Potential and flux reconstructions for *hp*-optimal a priori and a posteriori error estimates

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in collaboration with Théophile Chaumont-Frelet, Leszek Demkowicz, Alexandre Ern, Thirupathi Gudi, and Iain Smears

Inria Paris & Charles University Prague

Zurich, November 04, 2026





Outline

1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

Numerical approximation of PDEs

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- $H^1(\Omega)$: scalar-valued functions in $L^2(\Omega)$ with weak gradient ∇ in $\boldsymbol{L}^2(\Omega)$
- $\boldsymbol{H}(\mathrm{curl}, \Omega)$: vector-valued functions in $\boldsymbol{L}^2(\Omega)$ with weak curl $\nabla\times$ in $\boldsymbol{L}^2(\Omega)$
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- $\mathcal{T}_h$ a simplicial mesh of Ω with characteristic mesh size $h := \max_{K\in\mathcal{T}_h} h_K$

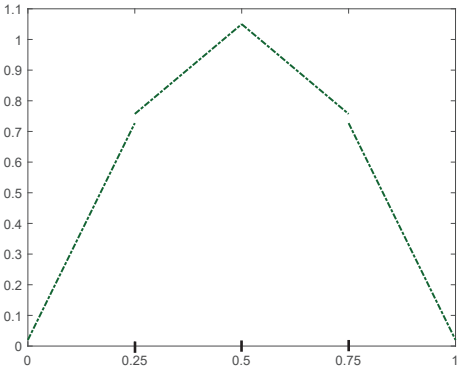
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- $\mathcal{P}_p(\mathcal{T}_h)$: piecewise polynomials of total degree $p \geq 0$
- $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)$: piecewise Nédélec polynomials of order $p \geq 0$
- $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)$: piecewise Raviart–Thomas polynomials of order $p \geq 0$

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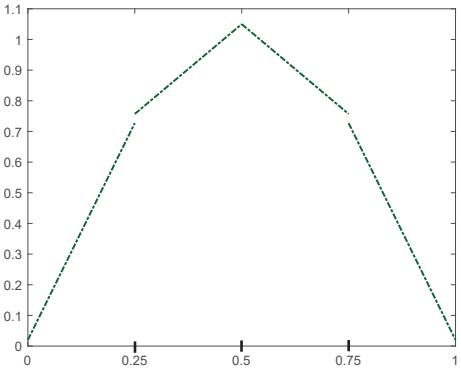
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Potential

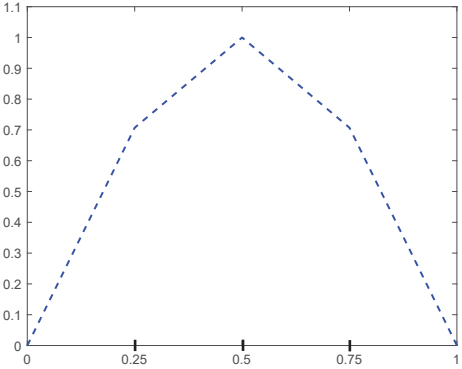


Potential ξ_h

Potential reconstruction

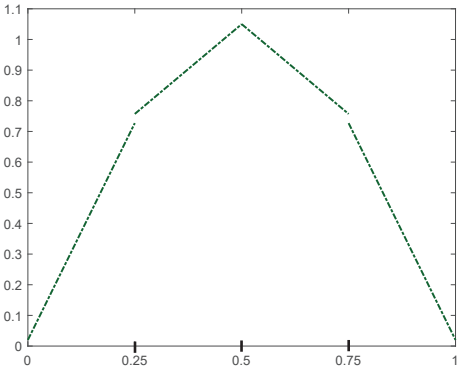


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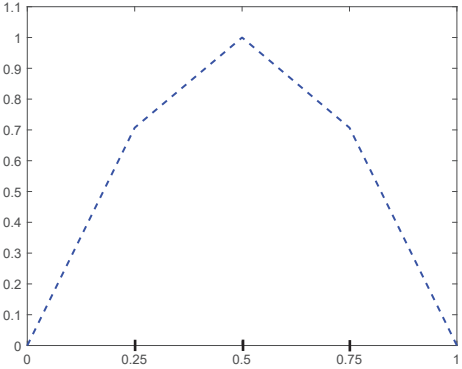


