

# Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors

Eric Cancès, Geneviève Dusson, Yvon Maday,  
Benjamin Stamm, **Martin Vohralík**

*Inria Paris & Ecole des Ponts*

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# Outline

- 1 Introduction
- 2 Laplace eigenvalue problem equivalences
  - Generic equivalences
  - Dual norm of the residual equivalences
  - Representation of the residual and eigenvalue bounds
- 3 A posteriori estimates
  - Eigenvalues
  - Eigenvectors
  - Comments
- 4 Application to conforming finite elements
- 5 Extension to nonconforming discretizations
- 6 Numerical experiments
- 7 Conclusions and future directions

# Laplace eigenvalue problem

## Setting

- $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , polygon/polyhedron

## Strong formulation

Find **eigenvector & eigenvalue pair**  $(u, \lambda)$  such that

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

## Weak formulation

Find  $(u_i, \lambda_i) \in V := H_0^1(\Omega) \times \mathbb{R}^+$ ,  $i \geq 1$ , with  $\|u_i\| = 1$ , such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \quad \forall v \in V.$$

## Comments

- we suppose here that all the **eigenvalues** are simple
- sign characterization  $(u_i, \chi_i) > 0$  with  $\chi_i \in L^2(\Omega)$  for the uniqueness of the eigenvectors

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# Previous results, eigenvalue bounds

- Armentano and Durán (2004), Plum (1997), Goerisch and He (1989), Still (1988), Kuttler and Sigillito (1978), Moler and Payne (1968), Fox and Rheinboldt (1966), Bazley and Fox (1961), Weinberger (1956), Forsythe (1955), Kato (1949)
- ...

# Previous results, guaranteed eigenvalue lower bounds

- Carstensen and Gedicke (2014) & Liu (2015):  $\oplus$  **guaranteed bound, arbitrarily coarse mesh**;  $\ominus$  a priori arguments (largest mesh element diameter), only lowest-order nonconforming FEs
- Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013):  $\oplus$  **general guaranteed bounds**;  $\ominus$  **condition on applicability, suboptimal convergence speed**
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# Previous results, eigenvector bounds

- Rannacher, Westenberger, Wollner (2010), Grubišić and Oval (2009), Durán, Padra, Rodríguez (2003), Heuveline and Rannacher (2002), Larson (2000), Maday and Patera (2000), Verfürth (1994) ...
- ... typically contain **uncomputable terms**, higher-order on fine enough meshes
- Wang, Chamoin, Ladevèze, Zhong (2016): bounds via the constitutive relation error framework (**almost guaranteed**)

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# Main results (conforming setting)

## Assumption A (Conforming variational solution)

There holds

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih} \quad (\Rightarrow \lambda_{1h} \geq \lambda_1)$

We bound

- $i$ -th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \sqrt{\lambda_i(u_{ih}, \lambda_{ih})}$$

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- 1  $i$ -th eigenvalue error

$$\lambda_{ih} - \lambda_i \leq \eta_i(u_{ih}, \lambda_{ih})^2$$

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- ✓  $C_{\text{eff},i}$  only depends on mesh shape regularity and on

$$\bar{C}_{ih} := 1 \text{ if } i = 1, \quad \bar{C}_{ih} := \max \left\{ \left( \frac{\lambda_{ih}}{\lambda_1} - 1 \right)^2, 1 \right\} \text{ if } i > 1.$$

- ✓ we give computable upper bounds on  $C_{\text{eff},i}$

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$i$ -th eigenvalue upper and lower bounds

$$\lambda_{ih} - \eta_i(u_{ih}, \lambda_{ih})^2 \leq \lambda_i \leq \lambda_{ih} - \tilde{\eta}_i(u_{ih}, \lambda_{ih})^2$$

# The pathway (conforming setting)

- 1 estimate the  $L^2(\Omega)$  error:

$$\|u_i - u_{ih}\| \leq \alpha_{ih}$$

- 2 prove equivalence of the eigenvalue & eigenvector errors:

$$\|\nabla(u_i - u_{ih})\|^2 \lesssim \lambda_{ih} - \lambda_i \leq \|\nabla(u_i - u_{ih})\|^2$$

- 3 prove equivalence of the eigenvector error & of the dual norm of the residual:

$$\|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1} \lesssim \|\nabla(u_i - u_{ih})\| \lesssim \|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1},$$

where

$$\langle \text{Res}(u_{ih}, \lambda_{ih}), v \rangle_{V', V} := \lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v) \quad v \in V$$

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$L^2(\Omega)$  bound

Lemma ( $L^2(\Omega)$  bound via a quadratic residual inequality)

Let Assumption A hold, let  $(u_j, u_{jh}) \geq 0$ , and let

$$\lambda_{i-1} < \lambda_{ih} \text{ when } i > 1, \lambda_{ih} < \lambda_{i+1}.$$

