Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors

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Outline

Laplace eigenvalue problem

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, polygon/polyhedron

Energy minimization

Find $u_1 \in V := H_0^1(\Omega)$ such that $(u_1, 1) > 0$ and

$$u_1 := \arg \min_{v \in V, \|v\|=1} \left\{ \frac{1}{2} \|\nabla v\|^2 \right\}.$$

Strong formulation

Find eigenvector & eigenvalue pair (u, λ) such that

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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Full problem

Weak formulation of the full problem

Find $(u_i, \lambda_i) \in V \times \mathbb{R}^+$, $i \geq 1$, with $\|u_i\| = 1$, such that

$$(\nabla u_i, \nabla v) = \lambda_i (u_i, v) \qquad \forall v \in V.$$

Comments

- sign characterization $(u_i, \chi_i) > 0$ with $\chi_i \in L^2(\Omega)$ for the uniqueness of the eigenvectors
- take $v = u_i$ as test function $\Rightarrow \|\nabla u_i\|^2 = \lambda_i$
- $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_i \to \infty$
- u_i, $i \geq 1$, form an orthonormal basis of $L^2(\Omega)$
- $u_i/\sqrt{\lambda_i}$, $i \geq 1$, form an orthonormal basis of V
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Previous results, eigenvalue bounds

- Armentano and Durán (2004), Plum (1997), Goerisch and He (1989), Still (1988), Kuttler and Sigillito (1978), Moler and Payne (1968), Fox and Rheinboldt (1966), Bazley and Fox (1961), Weinberger (1956), Forsythe (1955), Kato (1949)
- ...

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- Carstensen and Gedicke (2014) & Liu (2015): ⊕ guaranteed bound, arbitrarily coarse mesh; ⊖ a priori arguments (largest mesh element diameter), only lowest-order nonconforming FEs
- Hu, Huang, Lin (2014): ⊕ bounds in nonconforming FEs; ⊖ saturation assumption may be necessary
- Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013): ⊕ general guaranteed bounds; ⊖ condition on applicability, suboptimal convergence speed
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- Rannacher, Westenberger, Wollner (2010), Grubišić and Ovall (2009), Durán, Padra, Rodríguez (2003), Heuveline and Rannacher (2002), Larson (2000), Maday and Patera (2000), Verfürth (1994) ...
- ... typically contain uncomputable terms, higher-order on fine enough meshes
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The game

Assumption A (Conforming variational solution)

There holds

- $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}^+$
- $\|u_{ih}\| = 1$
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ $(\Rightarrow \lambda_{1h} \geq \lambda_1)$

We want to estimate

1. *i*-th eigenvalue error
2. *i*-th eigenvector energy error

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i(u_{ih}, \lambda_{ih})$$

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- $C_{\mathrm{eff},i}$ only depends on mesh shape regularity and on $\max\left\{\left(\frac{\lambda_i}{\lambda_{i-1}} - 1\right)^{-1}, \left(1 - \frac{\lambda_i}{\lambda_{i+1}}\right)^{-1}\right\} \frac{\lambda_i}{\lambda_1}$
- we give computable upper bounds on C_{eff}

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The pathway

1. estimate the $L^2(\Omega)$ error:

$$\|u_i - u_{ih}\| \leq \alpha_{ih}$$

2. prove equivalence of the eigenvalue & eigenvector errors:

$$C\|\nabla(u_i - u_{ih})\|^2 \leq \lambda_{ih} - \lambda_i \leq \|\nabla(u_i - u_{ih})\|^2$$

3. prove equivalence of the eigenvector error & of the dual norm of the residual:

$$\underline{C}\|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1} \leq \|\nabla(u_i - u_{ih})\| \leq \overline{C}\|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1},$$

where

$$\langle \mathrm{Res}(u_{ih}, \lambda_{ih}), v\rangle_{V',V} := \lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v) \qquad v \in V$$

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Lemma ($L^2(\Omega)$ bound via a quadratic residual inequality)

Let Assumption A hold, let $(u_i, u_{ih}) \geq 0$, *and let*

$$\lambda_{i-1} < \lambda_{ih} \text{ when } i > 1 \text{ , } \lambda_{ih} < \lambda_{i+1}.$$

Then

$$\|u_i - u_{ih}\| \leq \alpha_{ih} := \sqrt{2} C_{ih}^{-\frac{1}{2}} \|r_{(ih)}\|,$$

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Riesz representation of the residual $r_{(ih)} \in V$

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$L^2(\Omega)$ bound

Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$(r_{(ih)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla r_{(ih)}) = \frac{1}{\lambda_k}(\lambda_{ih}(u_{ih}, u_k) - (\nabla u_{ih}, \nabla u_k)) = \left(\frac{\lambda_{ih}}{\lambda_k} - 1\right)(u_{ih}, u_k)$$

