approach

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References

- A. Miraçi, J. Papež, M. Vohralík, A multilevel algebraic error estimator and the corresponding iterative solver with p-robust behavior, *SIAM J. Numer. Anal.* **58** (2020), 2856–2884.
- A. Miraçi, J. Papež, M. Vohralík, A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps, *SIAM J. Sci. Comput.* **43** (2021), S117–S145.
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Thank you for your attention!