Then

$$\|u_i - u_{ih}\| \leq \alpha_{ih} := \sqrt{2} C_{ih}^{-\frac{1}{2}} \|z_{(ih)}\|,$$

$$C_{ih} := \min \left\{ \left(1 - \frac{\lambda_{ih}}{\lambda_{i-1}}\right)^2, \left(1 - \frac{\lambda_{ih}}{\lambda_{i+1}}\right)^2 \right\}.$$

Riesz representation of the residual  $z_{(ih)} \in V$

$$\begin{aligned} (\nabla z_{(ih)}, \nabla v) &= \langle \text{Res}(u_{ih}, \lambda_{ih}), v \rangle_{V', V} \\ &= \lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v) \quad \forall v \in V \end{aligned}$$

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# $L^2(\Omega)$ bound

## Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$\begin{aligned}
 (z_{(ih)}, u_k) &= \frac{1}{\lambda_k} (\nabla u_k, \nabla z_{(ih)}) = \frac{1}{\lambda_k} (\lambda_{ih}(u_{ih}, u_k) - (\nabla u_{ih}, \nabla u_k)) \\
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Parseval equality for  $z_{(ih)}$

$$\|z_{(ih)}\|^2 =$$

by assumption  $\lambda_{l-1} < \lambda_{ih}$  when  $l > 1$ ,  $\lambda_{ih} < \lambda_{l+1}$  and definition of  $C_{ih}$

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Parseval equality for  $z_{(ih)}, u_k$  orthonormal basis:

$$\|z_{(ih)}\|^2 = \left( \frac{\lambda_{ih}}{\lambda_i} - 1 \right)^2 (u_{ih}, u_i)^2 + \sum_{k \geq 1, k \neq i} \left( 1 - \frac{\lambda_{ih}}{\lambda_k} \right)^2 (u_{ih} - u_i, u_k)^2$$

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## Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$\begin{aligned} (z_{(ih)}, u_k) &= \frac{1}{\lambda_k} (\nabla u_k, \nabla z_{(ih)}) = \frac{1}{\lambda_k} (\lambda_{ih}(u_{ih}, u_k) - (\nabla u_{ih}, \nabla u_k)) \\ &= \left( \frac{\lambda_{ih}}{\lambda_k} - 1 \right) (u_{ih}, u_k) \end{aligned}$$

Parseval equality for  $z_{(ih)}, u_k$  orthonormal basis:

$$\|z_{(ih)}\|^2 = \underbrace{\left( \frac{\lambda_{ih}}{\lambda_i} - 1 \right)^2 (u_{ih}, u_i)^2}_{\geq 0} + \sum_{k \geq 2} \underbrace{\left( 1 - \frac{\lambda_{ih}}{\lambda_k} \right)^2 (u_{ih} - u_i, u_k)^2}_{\geq C_{ih}}$$

by assumption  $\lambda_{i-1} < \lambda_{ih}$  when  $i > 1$ ,  $\lambda_{ih} < \lambda_{i+1}$  and definition of  $C_{ih}$

# $L^2(\Omega)$ bound via a quadratic residual inequality

## Sketch of the proof II.

Parseval equality for  $u_{ih} - u_i$ ,  $(u_{ih} - u_i, u_i) = -\frac{1}{2}\|u_i - u_{ih}\|^2$ :

$$\|\zeta_{(ih)}\|^2 \geq C_{ih}\|u_i - u_{ih}\|^2 - \frac{C_{ih}}{4}\|u_i - u_{ih}\|^4$$

taking  $e_{ih} := \|u_i - u_{ih}\|^2$ :

$$\frac{C_{ih}}{4}e_{ih}^2 - C_{ih}e_{ih} + \|\zeta_{(ih)}\|^2 \geq 0$$

assumption  $(u_i, u_{ih}) \geq 0$ , employing  $\|u_i\| = \|u_{ih}\| = 1$ :

$$e_{ih} = \|u_i - u_{ih}\|^2 = 2 - 2(u_i, u_{ih}) \leq 2,$$

conclusion:

$$\frac{C_{ih}}{2}e_{ih} \leq \|\zeta_{(ih)}\|^2 \Leftrightarrow \|u_i - u_{ih}\| \leq \sqrt{2}C_{ih}^{-\frac{1}{2}}\|\zeta_{(ih)}\|$$

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# Eigenvalue error equivalences

## Theorem (Eigenvalue error – eigenvector error equivalence)

*Under the above assumptions, there holds*

$$\|\nabla(u_i - u_{ih})\|^2 - \lambda_i \alpha_{ih}^2 \leq \lambda_{ih} - \lambda_i \leq \|\nabla(u_i - u_{ih})\|^2.$$

### Simple (known) proof

- there holds

$$\lambda_{ih} - \lambda_i = \|\nabla(u_{ih} - u_i)\|^2 - \lambda_i \|u_i - u_{ih}\|^2$$

- drop the second rhs term to obtain the upper bound
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# Eigenvector error equivalences

Theorem (Eigenvector error – dual norm of the residual equivalence)