Parseval equality for $r_{(ih)}$

$$\|r_{(ih)}\|^2 =$$

by assumption $\lambda_{i-1} < \lambda_{ih}$ when $i > 1$, $\lambda_{ih} < \lambda_{i+1}$ and definition of C_{ih}

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Parseval equality for $r_{(ih)}$, u_k orthonormal basis:

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $u_{ih} - u_i$, $(u_{ih} - u_i, u_i) = -\frac{1}{2}\|u_i - u_{ih}\|^2$:

$$\|r_{(ih)}\|^2 \geq \left(\frac{\lambda_{ih}}{\lambda_i} - 1\right)^2 (u_{ih}, u_i)^2 + C_{ih}\|u_i - u_{ih}\|^2 - \frac{C_{ih}}{4}\|u_i - u_{ih}\|^4$$

dropping the first term above, taking $e_{ih} := \|u_i - u_{ih}\|^2$:

$$\frac{C_{ih}}{4} e_{ih}^2 - C_{ih} e_{ih} + \|r_{(ih)}\|^2 \geq 0$$

assumption $(u_i, u_{ih}) \geq 0$, employing $\|u_i\| = \|u_{ih}\| = 1$:

$$e_{ih} = \|u_i - u_{ih}\|^2 = 2 - 2(u_i, u_{ih}) \leq 2,$$

conclusion:

$$\frac{C_{ih}}{2} e_{ih} \leq \|r_{(ih)}\|^2 \quad \Leftrightarrow \quad \|u_i - u_{ih}\| \leq \sqrt{2} C_{ih}^{-\frac{1}{2}} \|r_{(ih)}\|$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $u_{ih} - u_i$, $(u_{ih} - u_i, u_i) = -\frac{1}{2}\|u_i - u_{ih}\|^2$:

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dropping the first term above, taking $e_{ih} := \|u_i - u_{ih}\|^2$:

$$\frac{C_{ih}}{4} e_{ih}^2 - C_{ih} e_{ih} + \|\mathscr{r}(_{ih})\|^2 \geq 0$$

assumption $(u_i, u_{ih}) \geq 0$, employing $\|u_i\| = \|u_{ih}\| = 1$:

$$e_{ih} = \|u_i - u_{ih}\|^2 = 2 - 2(u_i, u_{ih}) \leq 2,$$

conclusion:

$$\frac{C_{ih}}{2} e_{ih} \leq \|\mathscr{r}(_{ih})\|^2 \quad \Leftrightarrow \quad \|u_i - u_{ih}\| \leq \sqrt{2} C_{ih}^{-\frac{1}{2}} \|\mathscr{r}(_{ih})\|$$

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Outline

First eigenvalue error equivalences

Theorem (Eigenvalue error – eigenvector error equivalence)

Under the above assumptions, there holds

$$\|\nabla(u_i - u_{ih})\|^2 - \lambda_i \alpha_{ih}^2 \leq \lambda_{ih} - \lambda_i \leq \|\nabla(u_i - u_{ih})\|^2.$$

Easy (known) proof

- there holds

$$\lambda_{ih} - \lambda_i = \|\nabla(u_{ih} - u_i)\|^2 - \lambda_i \|u_i - u_{ih}\|^2$$

- drop the second rhs term to obtain the upper bound
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First eigenvector error equivalences

Theorem (Eigenvector error – dual norm of the residual equivalence)

Under the above assumptions and if $\lambda_i \leq \lambda_{ih}$, there holds

$$\left(\frac{\|\nabla(u_i - u_{ih})\|^2}{\lambda_i} + \overline{C}_{ih}\right)^{-1} \|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1}^2$$
$$\leq \|\nabla(u_i - u_{ih})\|^2 \leq \|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1}^2 + 2\lambda_{ih}\alpha_{ih}^2,$$

where

$$\overline{C}_{ih} := 1 \text{ if } i = 1, \quad \overline{C}_{ih} := \max\left\{\left(\frac{\lambda_{ih}}{\lambda_1} - 1\right)^2, 1\right\} \text{ if } i > 1.$$

Outline

How to bound the dual residual norm?

