Equivalence of local- and global-best approximations and simple stable local commuting projectors in $\boldsymbol{H}(\mathrm{div}, \Omega)$

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Outline

1. Introduction: classical *a priori* error estimates for mixed finite element methods
2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence
4. Elementwise localized approximation estimates
5. Elementwise localized *a priori* error estimates
    - Mixed finite element methods
    - Least-squares mixed finite element methods
6. Tools (*p*-robustness)
    - Polynomial extension on a tetrahedron
    - Broken polynomial extension on a patch
7. Conclusions and outlook

Mixed finite elements for the Laplace equation

Laplace model problem

Find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

Dual mixed weak formulation

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that $(\boldsymbol{\sigma} = -\nabla u)$

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{v}) - (u, \nabla{\cdot}\boldsymbol{v}) &= 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega),\\ (\nabla{\cdot}\boldsymbol{\sigma}, q) &= (f, q) \quad \forall q \in L^2(\Omega) \end{aligned}$$

Mixed finite elements

Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \times \mathbb{P}_p(\mathcal{T})$, $p \geq 0$, s.t.

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Notation

- Ω: computational domain (open polygon/polyhedron)
- $\mathcal{T}$: simplicial mesh
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Classical *a priori* estimate via RTN interpolant

Theorem (Classical *a priori* estimate)

$$\underbrace{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|}_{\textit{MFE error}} = \underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|}_{\substack{\textit{global-best on } \Omega \\ \textit{normal trace-continuity constraint} \\ \textit{divergence constraint}}} \leq \|\boldsymbol{\sigma} - \underbrace{\boldsymbol{I}_p^{\mathrm{RTN}}(\boldsymbol{\sigma})}_{\substack{\in \boldsymbol{V}_h \\ \nabla\cdot = \Pi_p f}}\|$$

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- simple and local: for all $K \in \mathcal{T}$

$$(\boldsymbol{I}_p^{\mathrm{RTN}}(\boldsymbol{\sigma})\cdot\boldsymbol{n}_F, q_h)_F = (\boldsymbol{\sigma}\cdot\boldsymbol{n}_F, q_h)_F \quad \forall q_h \in \mathbb{P}_p(F),\ \forall F \in \mathcal{F}_K,$$

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- needs $\boldsymbol{\sigma}\cdot\boldsymbol{n}_F \in L^1(F)\ \forall F \in \mathcal{F}$: undefined for $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$
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$$\|\boldsymbol{I}_p^{\mathrm{RTN}}(\boldsymbol{\sigma})\| \not\lesssim \|\boldsymbol{\sigma}\|$$

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Advanced interpolants/stable local commuting projectors

- Buffa and Ciarlet (2001): small regularity but still not $\boldsymbol{H}(\mathrm{div}, \Omega)$
- Schöberl (2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): lower regularity but still not $\boldsymbol{H}(\mathrm{div}, \Omega)$
- Falk and Winther (2014)
- Christiansen (2015)
- Ern and Guermond (2017, 2018)
- ...

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Theorem (Equivalence in H^1_0, Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

Let $u \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathbb{P}_p(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}} \underbrace{\min_{v_h \in \mathbb{P}_p(K)} \|\nabla(u - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T} \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx_p$: up to a generic constant that only depends on space dimension d, shape-regularity of the mesh $\mathcal{T}$, and polynomial degree p

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

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Simple map $P_p : \boldsymbol{H}(\mathrm{div}, \Omega) \to \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

Definition (Map P_p by local projection and flux reconstruction)

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary.

1. Define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ as **elementwise L^2-orthogonal projection** of $\boldsymbol{\sigma}$

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}.$$

2. For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **MFE constrained minimization problem** on the patch $\mathcal{T}_{\boldsymbol{a}}$ around $\boldsymbol{a}$

Destuynder & Métivet (1999), Braess and Schöberl (2008)

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} \ldots} \| \psi_{\boldsymbol{a}} \boldsymbol{\xi}_h - \boldsymbol{v}_h \|_{\omega_{\boldsymbol{a}}}$$

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Definition (Map P_p by local projection and flux reconstruction)

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ and $p \geq 0$ be arbitrary.