Under the above assumptions and if  $\lambda_i \leq \lambda_{ih}$ , there holds

$$\begin{aligned} & \left( \frac{\|\nabla(u_i - u_{ih})\|^2}{\lambda_i} + \bar{C}_{ih} \right)^{-1} \|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1}^2 \\ & \leq \|\nabla(u_i - u_{ih})\|^2 \leq \|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1}^2 + 2\lambda_{ih}\alpha_{ih}^2, \end{aligned}$$

where

$$\bar{C}_{ih} := 1 \text{ if } i = 1, \quad \bar{C}_{ih} := \max \left\{ \left( \frac{\lambda_{ih}}{\lambda_1} - 1 \right)^2, 1 \right\} \text{ if } i > 1.$$

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# How to bound the dual residual norm?

## Dual norm of the residual

$$\|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1} = \sup_{v \in V, \|\nabla v\|=1} \{\lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v)\}$$

Guaranteed upper bound:  $\sigma_{ih} \in \mathbf{H}(\text{div}, \Omega)$  with  $\nabla \cdot \sigma_{ih} = \lambda_{ih} u_{ih}$

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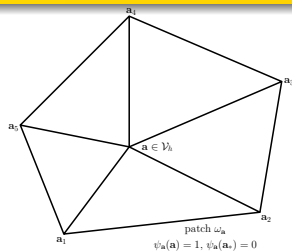
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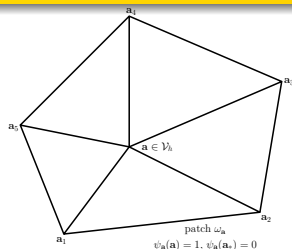
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# Equilibrated flux reconstruction



# Equilibrated flux reconstruction



## Equilibrated flux reconstruction (partition of unity cut-off)

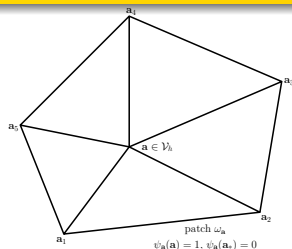
$$\sigma_{ih}^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \nabla \cdot \mathbf{v}_h = ?}$$

$$\| \underbrace{\psi_{\mathbf{a}}}_{\text{hat function}} \nabla u_{ih} + \mathbf{v}_h \|_{\omega_{\mathbf{a}}}$$

- $\sigma_{ih} := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_{ih}^{\mathbf{a}}$ , local minimizations
- $\sigma_{ih}$  is a  $\mathbf{H}(\text{div}, \Omega)$ -conforming lifting of the residual
- $\Rightarrow \nabla \cdot \sigma_{ih} = \lambda_{ih} u_{ih}$

Destuynder & Métivet (1999), Braess & Schöberl (2008)

# Equilibrated flux reconstruction



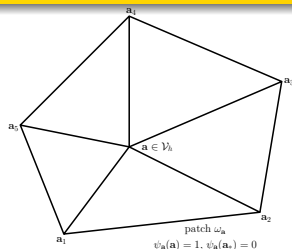
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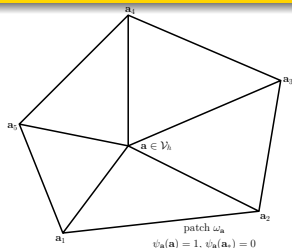
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# Assumptions

Assumption B (Galerkin orthogonality of the residual to  $\psi_{\mathbf{a}}$ )

There holds, for all  $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$ ,

$$\lambda_{ih}(\mathbf{u}_{ih}, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla \mathbf{u}_{ih}, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \text{Res}(\mathbf{u}_{ih}, \lambda_{ih}), \psi_{\mathbf{a}} \rangle_{V', V} = 0.$$

Assumption C (Shape regularity & piecewise polynomial form)

The meshes  $\mathcal{T}_h$  are shape regular. There holds

$\mathbf{u}_{ih} \in \mathbb{P}_p(\mathcal{T}_h)$ ,  $p \geq 1$ , and spaces  $\mathbf{V}_h \times Q_h$  are of degree  $p + 1$ .

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# Dual norm of the residual equivalences

## Theorem (Dual norm of the residual equivalences)

Let  $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}$  verifying Assumptions B and C be arbitrary. Then

$$\|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1} \leq \|\nabla u_{ih} + \sigma_{ih}\|,$$

as well as

$$\|\nabla u_{ih} + \sigma_{ih}\| \leq (d+1)C_{\text{st}}C_{\text{cont,PF}}\|\text{Res}(u_{ih}, \lambda_{ih})\|_{-1}.$$

- $C_{\text{st}}$  and  $C_{\text{cont,PF}}$  independent of the polynomial degree  $p$
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# Bounds on the Riesz representation of the residual

Lemma (Poincaré–Friedrichs bound on  $\|z_{ih}\|$ )

Let  $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}$  be arbitrary. There holds

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belong to  $H^{1+\delta}(\Omega)$ ,  $0 < \delta \leq 1$ , with