Dual norm of the residual

$$\|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1} = \sup_{v \in V,\, \|\nabla v\|=1} \langle \mathrm{Res}(u_{ih}, \lambda_{ih}), v \rangle_{V',V}$$
$$= \sup_{v \in V,\, \|\nabla v\|=1} \{\lambda_{ih}(u_{ih}, v) - (\nabla u_{ih}, \nabla v)\}$$

Guaranteed upper bound: $\boldsymbol{\sigma}_{ih} \in \mathbf{H}(\mathrm{div}, \Omega)$ **with** $\nabla \cdot \boldsymbol{\sigma}_{ih} = \lambda_{ih} u_{ih}$

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Equilibrated flux reconstruction

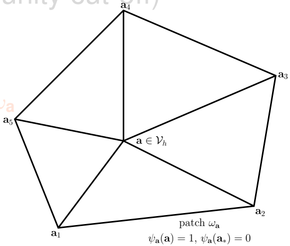
Ideal equilibrated flux reconstruction $(-\nabla u_{ih} \notin \mathbf{H}(\mathrm{div}, \Omega))$

$$\boldsymbol{\sigma}_{ih} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla \cdot \mathbf{v}_h = \lambda_{ih} u_{ih}} \|\nabla u_{ih} + \mathbf{v}_h\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega) \Rightarrow$ **global** minimization, too expensive

Equilibrated flux reconstruction (partition of unity cut-off)

$$\boldsymbol{\sigma}_{ih}^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla \cdot \mathbf{v}_h = ?} \|\underbrace{\psi_{\mathbf{a}}}_{\text{hat function}} \nabla u_{ih} + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$



- $\boldsymbol{\sigma}_{ih} := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_{ih}^{\mathbf{a}}$, **local** minimizations
- $\boldsymbol{\sigma}_{ih}$ is a $\mathbf{H}(\mathrm{div}, \Omega)$-conforming lifting of the residual

Destuynder & Métivet (1999), Braess & Schöberl (2008)

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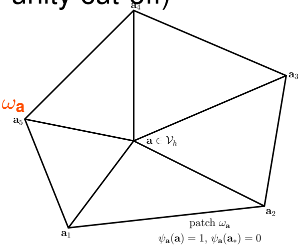

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Equilibrated fluxes by local Neumann problems

Definition (Local Neumann problems by mixed finite elements)

For all vertices $\mathbf{a} \in \mathcal{V}_h$, set

$$\mathbf{V}_h^{\mathbf{a}} := \{\mathbf{v}_h \in \mathbf{V}_h(\omega_{\mathbf{a}});\ \mathbf{v}_h \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}}\}, \qquad \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}},$$
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where $\mathbf{V}_h(\omega_{\mathbf{a}}) \times Q_h(\omega_{\mathbf{a}})$ are standard MFE spaces. Then prescribe $(\boldsymbol{\sigma}_{ih}^{\mathbf{a}}, p_h^{\mathbf{a}}) \in \mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$ by

$$(\boldsymbol{\sigma}_{ih}^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (p_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_{ih}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},$$
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Numerical assumptions

Assumption B (Galerkin orthogonality of the residual to $\psi_{\mathbf{a}}$)

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Assumption C (Shape regularity & piecewise polynomial form)

The meshes $\mathcal{T}_h$ are shape regular. There holds

$$u_{ih} \in \mathbb{P}_p(\mathcal{T}_h),\ p \geq 1, \text{ and spaces } \mathbf{V}_h \times Q_h \text{ are of degree } p+1.$$

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Dual norm of the residual equivalences

Theorem (Dual norm of the residual equivalences)

Let $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}$ verifying Assumptions B and C be arbitrary. Then

$$\|\mathrm{Res}(u_{ih}, \lambda_{ih})\|_{-1} \leq \|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\|,$$

as well as

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- C_{st} and $C_{\mathrm{cont,PF}}$ independent of the polynomial degree p
- we can compute upper bounds on C_{st} and $C_{\mathrm{cont,PF}}$

Ern & Vohralík (2015)

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Outline

Bounds on the Riesz representation of the residual

Lemma (Poincaré–Friedrichs bound on $\|r_{(ih)}\|$)

Let $(u_{ih}, \lambda_{ih}) \in V \times \mathbb{R}$ be arbitrary. There holds

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$$(\nabla \zeta_{(ih)}, \nabla v) = (r_{(ih)}, v) \qquad \forall v \in V$$

belong to $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, with

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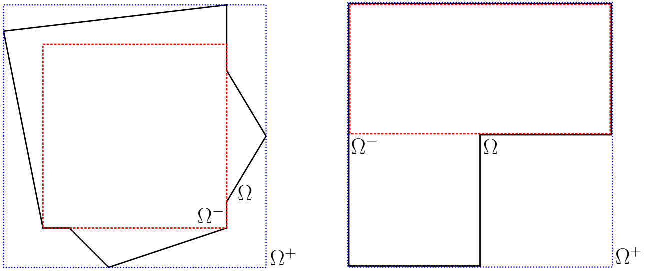
How to guarantee $\lambda_{i-1} < \lambda_{ih}$ when $i > 1$, $\lambda_{ih} < \lambda_{i+1}$?