1. Define discontinuous $\boldsymbol{\xi}_h \in \boldsymbol{RTN}_p(\mathcal{T})$ as **elementwise L^2-orthogonal projection** of $\boldsymbol{\sigma}$

$$\boldsymbol{\xi}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}.$$

2. For each vertex $\boldsymbol{a} \in \mathcal{V}$, solve the **MFE constrained minimization problem** on the patch $\mathcal{T}_{\boldsymbol{a}}$ around $\boldsymbol{a}$

Destuynder & Métivet (1999), Braess and Schöberl (2008)

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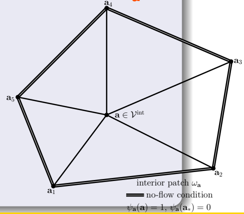
3. Combine

$$P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$$



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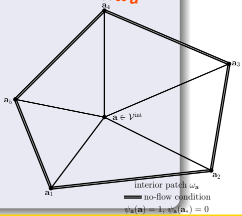

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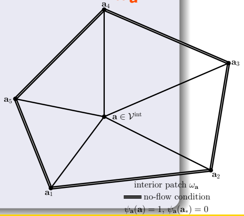

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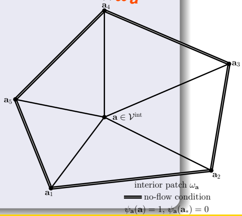

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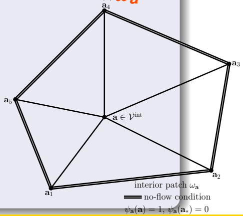

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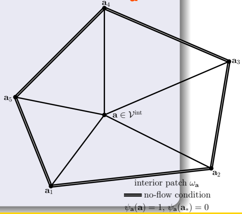

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go discrete (elementwise)

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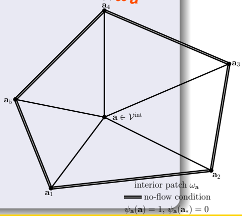

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Destuynder & Métivet (1999), Braess and Schöberl (2008)

smooth (patchwise)

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} := \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma}\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \left\| \boldsymbol{I}_p^{\mathrm{RTN}}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$$

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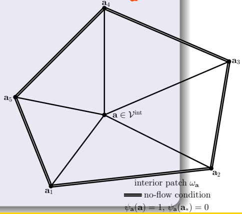

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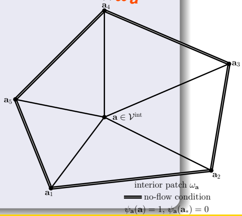
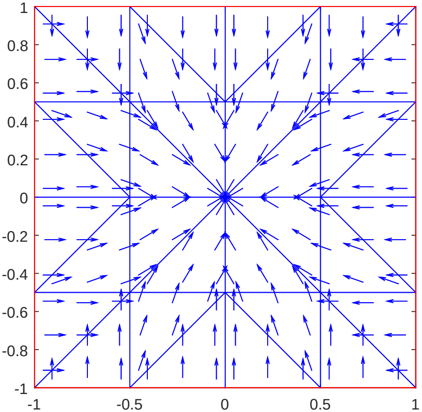


Equilibrated flux reconstruction in 2D



Flux ξ_h

$$\underbrace{\xi_h \in \boldsymbol{RTN}_p(\mathcal{T}),\ \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)}_{(\nabla\cdot\boldsymbol{\sigma},\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}^{\mathrm{int}}}$$

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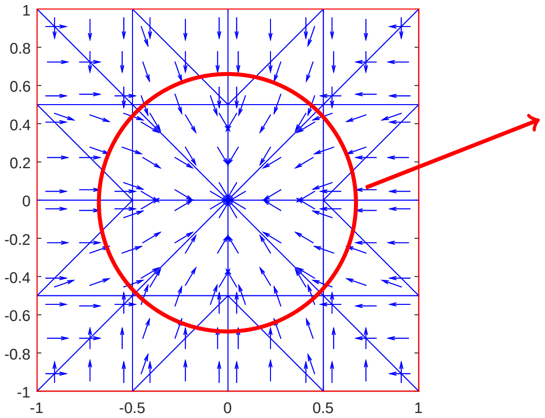