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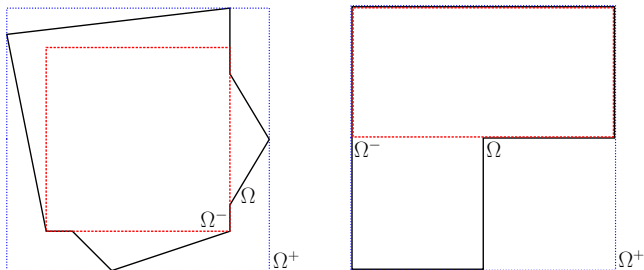
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# How to guarantee $\lambda_{i-1} < \lambda_{ih}$ when $i > 1$ & $\lambda_{ih} < \lambda_{i+1}$ ?

## Option 1: estimates of eigenvalues via domain inclusion



$$\begin{aligned} \Omega \subset \Omega^+ &\Rightarrow \lambda_k \geq \lambda_k(\Omega^+), \\ \Omega \supset \Omega^- &\Rightarrow \lambda_k \leq \lambda_k(\Omega^-), \end{aligned} \quad \forall k \geq 1$$

## Option 2: computational estimates on a rough mesh

- Carstensen and Gedicke (2014)
- Liu (2015)

Sign condition and practical  $L^2(\Omega)$  bound

Lemma (Sign condition and practical  $L^2(\Omega)$  bound)

Let  $0 < \underline{\lambda}_1 \leq \lambda_1$ ,  $\lambda_{i-1} \leq \bar{\lambda}_{i-1} < \lambda_{ih}$ ,  $\lambda_{ih} < \underline{\lambda}_{i+1} \leq \lambda_{i+1}$ , and set

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Let  $(u_{ih}, \chi_i) > 0$  and request

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Then

$$(u_i, u_{ih}) \geq 0, \quad \|u_i - u_{ih}\| \leq \bar{\alpha}_{ih}.$$

$$c_{ih} := \max \left\{ \left( \frac{\lambda_{ih}}{\bar{\lambda}_{i-1}} - 1 \right)^{-1}, \left( 1 - \frac{\lambda_{ih}}{\underline{\lambda}_{i+1}} \right)^{-1} \right\}.$$

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# Guaranteed bounds for the $i$ -th eigenvalue

## Theorem (Eigenvalue bounds)

Under the above assumptions, there holds

$$\lambda_{ih} - \lambda_i \leq \eta_i^2,$$

where

*any smallness assumption can be avoided*

$$\text{case A: } \eta_i^2 := \overbrace{(1 + 4\tilde{c}_{ih}^2 \lambda_{ih})} \|\nabla u_{ih} + \sigma_{ih}\|^2,$$

*verifiable smallness condition, \(\surd\)*

$$\text{case B: } \eta_i^2 := c_{ih}^2 \overbrace{\left(1 - \frac{\bar{\lambda}_i}{\lambda_1} \frac{\bar{\alpha}_{ih}^2}{4}\right)^{-1}} \|\nabla u_{ih} + \sigma_{ih}\|^2,$$

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# Guaranteed bounds for the $i$ -th eigenvector

## Theorem (Eigenvector bounds)

*Under the assumptions of the eigenvalue theorem,*

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i.$$

*Moreover,*

$$\|\nabla u_{ih} + \sigma_{ih}\| \leq (d+1) C_{\text{st}} C_{\text{cont,PF}} \underbrace{\left( \frac{\|\nabla(u_i - u_{ih})\|^2}{\lambda_i} + \bar{C}_{ih} \right)^{\frac{1}{2}}}_{\downarrow \bar{C}_{ih}^{\frac{1}{2}}}$$

$$\|\nabla(u_i - u_{ih})\|.$$

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# Comments

## Eigenvalue bounds

- **guaranteed**
- **optimally convergent**

## Eigenvector bounds

- **efficient** and **polynomial-degree robust**
- $\|\nabla u_{ih} + \sigma_{ih}\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_{ih} + \sigma_{ih}\|_K^2 \Rightarrow$  **adaptivity-ready**
- **maximal overestimation guaranteed**

## Three settings

- A no applicability condition (fine mesh, approximate solution)
- B improvements for explicit, a posteriori verifiable conditions
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## Inexact eigenvalue algebraic solvers

- discretization and algebraic error flux reconstruction
- adaptive stopping criteria via error components balance

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# Application to conforming finite elements

## Finite element method

Find  $(u_{ih}, \lambda_{ih}) \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V \times \mathbb{R}^+$  with  $(u_{ih}, u_{jh}) = \delta_{ij}$ ,  $1 \leq i, j \leq \dim V_h$ , and  $(u_{ih}, \chi_i) > 0$ ,  $p \geq 1$ , such that,

$$(\nabla u_{ih}, \nabla v_h) = \lambda_{ih}(u_{ih}, v_h) \quad \forall v_h \in V_h.$$