Option 1: estimates of eigenvalues via domain inclusion



$$\begin{aligned} \Omega \subset \Omega^+ \quad &\Rightarrow \quad \lambda_k \geq \lambda_k(\Omega^+), \\ \Omega \supset \Omega^- \quad &\Rightarrow \quad \lambda_k \leq \lambda_k(\Omega^-), \end{aligned} \qquad \forall k \geq 1$$

Option 2: computational estimates on a rough mesh

- Carstensen and Gedicke (2014)
- Liu (2015)

Sign condition and practical $L^2(\Omega)$ bound

Lemma (Sign condition and practical $L^2(\Omega)$ bound)

Let $0 < \underline{\lambda}_1 \le \lambda_1$, $\lambda_{i-1} \le \overline{\lambda}_{i-1} < \lambda_{ih}$, $\lambda_{ih} < \underline{\lambda}_{i+1} \le \lambda_{i+1}$, *and set*

$$\tilde{c}_{ih} := \max\left\{ \overline{\lambda}_{i-1}^{-\frac{1}{2}} \left(\frac{\lambda_{ih}}{\overline{\lambda}_{i-1}} - 1 \right)^{-1}, \underline{\lambda}_{i+1}^{-\frac{1}{2}} \left(1 - \frac{\lambda_{ih}}{\underline{\lambda}_{i+1}} \right)^{-1} \right\}.$$

Let $(u_{ih}, \chi_i) > 0$ *and request*

$$\overline{\alpha}_{ih} := \sqrt{2}\tilde{c}_{ih} \|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\| \le \min\left\{ \left(\frac{2\underline{\lambda}_1}{\lambda_{ih}} \right)^{\frac{1}{2}}, \|\chi_i\|^{-1}(u_{ih}, \chi_i) \right\}.$$

Then

$$(u_i, u_{ih}) \ge 0, \qquad \|u_i - u_{ih}\| \le \overline{\alpha}_{ih}.$$

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Outline

Outline

Guaranteed bounds for the *i*-th eigenvalue

Theorem (Eigenvalue bounds)

Under the above assumptions, there holds

$$\lambda_{ih} - \lambda_i \leq \eta_i^2,$$

where

$$\text{case A: } \eta_i^2 := \overbrace{(1 + 4\tilde{c}_{ih}^2 \lambda_{ih})}^{\text{any smallness assumption can be avoided}} \|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\|^2,$$

$$\text{case B: } \eta_i^2 := c_{ih}^2 \overbrace{\left(1 - \frac{\overline{\lambda}_i}{\underline{\lambda}_1} \frac{\overline{\alpha}_{ih}^2}{4}\right)^{-1}}^{\searrow 1} \|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\|^2,$$

$$\text{case C: } \eta_i^2 := \overbrace{(1 + 4c_{ih}^2 \lambda_{ih} (C_{\mathrm{I}} C_{\mathrm{S}} h^\delta)^2)}^{\text{under elliptic regularity, } \searrow 1} \|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\|^2.$$

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Guaranteed bounds for the i-th eigenvector

Theorem (Eigenvector bounds)

Under the assumptions of the eigenvalue theorem,

$$\|\nabla(u_i - u_{ih})\| \leq \eta_i.$$

Moreover,

$$\|\nabla u_{ih} + \boldsymbol{\sigma}_{ih}\| \leq (d+1) C_{\mathrm{st}} C_{\mathrm{cont,PF}} \underbrace{\left(\frac{\|\nabla(u_i - u_{ih})\|^2}{\lambda_i} + \overline{C}_{ih} \right)^{\frac{1}{2}}}_{\searrow \overline{C}_{ih}^{\frac{1}{2}}} \|\nabla(u_i - u_{ih})\|.$$

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Comments

Eigenvalue bounds

- **guaranteed**
- **optimally convergent**

Eigenvector bounds

- **efficient** and **polynomial-degree robust**
- $\|\nabla u_{ih} + \sigma_{ih}\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_{ih} + \sigma_{ih}\|_K^2 \Rightarrow$ **adaptivity-ready**
- **maximal overestimation guaranteed**

Three settings

- no applicability condition (fine mesh, approximate solution)
- improvements for explicit, a posteriori verifiable conditions
- multiplicative factor goes to one under elliptic regularity

Inexact eigenvalue algebraic solvers

- discretization and algebraic error flux reconstruction
- adaptive stopping criteria via error components balance

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Improved eigenvalue upper bounds

$H^1_0(\Omega)$-conforming residual lifting (Babuška & Strouboulis (2001), Repin (2008))

Definition (Conforming local Neumann problems: lifted residual)