Potential reconstruction s_h

Potential reconstruction

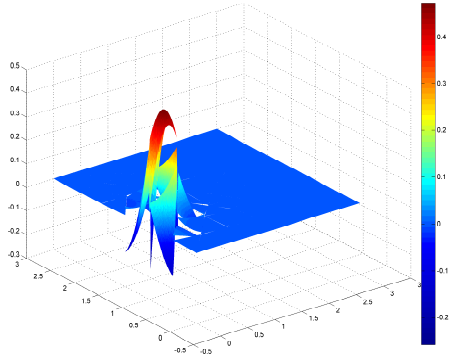


Potential $\xi_h \in \mathcal{P}_1(\mathcal{T}_h)$

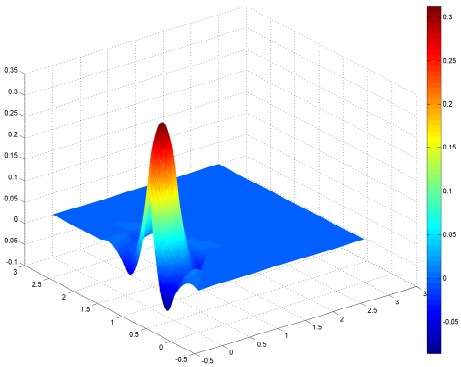


Potential reconstruction s_h
$\in \mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Potential reconstruction

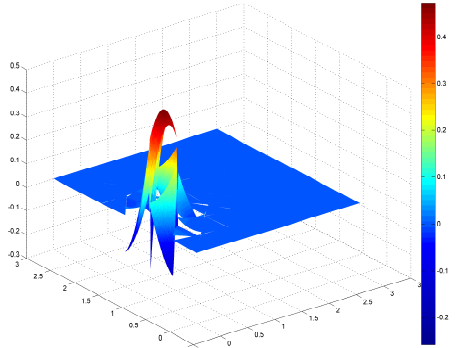


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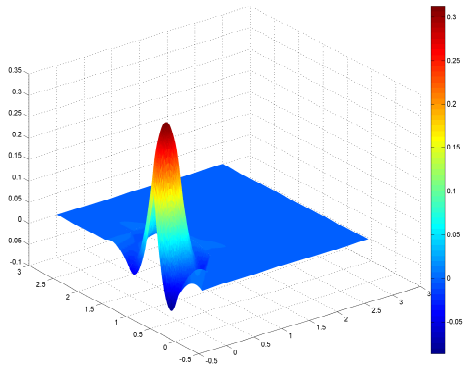


Potential reconstruction s_h

Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H^1(\Omega)$$

Approximation of the potential reconstruction

Theorem (Global approximation; $p' = p + 1$, Ern & Vohralík (2015, 2020))

Up to a jump term, s_h is closer to ξ_h than any $u \in H^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

- $\lesssim$: up to a generic constant that only depends on the shape-regularity of the mesh $\mathcal{T}_h$

s_h **so good that no** $u \in H^1(\Omega)$ **can do better**

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- $\lesssim_p$: up to a generic constant that only depends on the shape-regularity of the mesh $\mathcal{T}_h$ and the polynomial degree p

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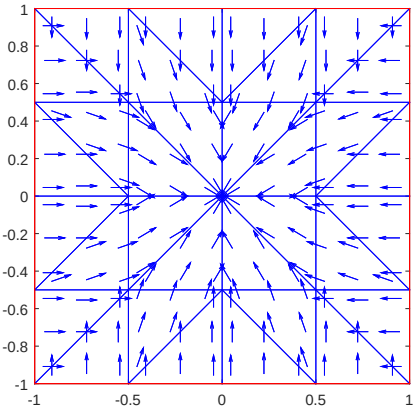
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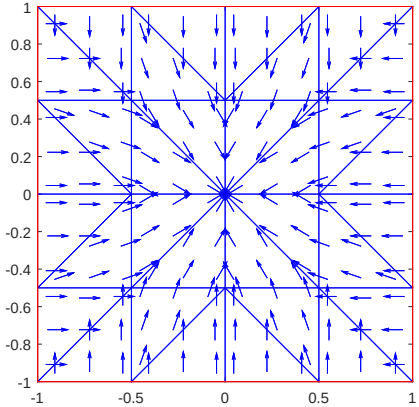
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flux