## Assumptions verification

- ✓  $V_h \subset V$
- ✓  $\|u_{ih}\| = 1$  and  $(u_{ih}, \chi_i) > 0$  by definition
- ✓  $\|\nabla u_{ih}\|^2 = \lambda_{ih}$  follows by taking  $v_h = u_{ih}$  ( $\Rightarrow$  Assumption A)
- ✓ Assumption B follows upon taking  $v_h = \psi_a \in V_h$
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# Nonconforming discretizations

## Nonconforming setting

- $u_{ih} \notin V$ ,  $\|u_{ih}\| \neq 1$
- $\|\nabla u_{ih}\|^2 \neq \lambda_{ih}$

## Main tool

- conforming eigenvector reconstruction

$$s_{ih}^a := \arg \min_{v_h \in W_h^a \subset H_0^1(\omega_a)} \|\nabla(\psi_a u_{ih} - v_h)\|_{\omega_a}, \quad S_{ih} := \sum_{a \in \mathcal{V}_h} s_{ih}^a$$

## Unified framework

- ✓ conforming finite elements
- ✓ nonconforming finite elements
- ✓ discontinuous Galerkin elements
- ✓ mixed finite elements

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# Unit square

## Setting

- $\Omega = (0, 1)^2$
- $\lambda_1 = 2\pi^2$ ,  $\lambda_2 = 5\pi^2$  known explicitly
- $u_1(x, y) = \sin(\pi x) \sin(\pi y)$  known explicitly

## Parameters

- convex domain:  $C_S = 1$ ,  $\delta = 1$ ,  $C_1 \approx 1/\sqrt{8}$
- auxiliary bounds  $\underline{\lambda}_1 = 1.5\pi^2$ ,  $\underline{\lambda}_2 = 4.5\pi^2$

## Effectivity indices

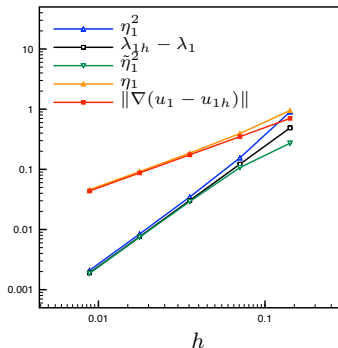
- recall  $\tilde{\eta}_i^2 \leq \lambda_{ih} - \lambda_i \leq \eta_i^2$

$$l_{\lambda, \text{eff}}^{\text{lb}} := \frac{\lambda_{ih} - \lambda_i}{\tilde{\eta}_i^2}, \quad l_{\lambda, \text{eff}}^{\text{ub}} := \frac{\eta_i^2}{\lambda_{ih} - \lambda_i}$$

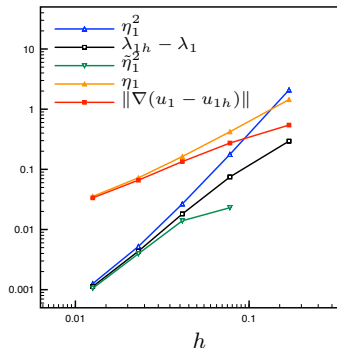
- recall  $\|\nabla(u_i - u_{ih})\| \leq \eta_i$

$$l_{u, \text{eff}}^{\text{ub}} := \frac{\eta_i}{\|\nabla(u_i - u_{ih})\|}$$

# Conforming finite elements



Structured meshes



Unstructured meshes

# Conforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda,\text{eff}}^{\text{lb}}$	$I_{\lambda,\text{eff}}^{\text{ub}}$	$E_{\lambda,\text{rel}}$	$I_{u,\text{eff}}^{\text{ub}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

## Structured meshes

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda,\text{eff}}^{\text{lb}}$	$I_{\lambda,\text{eff}}^{\text{ub}}$	$E_{\lambda,\text{rel}}$	$I_{u,\text{eff}}^{\text{ub}}$
10	0.1698	143	19.7392	20.0336	18.8265	—	—	4.10	—	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

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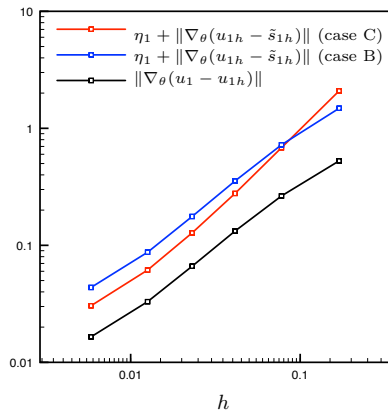
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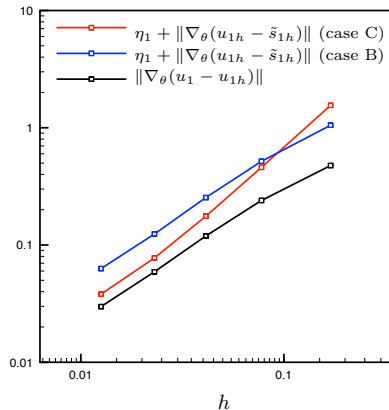
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## Unstructured meshes