For each $\mathbf{a} \in \mathcal{V}_h$, define $r^{\mathbf{a}}_{ih} \in X^{\mathbf{a}}_h \subset H^1(\omega_{\mathbf{a}})$ by

$$(\nabla r^{\mathbf{a}}_{ih}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \mathrm{Res}(u_{ih}, \lambda_{ih}), \psi_{\mathbf{a}} v_h \rangle_{V',V} \qquad \forall v_h \in X^{\mathbf{a}}_h.$$

Then set

$$r_{ih} := \sum_{\mathbf{a}\in\mathcal{V}_h} \psi_{\mathbf{a}} r^{\mathbf{a}}_{ih} \in V = H^1_0(\Omega).$$

Guaranteed lower bound on the residual

$$\sup_{v\in V,\, \|\nabla v\|=1} \langle \mathrm{Res}(u_{ih}, \lambda_{ih}), v \rangle_{V',V} \geq \frac{\langle \mathrm{Res}(u_{ih}, \lambda_{ih}), r_{ih} \rangle_{V',V}}{\|\nabla r_{ih}\|}$$

$$= \frac{\sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathrm{Res}(u_{ih}, \lambda_{ih}), \psi_{\mathbf{a}} r^{\mathbf{a}}_{ih} \rangle_{V',V}}{\|\nabla r_{ih}\|} = \frac{\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}_{ih}\|^2_{\omega_{\mathbf{a}}}}{\|\nabla r_{ih}\|} \geq \frac{\left\{\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}_{ih}\|^2_{\omega_{\mathbf{a}}}\right\}^{\frac{1}{2}}}{(d+1)^{\frac{1}{2}} C_{\mathrm{cont,PF}}}$$

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Theorem (Improved eigenvalue upper bounds (cases A & C))

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$$\lambda_i \le \lambda_{ih} - \tilde{\eta}_i^2,$$

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Outline

Application to conforming finite elements

Finite element method

Find $(u_{ih}, \lambda_{ih}) \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V \times \mathbb{R}^+$ with $(u_{ih}, u_{jh}) = \delta_{ij}$, $1 \le i,j \le \dim V_h$, and $(u_{ih}, \chi_i) > 0$, $p \ge 1$, such that,

$$(\nabla u_{ih}, \nabla v_h) = \lambda_{ih}(u_{ih}, v_h) \qquad \forall v_h \in V_h.$$

Assumptions verification

- $V_h \subset V$
- $\|u_{ih}\| = 1$ and $(u_{ih}, \chi_i) > 0$ by definition
- $\|\nabla u_{ih}\|^2 = \lambda_{ih}$ follows by taking $v_h = u_{ih}$ ($\Rightarrow$ Assumption A)
- Assumption B follows upon taking $v_h = \psi_{\mathbf{a}} \in V_h$
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Outline

European Research Council

Nonconforming discretizations

Nonconforming setting

- $u_{ih} \notin V$, $\|u_{ih}\| \neq 1$
- $\|\nabla u_{ih}\|^2 \neq \lambda_{ih}$

Main tool

- conforming eigenvector reconstruction

$$s_{ih}^{\mathbf{a}} := \arg \min_{v_h \in W_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_{ih} - v_h)\|_{\omega_{\mathbf{a}}}, \qquad s_{ih} := \sum_{\mathbf{a} \in \mathcal{V}_h} s_{ih}^{\mathbf{a}}$$

Unified framework

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin elements
- mixed finite elements

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Outline

Unit square

Setting

- $\Omega = (0,1)^2$
- $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$ known explicitly
- $u_1(x,y) = \sin(\pi x)\sin(\pi y)$ known explicitly

Parameters

- convex domain: $C_{\rm S} = 1$, $\delta = 1$, $C_{\rm I} \approx 1/\sqrt{8}$
- auxiliary bounds $\underline{\lambda_1} = 1.5\pi^2$, $\underline{\lambda_2} = 4.5\pi^2$

Effectivity indices

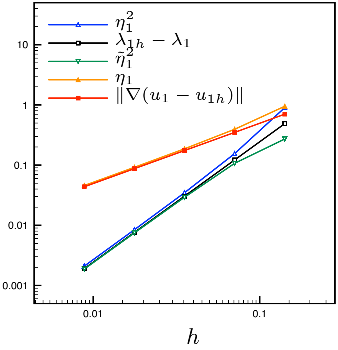
- recall $\tilde\eta_i^2 \le \lambda_{ih} - \lambda_i \le \eta_i^2$

$$I^{\rm lb}_{\lambda,\rm eff} := \frac{\lambda_{ih} - \lambda_i}{\tilde\eta_i^2}, \qquad I^{\rm ub}_{\lambda,\rm eff} := \frac{\eta_i^2}{\lambda_{ih} - \lambda_i}$$