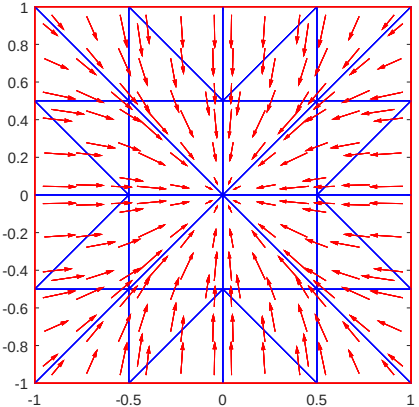


Flux ξ_h

flux reconstruction

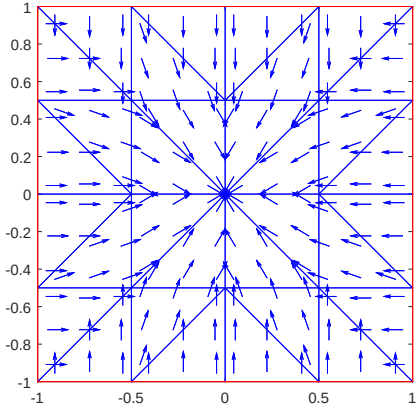


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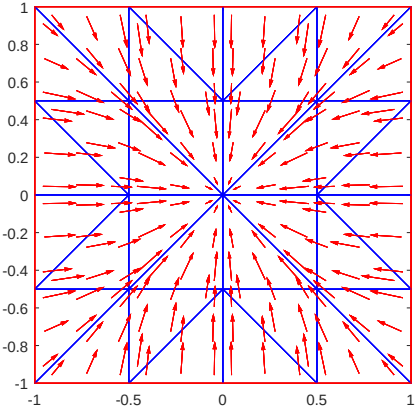


flux reconstruction σ_h

flux reconstruction



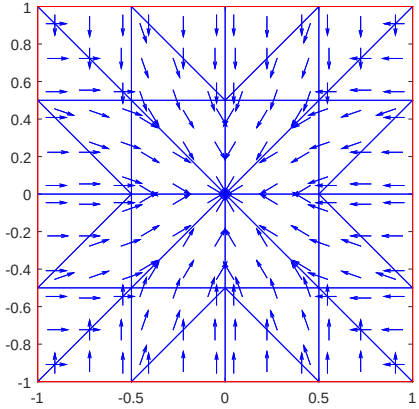
Flux $\boldsymbol{\xi}_h$



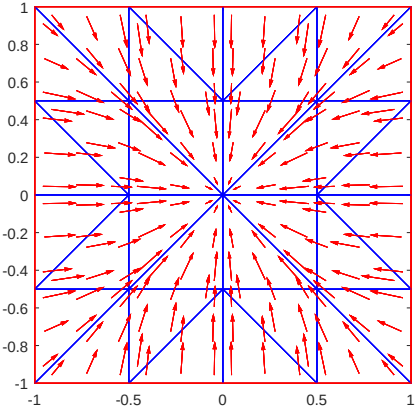
flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h)} \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div}, \Omega)$$