# Nonconforming finite elements & DG's



Nonconforming finite elements



Discontinuous Galerkin

# Nonconforming finite elements & DG's

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\ell_{u,eff}^{ub}$
10	0.1414	320	19.7392	19.6850	18.8966	19.8262	4.80e-02	2.68
20	0.0707	1240	19.7392	19.7257	19.6495	19.7616	5.69e-03	2.11
40	0.0354	4880	19.7392	19.7358	19.7246	19.7448	1.02e-03	1.91
80	0.0177	19360	19.7392	19.7384	19.7361	19.7406	2.29e-04	1.85
160	0.0088	77120	19.7392	19.7390	19.7385	19.7396	5.53e-05	1.83
320	0.0044	307840	19.7392	19.7392	19.7390	19.7393	1.37e-05	1.83

## Nonconforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\ell_{u,eff}^{ub}$
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20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

## SIP discontinuous Galerkin

# Nonconforming finite elements & DG's

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\Gamma_{u,eff}^{ub}$
10	0.1414	320	19.7392	19.6850	18.8966	19.8262	4.80e-02	2.68
20	0.0707	1240	19.7392	19.7257	19.6495	19.7616	5.69e-03	2.11
40	0.0354	4880	19.7392	19.7358	19.7246	19.7448	1.02e-03	1.91
80	0.0177	19360	19.7392	19.7384	19.7361	19.7406	2.29e-04	1.85
160	0.0088	77120	19.7392	19.7390	19.7385	19.7396	5.53e-05	1.83
320	0.0044	307840	19.7392	19.7392	19.7390	19.7393	1.37e-05	1.83

## Nonconforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\Gamma_{u,eff}^{ub}$
10	0.1698	732	19.7392	19.9432	17.8788	19.9501	1.10e-01	3.26
20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

## SIP discontinuous Galerkin



# Nonconforming finite elements & DG's

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\Gamma_{U,eff}^{ub}$
10	0.1414	320	19.7392	19.6850	18.8966	19.8262	4.80e-02	2.68
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320	0.0044	307840	19.7392	19.7392	19.7390	19.7393	1.37e-05	1.83

## Nonconforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\Gamma_{U,eff}^{ub}$
10	0.1698	732	19.7392	19.9432	17.8788	19.9501	1.10e-01	3.26
20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

## SIP discontinuous Galerkin

# L-shaped domain

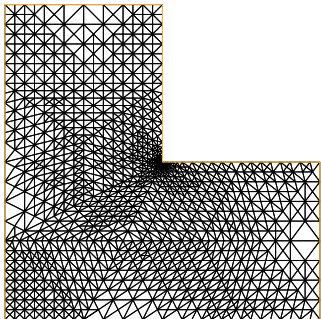
## Setting

- $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$
- $\lambda_1 \approx 9.6397238440$

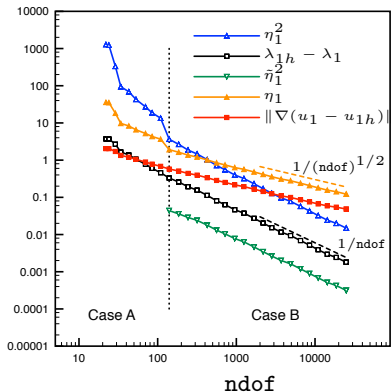
## Parameters

- auxiliary bounds  $\underline{\lambda}_1 = \pi^2/2$  and  $\underline{\lambda}_2 = 5\pi^2/4$  by inclusion into the square  $(-1, 1)^2$

# Conforming finite elements



Adaptively refined mesh



Errors and estimators

# Conforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{U, \text{eff}}^{\text{ub}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
50	0.0634	2286	9.6397	9.6783	9.2904	9.6726	6.77	10.06	4.03e-02	3.19
100	0.0397	8691	9.6397	9.6536	9.5173	9.6515	6.61	9.84	1.40e-02	3.17
200	0.0185	34206	9.6397	9.6448	9.5946	9.6440	6.59	9.85	5.14e-03	3.20
400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

## Unstructured meshes

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda, \text{eff}}^{\text{lb}}$	$I_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$I_{U, \text{eff}}^{\text{ub}}$
10	140	9.6397	9.9700	6.3175	9.9260	7.50	11.06	4.44e-01	3.31
15	561	9.6397	9.7207	9.0035	9.7075	6.17	8.86	7.53e-02	2.98
20	2188	9.6397	9.6601	9.4887	9.6566	5.88	8.43	1.75e-02	2.88
25	8513	9.6397	9.6449	9.6019	9.6440	5.77	8.31	4.37e-03	2.75
30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

## Adaptively refined meshes

# Conforming finite elements

$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$l_{\lambda, \text{eff}}^{\text{lb}}$	$l_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$l_{u, \text{eff}}^{\text{ub}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
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400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