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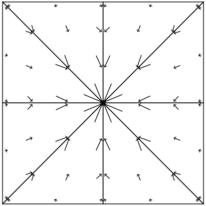

$\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h$

$$\underbrace{\xi_h \in \boldsymbol{RTN}_p(\mathcal{T}), \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega)}_{(\nabla \cdot \boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\xi}_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}^{\rm int}}$$

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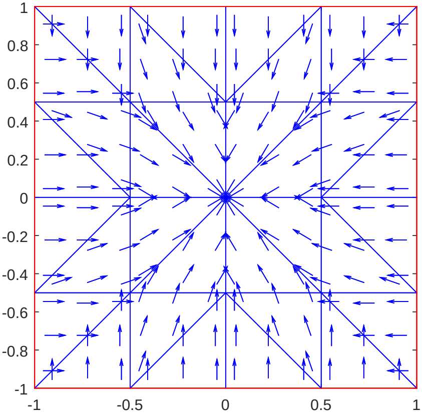
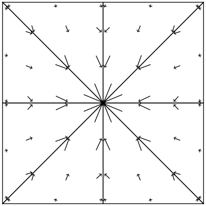


Flux ξ_h



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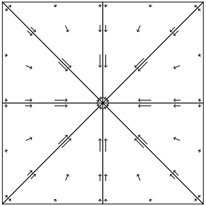

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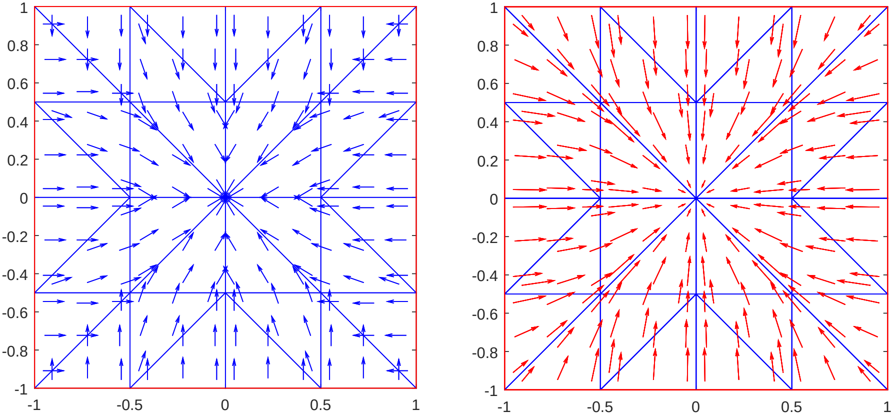

Flux ξ_h Equilibrated flux reconstruction σ_h

$$\underbrace{\xi_h \in \boldsymbol{RTN}_p(\mathcal{T}), \nabla\cdot\boldsymbol{\sigma} \in L^2(\Omega)}_{(\nabla\cdot\boldsymbol{\sigma},\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_p \nabla\cdot\boldsymbol{\sigma}$$

Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ *from* construction *is locally defined,*

$\nabla\cdot(P_p\boldsymbol{\sigma}) = \Pi_p(\nabla\cdot\boldsymbol{\sigma})$ *commuting,*

$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma}$ *if* $\boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ *projector,*

$$\|P_p\boldsymbol{\sigma}\| \lesssim \|\boldsymbol{\sigma}\| + \left\{\sum_{K\in\mathcal{T}} \frac{h_K^2}{p^2}\|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2\right\}^{1/2}$$ *stable up to osc*

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Stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

Theorem (Stable local commuting projector, Ern, Gudi, Smears, & V. (2019)**)**

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,* $P_p\boldsymbol{\sigma} := \boldsymbol{\sigma}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *from* ▸ construction *is locally defined,*

$\nabla{\cdot}(P_p\boldsymbol{\sigma}) = \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})$ *commuting,*