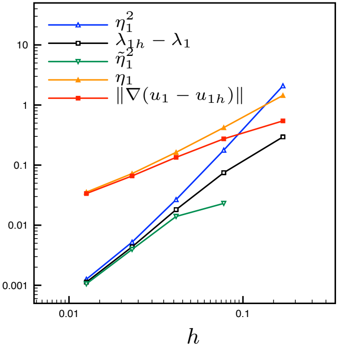
- recall $\|\nabla(u_i - u_{ih})\| \le \eta_i$

$$I^{\rm ub}_{u,\rm eff} := \frac{\eta_i}{\|\nabla(u_i - u_{ih})\|}$$

Conforming finite elements



Structured meshes



Unstructured meshes

Conforming finite elements

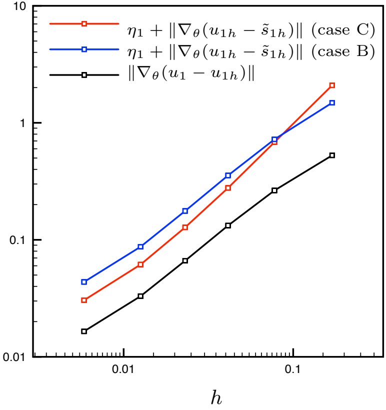
N	h	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda,\mathrm{eff}}^{\mathrm{lb}}$	$I_{\lambda,\mathrm{eff}}^{\mathrm{ub}}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Structured meshes

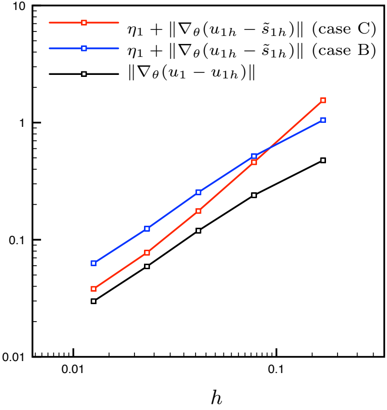
N	h	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I_{\lambda,\mathrm{eff}}^{\mathrm{lb}}$	$I_{\lambda,\mathrm{eff}}^{\mathrm{ub}}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Unstructured meshes

Nonconforming finite elements & DG's



Nonconforming finite elements



Discontinuous Galerkin

Nonconforming finite elements & DG's

N	h	ndof	λ_1	λ_{1h}	$\|\nabla \tilde{s}_{1h}\|^2 - \eta_1^2$	$\|\nabla \tilde{s}_{1h}\|^2$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1414	320	19.7392	19.6850	18.8966	19.8262	4.80e-02	2.68
20	0.0707	1240	19.7392	19.7257	19.6495	19.7616	5.69e-03	2.11
40	0.0354	4880	19.7392	19.7358	19.7246	19.7448	1.02e-03	1.91
80	0.0177	19360	19.7392	19.7384	19.7361	19.7406	2.29e-04	1.85
160	0.0088	77120	19.7392	19.7390	19.7385	19.7396	5.53e-05	1.83
320	0.0044	307840	19.7392	19.7392	19.7390	19.7393	1.37e-05	1.83

Nonconforming finite elements

N	h	ndof	λ_1	λ_{1h}	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I_{u,\mathrm{eff}}^{\mathrm{ub}}$
10	0.1698	732	19.7392	19.9432	17.8788	19.9501	1.10e-01	3.26
20	0.0776	2892	19.7392	19.7928	19.6264	19.7939	8.50e-03	1.91
40	0.0413	11364	19.7392	19.7526	19.7295	19.7529	1.18e-03	1.47
80	0.0230	45258	19.7392	19.7425	19.7381	19.7426	2.28e-04	1.31
160	0.0126	182070	19.7392	19.7400	19.7390	19.7401	5.35e-05	1.28

SIP discontinuous Galerkin

L-shaped domain

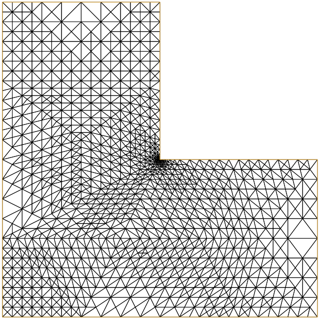
Setting

- $\Omega := (-1,1)^2 \setminus [0,1] \times [-1,0]$
- $\lambda_1 \approx 9.6397238440$