flux reconstruction



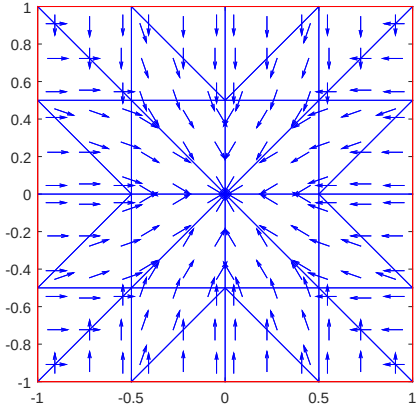
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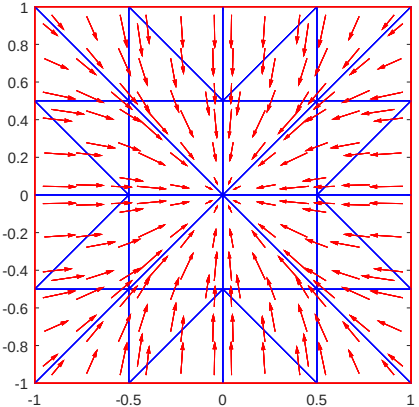
flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})\omega_{\boldsymbol{a}}+(\boldsymbol{\xi}_h,\nabla\psi_{\boldsymbol{a}})\omega_{\boldsymbol{a}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega)$$

Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

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Approximation of the flux reconstruction

Theorem (Global approximation; $p' = p+1$, Braess, Pillwein, Schöberl (2009; 2D), Ern & Vohralík (2020; 3D)**)**

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ such that $\nabla\cdot\boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_h^p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$ can do better

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Optimal a posteriori error estimates

Theorem (Guaranteed upper bound, efficiency, and *p*-robustness)

Numerical approximation of the div–grad, curl–curl, and grad–div PDEs, using **potential** *and* **equilibrated flux reconstructions**:

$$
\begin{aligned}
\overbrace{\|\nabla(u-u_h)\|}^{\text{unknown error}} &\le \overbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|}^{\text{computable estimator}} \lesssim \overbrace{\|\nabla(u-u_h)\|}^{\text{unknown error}} && H^1(\Omega),\\
\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\| &\le \|\nabla\times\boldsymbol{A}_h - \boldsymbol{\sigma}_h\| \lesssim \|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\| && \boldsymbol{H}(\mathrm{curl},\Omega),\\
\|\nabla\cdot(\boldsymbol{\xi}-\boldsymbol{\xi}_h)\| &= \|\nabla\cdot\boldsymbol{\xi}_h - s_h\| && \boldsymbol{H}(\mathrm{div},\Omega).
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Prager and Synge (1947), Ladevèze (1975), Babuška & Rheinboldt (1987), Verfürth (1989), Ainsworth & Oden (1993), Dari, Durán, Padra, & Vampa (1996), Destuynder & Métivet (1999), Ainsworth (2005), Vejchodský (2006), Kim (2007), Braess & Schöberl (2008), Vohralík (2008), Braess, Pillwein, Schöberl (2009), Ern & Vohralík (2013, 2015, 2020), Gedicke, Geevers, Perugia, & Schöberl (2021), Chaumont-Frelet & Vohralík (2023)

H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	1.17
$\approx h_0/2$				
$\approx h_0/4$				
$\approx h_0/8$				
h_0	2			
$\approx h_0/2$				
$\approx h_0/4$				
$\approx h_0/8$				
h_0	3			
$\approx h_0/2$				
$\approx h_0/4$				
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h_0	4			
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$\approx h_0/8$				

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$\approx h_0/4$		7.0%	6.5%	
$\approx h_0/8$		3.3%	3.1%	
$\approx h_0/2$	2	9.5×10^{-1}%	8.2×10^{-1}%	
$\approx h_0/4$	3	5.6×10^{-2}%	5.0×10^{-2}%	
$\approx h_0/8$	4	3.9×10^{-3}%	3.5×10^{-3}%	

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h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	1.17
$\approx h_0/2$		$1.4 \times 10^{1}\%$	$1.3 \times 10^{1}\%$	1.09
$\approx h_0/4$		7.0%	6.6%	1.06
$\approx h_0/8$		3.5%	3.1%	1.03
$\approx h_0/2$	2	$9.5 \times 10^{-1}\%$	$9.2 \times 10^{-1}\%$	1.03
$\approx h_0/4$	3	$5.8 \times 10^{-2}\%$	$5.9 \times 10^{-2}\%$	0.98
$\approx h_0/8$	4	$5.9 \times 10^{-3}\%$	$5.8 \times 10^{-3}\%$	1.02