$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma}$ *if* $\boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$ *projector,*

$$\|P_p\boldsymbol{\sigma}\| \lesssim_p \|\boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} h_K^2 \|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K^2 \right\}^{1/2}$$ *stable up to osc.*

Comments

- P_p defined on $\boldsymbol{H}(\mathrm{div}, \Omega)$
- $\lesssim_p$: only depends on d, shape-regularity of $\mathcal{T}$, and p
- $h_K\|\nabla{\cdot}\boldsymbol{\sigma} - \Pi_p(\nabla{\cdot}\boldsymbol{\sigma})\|_K$: data oscillation term common in *a posteriori* analysis, disappears when $\nabla{\cdot}\boldsymbol{\sigma}$ is a piecewise p-degree polynomial

Proof: local problems, commutativity

- recall $\xi_h \in \boldsymbol{RTN}_p(\mathcal{T})$ is elementwise $\boldsymbol{L}^2$-orthogonal projection of $\boldsymbol{\sigma}$

$$\xi_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 \qquad \forall K \in \mathcal{T}$$

- since $\nabla\psi_{\boldsymbol{a}} \in \boldsymbol{RTN}_p(K)$, $\forall \boldsymbol{a} \in \mathcal{V}_K$, $p \geq 0$,

$$(\boldsymbol{\sigma} - \xi_h, \nabla\psi_{\boldsymbol{a}})_K = 0 \qquad \forall K \in \mathcal{T}$$

- since $\boldsymbol{\sigma}|_{\omega_{\boldsymbol{a}}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ and $\psi_{\boldsymbol{a}} \in H_0^1(\omega_{\boldsymbol{a}})$ ($\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$), Green theorem

$$(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0$$

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$$(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\sigma}, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \Rightarrow \boxed{(\nabla\cdot\boldsymbol{\sigma}, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\xi_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}}$$

- implies well-posedness of

- partition of unity implies commutativity

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- implies **well-posedness** of

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} := \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla{\cdot}\boldsymbol{v}_h = \Pi_p(\nabla{\cdot}\boldsymbol{\sigma}\psi_{\boldsymbol{a}} + \xi_h{\cdot}\nabla\psi_{\boldsymbol{a}})}} \left\| \boldsymbol{I}_p^{\mathrm{RTN}}(\psi_{\boldsymbol{a}} \xi_h) - \boldsymbol{v}_h \right\|_{\omega_{\boldsymbol{a}}}$$

- partition of unity $\sum_{\boldsymbol{a} \in \mathcal{V}} \psi_{\boldsymbol{a}} = 1$ implies **commutativity**

$$\nabla{\cdot}\boldsymbol{\sigma}_h = \sum_{\boldsymbol{a} \in \mathcal{V}} \nabla{\cdot}\boldsymbol{\sigma}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}} \Pi_p(\nabla{\cdot}\boldsymbol{\sigma}\psi_{\boldsymbol{a}} + \xi_h{\cdot}\nabla\psi_{\boldsymbol{a}}) = \Pi_p \nabla{\cdot}\boldsymbol{\sigma}$$

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Proof: stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern, V. (2016; 3D), using ▸ Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma}\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_p^{\mathrm{RTN}}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = \Pi_p(\nabla\cdot\boldsymbol{\sigma}\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_p^{\mathrm{RTN}}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Corollary (Global stability)

$P_p\boldsymbol{\sigma} = \boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ up to divergence oscillation:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\xi}_h - \boldsymbol{\sigma})\|_K^2 \right\}^{1/2}.$$

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Proof: projection and $\boldsymbol{L}^2$ stability of the map P_p

Projection property of P_p

if $\boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, then $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from ▸ construction, global ▸ $\boldsymbol{H}(\mathrm{div})$ stability

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$$\Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\xi}_h \Rightarrow \boldsymbol{\sigma}_h = \boldsymbol{\sigma}$$

$\boldsymbol{L}^2$ stability of the map P_p up to oscillation

triangle inequality $\|\boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\boldsymbol{\sigma}\|$, triangle inequality

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \leq \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|,$$

and improved stability

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}} h_K^2 \|\nabla \cdot \boldsymbol{\sigma} - \Pi_p(\nabla \cdot \boldsymbol{\sigma})\|_K^2 \right\}^{1/2}$$

together with $\|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| \leq \|\boldsymbol{\sigma}\|$