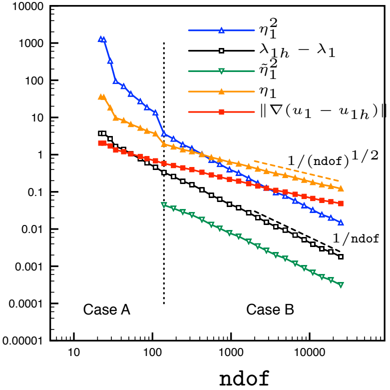
Parameters

- auxiliary bounds $\underline{\lambda_1} = \pi^2/2$ and $\underline{\lambda_2} = 5\pi^2/4$ by inclusion into the square $(-1,1)^2$

Conforming finite elements



Adaptively refined mesh



Errors and estimators

Conforming finite elements

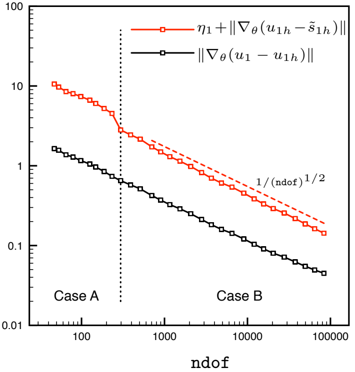
N	h	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
25	0.1263	556	9.6397	9.7637	8.3825	9.7473	7.57	11.14	1.51e-01	3.35
50	0.0634	2286	9.6397	9.6783	9.2904	9.6726	6.77	10.06	4.03e-02	3.19
100	0.0397	8691	9.6397	9.6536	9.5173	9.6515	6.61	9.84	1.40e-02	3.17
200	0.0185	34206	9.6397	9.6448	9.5946	9.6440	6.59	9.85	5.14e-03	3.20
400	0.0094	136062	9.6397	9.6416	9.6226	9.6413	6.68	9.96	1.94e-03	3.33

Unstructured meshes

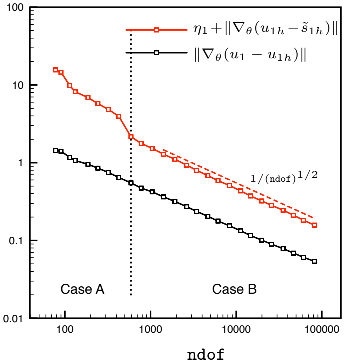
Level	ndof	λ_1	λ_{1h}	$\lambda_{1h} - \eta_1^2$	$\lambda_{1h} - \tilde{\eta}_1^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	140	9.6397	9.9700	6.3175	9.9260	7.50	11.06	4.44e-01	3.31
15	561	9.6397	9.7207	9.0035	9.7075	6.17	8.86	7.53e-02	2.98
20	2188	9.6397	9.6601	9.4887	9.6566	5.88	8.43	1.75e-02	2.88
25	8513	9.6397	9.6449	9.6019	9.6440	5.77	8.31	4.37e-03	2.75
30	24925	9.6397	9.6415	9.6266	9.6412	5.73	8.26	1.51e-03	2.51

Adaptively refined meshes

Nonconforming finite elements & DG's



Nonconforming finite elements



Discontinuous Galerkin

Nonconforming finite elements & DG's

Level	ndof	λ_1	λ_{1h}	$\|\nabla \tilde{s}_{1h}\|^2 - \eta_1^2$	$\|\nabla \tilde{s}_{1h}\|^2$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
5	98	9.6397	8.9699	-29.6187	9.9072	–	6.36
10	296	9.6397	9.4403	4.8193	9.7445	6.76e-01	4.32
15	1161	9.6397	9.5868	8.6628	9.6646	1.09e-01	3.99
20	4860	9.6397	9.6275	9.4310	9.6457	2.25e-02	3.81
25	20429	9.6397	9.6369	9.5925	9.6411	5.06e-03	3.62
30	83472	9.6397	9.6390	9.6284	9.6401	1.21e-03	3.18

Nonconforming finite elements

Level	ndof	λ_1	λ_{1h}	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2} - \eta_1^2$	$\frac{\|\nabla s_{1h}\|^2}{\|s_{1h}\|^2}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
5	186	9.6397	10.2136	-30.6026	10.3629	–	7.19
10	777	9.6397	9.8154	7.2388	9.8388	3.04e-01	3.75
15	3453	9.6397	9.6865	9.1572	9.6902	5.66e-02	3.38
20	14706	9.6397	9.6509	9.5335	9.6517	1.23e-02	3.23
25	61137	9.6397	9.6425	9.6144	9.6426	2.93e-03	3.00

SIP discontinuous Galerkin

Higher eigenvalues: unit triangle & L-shaped domain

Setting – unit triangle

- Ω: triangle with vertices $(0,0),(1,0),(0,1)$
- nonconforming finite elements, coarse mesh (2145 triang.):

$$\underline{\lambda}_1 = 49.2883,\ \underline{\lambda}_2 = 98.4296,\ \underline{\lambda}_3 = 127.937,\ \underline{\lambda}_4 = 166.975,\ \underline{\lambda}_5 = 196.439$$