H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	**1.17**
$\approx h_0/2$		$1.4 \times 10^{1}\%$	$1.3 \times 10^{1}\%$	1.09
$\approx h_0/4$		7.0%	6.6%	1.06
$\approx h_0/8$		3.3%	3.1%	1.04
$\approx h_0/2$	2	$9.5 \times 10^{-1}\%$	$9.2 \times 10^{-1}\%$	1.04
$\approx h_0/4$	3	$5.6 \times 10^{-2}\%$	$5.5 \times 10^{-2}\%$	1.01
$\approx h_0/8$	4	$3.3 \times 10^{-3}\%$	$3.2 \times 10^{-3}\%$	1.01

H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	**1.17**
$\approx h_0/2$		$1.4 \times 10^{1}\%$	$1.3 \times 10^{1}\%$	**1.09**
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H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	**1.17**
$\approx h_0/2$		$1.4 \times 10^{1}\%$	$1.3 \times 10^{1}\%$	**1.09**
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H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
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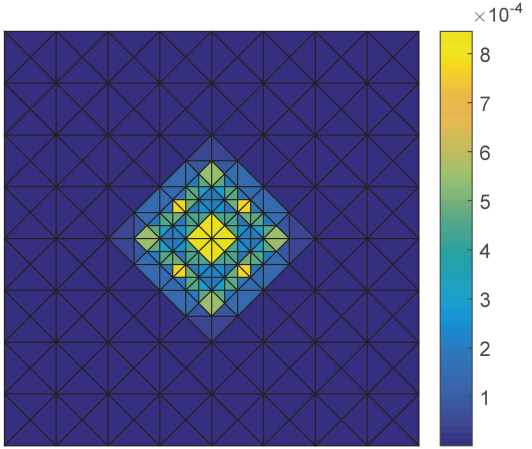
H^1 setting: How **large** is the error?

h	p	relative error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	relative error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	effectivity index $\frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
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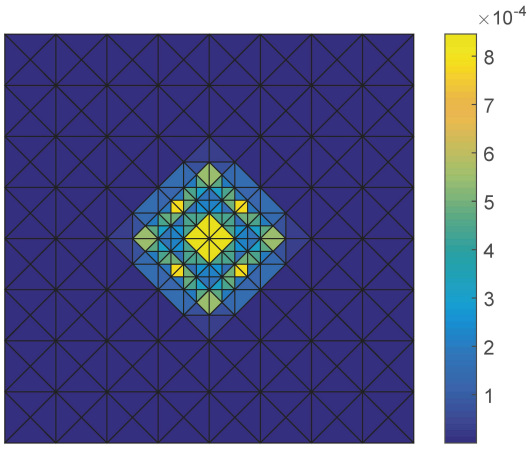
A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)



H^1 setting: where (in space) is the error **localized**?



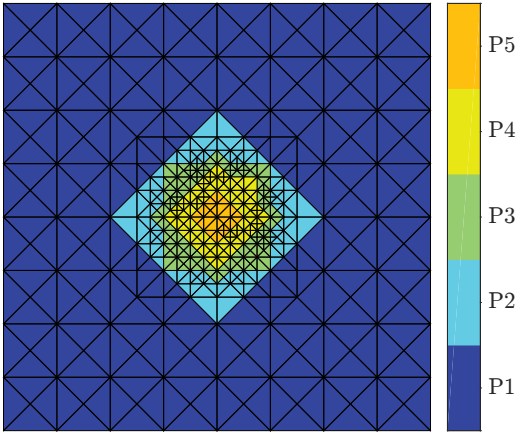
Estimated error distribution $\eta_K(u_h)$



Exact error distribution $\|\nabla(u - u_h)\|_K$

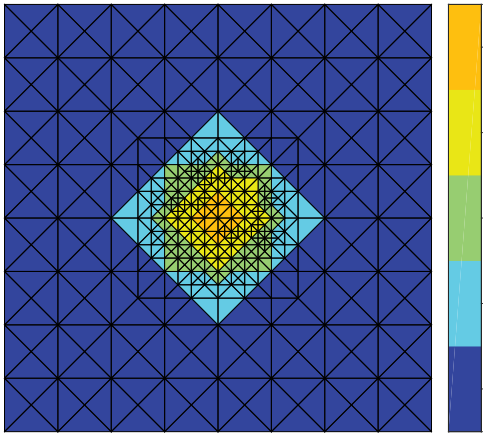
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