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if $\boldsymbol{\sigma} \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, then $\boldsymbol{\xi}_h = \boldsymbol{\sigma}$ from construction, global $\boldsymbol{H}(\mathrm{div})$ stability

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Outline

Global-best approx. ≈ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot \boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}_{\text{global-best on } \Omega}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K\in\mathcal{T}} \underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}_{\text{local-best on each } K}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- the right number (a priori) much smaller than the left one
- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}$, and p
- no need of interpolate for optimal error bounds

Global-best approx. ≈ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

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Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K\in\mathcal{T}} \underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- the right number (a priori) much smaller than the left one
- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}$, and p
- no need of interpolate for optimal error bounds

Global-best approx. $\approx$ local-best approx. in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2019))

Let $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

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Proof ideas

- global ▸ $\boldsymbol{H}(\mathrm{div})$ stability

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\xi}_h - \boldsymbol{\sigma})\|_K^2 \right\}^{1/2}$$

- bound on minimum, triangle inequality

$$\min_{\substack{\boldsymbol{v}_h\in \boldsymbol{RTN}_p(\mathcal{T})\cap\boldsymbol{H}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{v}_h=\Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \le \|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\| + \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\|$$

$$\lesssim_p \left\{ \sum_{K\in\mathcal{T}} \left[\|\boldsymbol{\sigma} - \boldsymbol{\xi}_h\|_K^2 + h_K^2\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\xi}_h)\|_K^2\right] \right\}^{1/2}$$

- $[\boldsymbol{L}^2(K)]^d$-orthogonal projection consequence

$$\|\boldsymbol{\sigma}-\boldsymbol{\xi}_h\|_K^2+h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\xi}_h)\|_K^2 \lesssim_p \min_{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2+h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{v}_h)\|_K^2\right]$$

- divergence constraint

$$\sum_{K\in\mathcal{T}} h_K^2\|\nabla\cdot\boldsymbol{\sigma}-\Pi_p(\nabla\cdot\boldsymbol{\sigma})\|_K^2 \le \sum_{K\in\mathcal{T}} \min_{\boldsymbol{v}_h\in\boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma}-\boldsymbol{v}_h\|_K^2+h_K^2\|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{v}_h)\|_K^2\right]$$

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Outline

Approximation optimal in h for any regularity

Corollary (Elementwise localized approximation estimate)

For any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on any* $K \in \mathcal{T}$,

$$\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K), \; s > 0,$$

there holds

$$\left\{ \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \right\}^{\frac{1}{2}}$$

$$\lesssim_p \left\{ \sum_{K\in\mathcal{T}} \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \right\}^{\frac{1}{2}}$$

$$\lesssim_{p,\boldsymbol{\sigma}} h^{\min\{p+1,s\}}.$$

- $\lesssim_{p,\boldsymbol{\sigma}}$: only depends on d, shape-regularity of $\mathcal{T}$, p, and $\left\{ \sum_{K\in\mathcal{T}} |\boldsymbol{\sigma}|^2_{\boldsymbol{H}^{\min\{p+1,s\}}(K)} \right\}^{1/2}$

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Approximation optimal in h and p for small regularity

Corollary (Elementwise localized approximation estimate)

Let $p \geq 1$*. For any* $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on any* $K \in \mathcal{T}$*,*

$$\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K), \ s \in (0, p],$$

there holds

$$\left\{ \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot \boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} \frac{h_K^2}{p^2} \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \right\}^{\frac{1}{2}}$$

$$\lesssim \left\{ \sum_{K\in\mathcal{T}} \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_{p-1}(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{p^2} \|\nabla\cdot\boldsymbol{\sigma} - \Pi_{p-1}(\nabla\cdot\boldsymbol{\sigma})\|_K^2 \right\}^{\frac{1}{2}}$$

$$\lesssim_{\boldsymbol{\sigma}} \frac{h^s}{p^s}.$$

- $\lesssim_{\boldsymbol{\sigma}}$: only d, reg. of $\mathcal{T}$, and $\left\{ \sum_{K\in\mathcal{T}} |\boldsymbol{\sigma}|^2_{\boldsymbol{H}^s(K)} \right\}^{1/2}$
- p-robust (hp-optimal) estimate