## Unstructured meshes

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$l_{\lambda, \text{eff}}^{\text{lb}}$	$l_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$l_{u, \text{eff}}^{\text{ub}}$
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30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

## Adaptively refined meshes

# Conforming finite elements

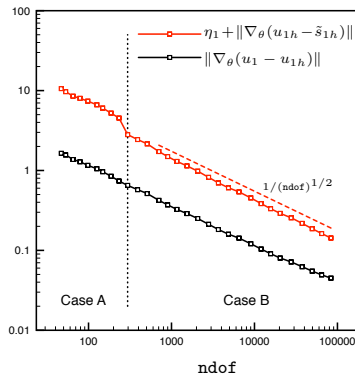
$N$	$h$	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$l_{\lambda, \text{eff}}^{\text{lb}}$	$l_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$l_{u, \text{eff}}^{\text{ub}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
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400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

## Unstructured meshes

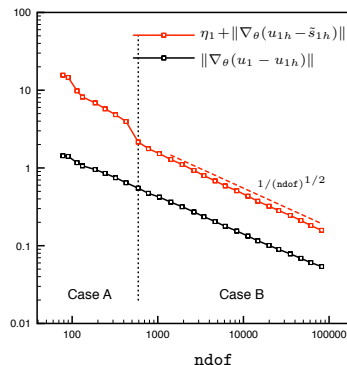
Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$l_{\lambda, \text{eff}}^{\text{lb}}$	$l_{\lambda, \text{eff}}^{\text{ub}}$	$E_{\lambda, \text{rel}}$	$l_{u, \text{eff}}^{\text{ub}}$
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30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

## Adaptively refined meshes

# Nonconforming finite elements & DG's



Nonconforming finite elements



Discontinuous Galerkin

# Nonconforming finite elements & DG's

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\ \nabla \tilde{s}_{1h}\ ^2 - \eta_1^2$	$\ \nabla \tilde{s}_{1h}\ ^2$	$E_{\lambda,rel}$	$I_{u,eff}^{ub}$
5	98	9.6397	8.9699	-29.6187	9.9072	-	6.36
10	296	9.6397	9.4403	4.8193	9.7445	6.76e-01	4.32
15	1161	9.6397	9.5868	8.6628	9.6646	1.09e-01	3.99
20	4860	9.6397	9.6275	9.4310	9.6457	2.25e-02	3.81
25	20429	9.6397	9.6369	9.5925	9.6411	5.06e-03	3.62
30	83472	9.6397	9.6390	9.6284	9.6401	1.21e-03	<b>3.18</b>

## Nonconforming finite elements

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$I_{u,eff}^{ub}$
5	186	9.6397	10.2136	-30.6026	10.3629	-	7.19
10	777	9.6397	9.8154	7.2388	9.8388	3.04e-01	3.75
15	3453	9.6397	9.6865	9.1572	9.6902	5.66e-02	3.38
20	14706	9.6397	9.6509	9.5335	9.6517	1.23e-02	3.23
25	61137	9.6397	9.6425	9.6144	9.6426	2.93e-03	<b>3.00</b>

## SIP discontinuous Galerkin



# Nonconforming finite elements & DG's

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\ \nabla \tilde{s}_{1h}\ ^2 - \eta_1^2$	$\ \nabla \tilde{s}_{1h}\ ^2$	$E_{\lambda,rel}$	$\Gamma_{U,eff}^{ub}$
5	98	9.6397	8.9699	-29.6187	9.9072	-	6.36
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20	4860	9.6397	9.6275	9.4310	9.6457	2.25e-02	3.81
25	20429	9.6397	9.6369	9.5925	9.6411	5.06e-03	3.62
30	83472	9.6397	9.6390	9.6284	9.6401	1.21e-03	<b>3.18</b>

## Nonconforming finite elements

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$\Gamma_{U,eff}^{ub}$
5	186	9.6397	10.2136	-30.6026	10.3629	-	7.19
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25	61137	9.6397	9.6425	9.6144	9.6426	2.93e-03	<b>3.00</b>

## SIP discontinuous Galerkin

# Nonconforming finite elements & DG's

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\ \nabla \tilde{s}_{1h}\ ^2 - \eta_1^2$	$\ \nabla \tilde{s}_{1h}\ ^2$	$E_{\lambda,rel}$	$I_{u,eff}^{ub}$
5	98	9.6397	8.9699	-29.6187	9.9072	-	6.36
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30	83472	9.6397	9.6390	9.6284	9.6401	1.21e-03	<b>3.18</b>

## Nonconforming finite elements

Level	ndof	$\lambda_1$	$\lambda_{1h}$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2} - \eta_1^2$	$\frac{\ \nabla s_{1h}\ ^2}{\ s_{1h}\ ^2}$	$E_{\lambda,rel}$	$I_{u,eff}^{ub}$
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25	61137	9.6397	9.6425	9.6144	9.6426	2.93e-03	<b>3.00</b>