H^1 setting: *hp* adaptivity, (**smooth** solution)

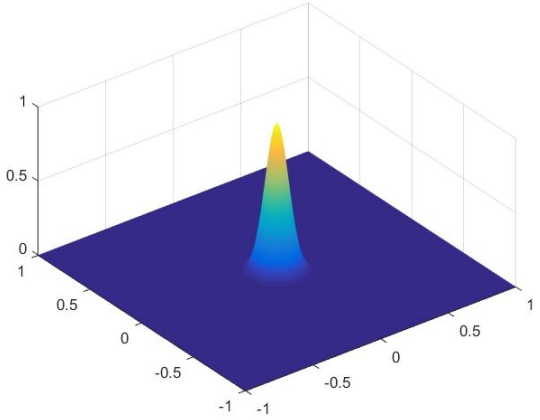


Mesh $\mathcal{T}_h$ and pol. degrees p_K

H^1 setting: *hp* adaptivity, (**smooth** solution)



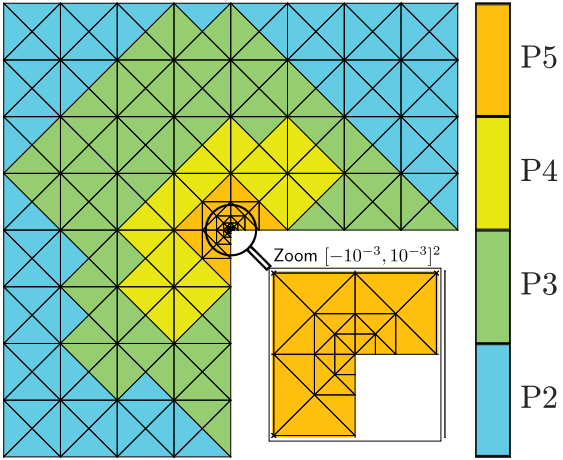
Mesh $\mathcal{T}_h$ and pol. degrees p_K



Exact solution

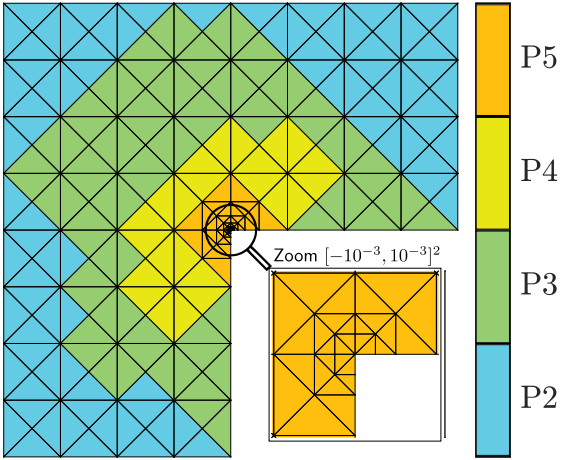
P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

H^1 setting: *hp* adaptivity, (**singular** solution)

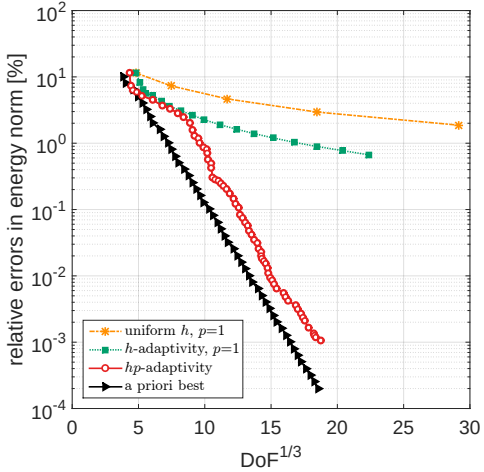


Mesh $\mathcal{T}_h$ and polynomial degrees p_K

H^1 setting: *hp* adaptivity, (**singular** solution)