Approximation optimal in h and p for small regularity

Corollary (Elementwise localized approximation estimate)

Let $p \geq 1$. *For any* $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *s.t., locally on any* $K \in \mathcal{T}$,

$$\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K), \; s \in (0, p],$$

there holds

$$\left\{ \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}} \frac{h_K^2}{p^2} \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right] \right\}^{\frac{1}{2}}$$

$$\lesssim \left\{ \sum_{K\in\mathcal{T}} \min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_{p-1}(K)} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{p^2} \|\nabla\cdot\boldsymbol{\sigma} - \Pi_{p-1}(\nabla\cdot\boldsymbol{\sigma})\|_K^2 \right\}^{\frac{1}{2}}$$

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Outline

1. Introduction: classical *a priori* error estimates for mixed finite element methods
2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence
4. Elementwise localized approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
6. Tools (p-robustness)
   - Polynomial extension on a tetrahedron
   - Broken polynomial extension on a patch
7. Conclusions and outlook

Outline

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Corollary (Localized *a priori* estimate for mixed finite elements)

Let $\boldsymbol{\sigma}$ be the weak solution and $\boldsymbol{\sigma}_h$ its MFE approximation. Then

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint}}}$$

$$\lesssim_p \left\{ \sum_{K\in\mathcal{T}} \underbrace{\min_{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K)} \left[\|\boldsymbol{\sigma} - \boldsymbol{v}_h\|_K^2 + h_K^2 \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{v}_h)\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint}}} \right\}^{\frac{1}{2}}$$

$$\lesssim_{p,\boldsymbol{\sigma}} h^{\min\{p+1,s\}}$$

Outline

Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Mixed least-squares weak formulation

Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned}(\boldsymbol{\sigma} + \nabla u, \nabla v) &= 0 && \forall v \in H_0^1(\Omega),\\ (\nabla\cdot\boldsymbol{\sigma}, \nabla\cdot\boldsymbol{p}) + (\boldsymbol{\sigma} + \nabla u, \boldsymbol{p}) &= (f, \nabla\cdot\boldsymbol{p}) && \forall \boldsymbol{p} \in \boldsymbol{H}(\mathrm{div}, \Omega).\end{aligned}$$

Least-squares mixed finite elements

Let $\boldsymbol{V}_h := \boldsymbol{RTN}_p(\mathcal{T}) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$, $V_h := \mathbb{P}_q(\mathcal{T}) \cap H_0^1(\Omega)$, $q \geq 1$. Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{V}_h \times V_h$ such that

$$\begin{aligned}(\boldsymbol{\sigma}_h + \nabla u_h, \nabla v_h) &= 0 && \forall v_h \in V_h,\\ (\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\boldsymbol{p}_h) + (\boldsymbol{\sigma}_h + \nabla u_h, \boldsymbol{p}_h) &= (f, \nabla\cdot\boldsymbol{p}_h) && \forall \boldsymbol{p}_h \in \boldsymbol{V}_h.\end{aligned}$$

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Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}(\mathrm{div}, \Omega) \times H_0^1(\Omega)$ such that

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Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Lemma (A priori bound for least-squares mixed finite elements)

There exists a positive constant C only depending on Ω s.t.

$$
\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\nabla(u - u_h)\| \\
&\leq C\left(\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| + \min_{v_h \in \mathbb{P}_q(\mathcal{T}) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\| \right),
\end{aligned}
$$

$$
\begin{aligned}
\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2 &\leq \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|^2 + \frac{3}{2}\|\nabla(u - u_h)\|^2 \\
&+ \frac{3}{2} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2.
\end{aligned}
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combine with $\boldsymbol{H}(\mathrm{div},\Omega)$ local-global-best and $H^1_0(\Omega)$ local-global-best:

Corollary (Localized *a priori* estimate for least-squares MFEs)

Let $\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K)$, $s > 0$, and $u|_K \in H^{1+t}(K)$, $t > 0$, $\forall K \in \mathcal{T}$. Then

$$
\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\| + \|\nabla(u-u_h)\| + \|\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\| \lesssim_{p,\boldsymbol{\sigma},u} h^{\min\{p+1,s\}} + h^{\min\{q,t\}}.
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There exists a positive constant C only depending on Ω s.t.