- convex domain: $C_{\mathrm{S}} = 1$, $\delta = 1$, $C_{\mathrm{I}} \approx 1/\sqrt{8}$
- case C

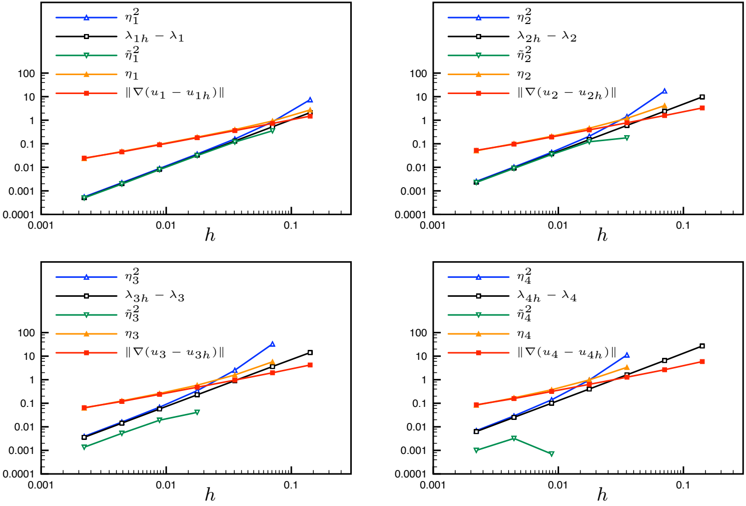
Setting – L shaped domain

- $\Omega := (-1,1)^2 \setminus [0,1] \times [-1,0]$
- nonconforming finite elements, coarse mesh (3201 triang.):

$$\underline{\lambda}_1 = 9.60692,\ \underline{\lambda}_2 = 15.1695,\ \underline{\lambda}_3 = 19.6932,\ \underline{\lambda}_4 = 29.4166,\ \underline{\lambda}_5 = 31.7363$$

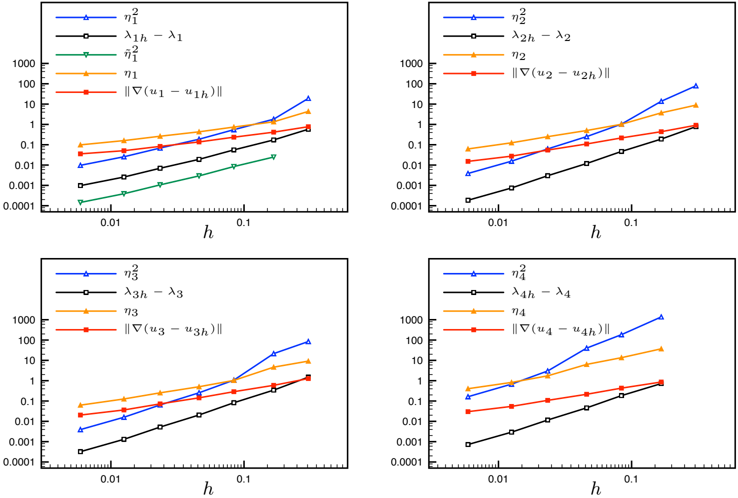
- case A/B

Unit triangle & conforming finite elements



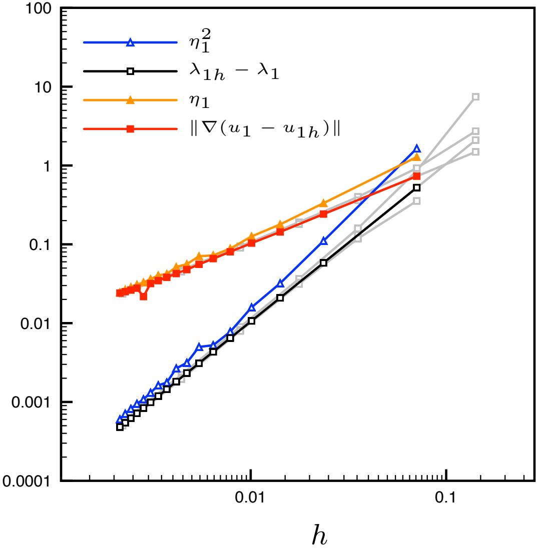
First four eigenvalues

L-shaped domain & conforming finite elements



First four eigenvalues

Unit triangle & conforming finite elements



Inexact solver

Outline

Conclusions and future directions

Conclusions

- guaranteed upper and lower bounds for the i-th eigenvalue
- guaranteed and polynomial-degree robust bounds for the associated eigenvector
- general framework

Ongoing work

- extension to nonlinear eigenvalue problems

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Thank you for your attention!