## SIP discontinuous Galerkin

# Higher eigenvalues: unit triangle & L-shaped domain

## Setting – unit triangle

- $\Omega$ : triangle with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$
- nonconforming finite elements, coarse mesh (2145 triang.):

$$\underline{\lambda}_1 = 49.2883, \quad \underline{\lambda}_2 = 98.4296, \quad \underline{\lambda}_3 = 127.937, \quad \underline{\lambda}_4 = 166.975, \quad \underline{\lambda}_5 = 196.439$$

- convex domain:  $C_S = 1$ ,  $\delta = 1$ ,  $C_I \approx 1/\sqrt{8}$
- case C

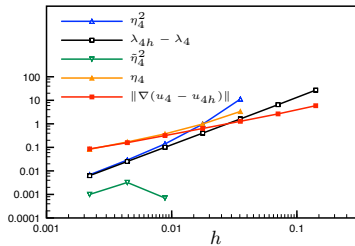
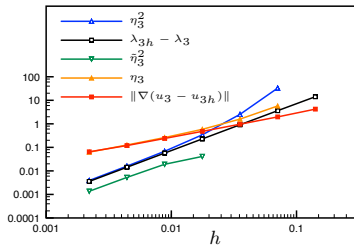
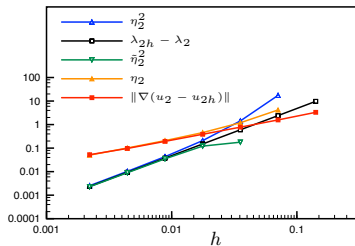
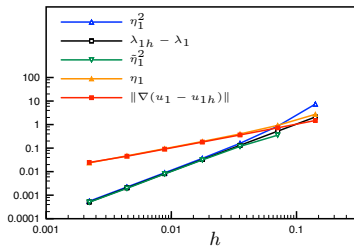
## Setting – L shaped domain

- $\Omega := (-1, 1)^2 \setminus [0, 1] \times [-1, 0]$
- nonconforming finite elements, coarse mesh (3201 triang.):

$$\underline{\lambda}_1 = 9.60692, \quad \underline{\lambda}_2 = 15.1695, \quad \underline{\lambda}_3 = 19.6932, \quad \underline{\lambda}_4 = 29.4166, \quad \underline{\lambda}_5 = 31.7363$$

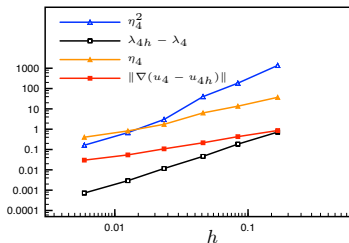
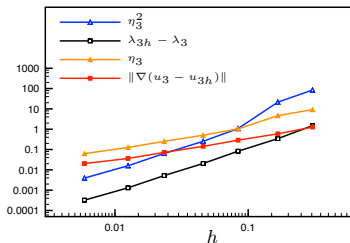
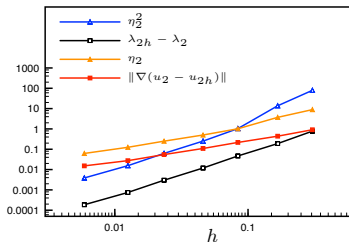
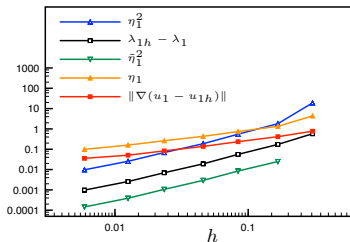
- case A/B

# Unit triangle & conforming finite elements



First four eigenvalues

# L-shaped domain & conforming finite elements



First four eigenvalues

# Outline

- 1 Introduction
- 2 Laplace eigenvalue problem equivalences
  - Generic equivalences
  - Dual norm of the residual equivalences
  - Representation of the residual and eigenvalue bounds
- 3 A posteriori estimates
  - Eigenvalues
  - Eigenvectors
  - Comments
- 4 Application to conforming finite elements
- 5 Extension to nonconforming discretizations
- 6 Numerical experiments
- 7 Conclusions and future directions

# Conclusions and future directions

## Conclusions

- guaranteed upper and lower bounds for the  $i$ -th eigenvalue
- guaranteed and polynomial-degree robust bounds for the associated eigenvector
- general framework

## Ongoing work

- extension to nonlinear eigenvalue problems

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- guaranteed upper and lower bounds for the  $i$ -th eigenvalue
- guaranteed and polynomial-degree robust bounds for the associated eigenvector
- general framework

## Ongoing work

- extension to nonlinear eigenvalue problems



# Bibliography

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- CANCÈS E., DUSSON G., MADAY Y., STAMM B., VOHRALÍK M., Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations, *SIAM J. Numer. Anal.*, accepted for publication.
- CANCÈS E., DUSSON G., MADAY Y., STAMM B., VOHRALÍK M., Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: a unified framework, HAL Preprint 01483461.

**Thank you for your attention!**