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There exists a positive constant C only depending on Ω s.t.

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\nabla(u - u_h)\| \leq C\left(\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\| + \min_{v_h \in \mathbb{P}_q(\mathcal{T}) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\| \right),$$

$$\|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2 \leq \|\nabla\cdot\boldsymbol{\sigma} - \Pi_p(\nabla\cdot\boldsymbol{\sigma})\|^2 + \frac{3}{2}\|\nabla(u - u_h)\|^2 + \frac{3}{2} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{\sigma})}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|^2.$$

combine with ▸ $\boldsymbol{H}(\mathrm{div},\Omega)$ local-global-best and ▸ $H_0^1(\Omega)$ local-global-best:

Corollary (Localized *a priori* estimate for least-squares MFEs)

Let $\boldsymbol{\sigma}|_K \in \boldsymbol{H}^s(K)$, $s > 0$, and $u|_K \in H^{1+t}(K)$, $t > 0$, $\forall K \in \mathcal{T}$. Then

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\nabla(u - u_h)\| + \|\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| \lesssim_{p,\boldsymbol{\sigma},u} h^{\min\{p+1,s\}} + h^{\min\{q,t\}}.$$

Outline

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2. Simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
3. Global-best – local-best equivalence
4. Elementwise localized approximation estimates
5. Elementwise localized *a priori* error estimates
   - Mixed finite element methods
   - Least-squares mixed finite element methods
6. Tools (p-robustness)
   - Polynomial extension on a tetrahedron
   - Broken polynomial extension on a patch
7. Conclusions and outlook

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Polynomial extension on a tetrahedron

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying the compatibility $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

Polynomial extension on a tetrahedron

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying the compatibility $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \; \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \; \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K .$$

Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Polynomial extension on a tetrahedron

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016))

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. *Let* $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, *satisfying the compatibility* $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ *if* $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. *Then*

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

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Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & McIntosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2016)**)**

Let $p \geq 0$, $K \in \mathcal{T}$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying the compatibility $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\|\varphi_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Outline

Broken polynomial extension on a patch

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2016; 3D))

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Outline

Conclusions and outlook

Conclusions

- a simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div}, \Omega)$
- global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div}, \Omega)$
- optimal localized approximation estimates
- optimal localized *a priori* error estimates for mixed finite elements and least-squares mixed finite elements
- *p*-robust estimates optimal for *hp* methods and low regularity solutions

Ongoing work

- extensions to other settings

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References

- ERN A., GUDI T., SMEARS I., VOHRALÍK M., Equivalence of local- and global-best approximations and a simple stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$. Applications to elementwise *a priori* error estimates for mixed finite element and least-squares methods under minimal regularity, in preparation, 2019.

- ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, *SIAM J. Numer. Anal.* **53** (2015), 1058–1081.

- ERN A., VOHRALÍK M., Stable broken H^1 and $\boldsymbol{H}(\mathrm{div})$ polynomial extensions for polynomial-degree-robust potential and flux reconstruction in three space dimensions, HAL Preprint 01422204, 2016.

- ERN A., SMEARS I., VOHRALÍK M., Discrete p-robust $\boldsymbol{H}(\mathrm{div})$-liftings and a posteriori estimates for elliptic problems with H^{-1} source terms, *Calcolo* **54** (2017), 1009–1025.

Thank you for your attention!

Potentials

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009))

Let $p \geq 1$, $K \in \mathcal{T}$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} .$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials

Theorem (Broken H^1 polynomial extension on a patch Ern & V. (2015, 2016)**)**

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}^{\mathrm{int}}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility

$$\begin{aligned} r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} &= 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}, \\ \sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}. \end{aligned}$$

Then

$$\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}}) \\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.$$