Mesh $\mathcal{T}_h$ and polynomial degrees p_K

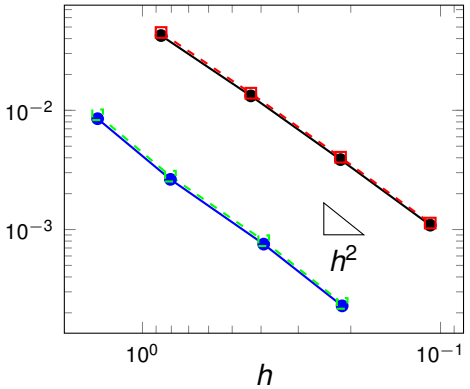


Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

$\boldsymbol{H}(\mathrm{curl})$ setting: $\boldsymbol{H}^3$ solution, uniform h-refinement

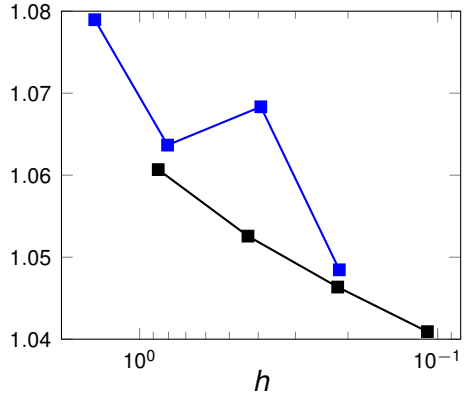
$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



error estimate, $p=1$

error estimate, $p=2$

Effectivity index $\eta/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



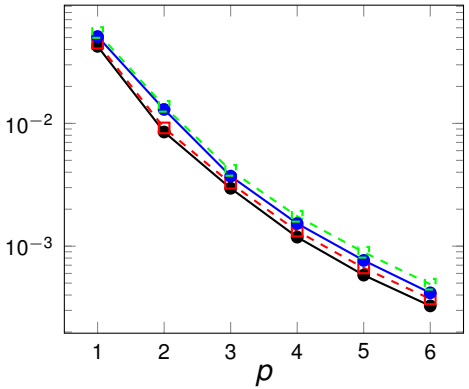
effectivity index, $p=1$

effectivity index, $p=2$

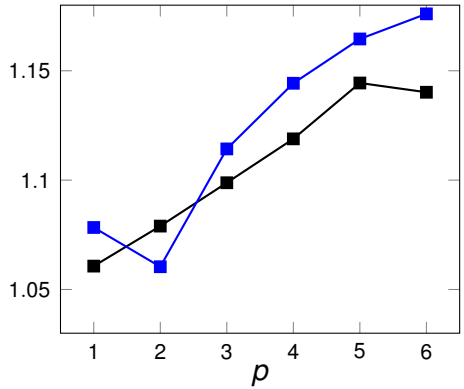
T. Chaumont-Frelet, M. Vohralík, SIAM Journal on Numerical Analysis (2023)

$\boldsymbol{H}(\mathrm{curl})$ setting: $\boldsymbol{H}^3$ solution, uniform p-refinement

$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



Effectivity index $\eta/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



error estimate, struct. mesh
error estimate, unstruct. mesh
effectivity index, struct. mesh
effectivity index, unstruct. mesh

T. Chaumont-Frelet, M. Vohralík, SIAM Journal on Numerical Analysis (2023)

Outline

1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

De Rham diagram

Commuting de Rham diagram

$$H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} L^2_*(\Omega)$$

De Rham diagram

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$$\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) \xrightarrow{\nabla} \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \xrightarrow{\nabla\times} \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \xrightarrow{\nabla\cdot} \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)$$

De Rham diagram

Commuting de Rham diagram

$$\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}$$

De Rham diagram

Commuting de Rham diagram

$$\begin{array}{ccccccc} H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

Whitney (1957), Raviart & Thomas (1977), Nédélec (1980), Bossavit (1998), Schöberl (2001, 2005), Hiptmair (2002), Christiansen & Winther (2008), Bespalov & Heuer (2011), Falk & Winther (2014), Ern & Guermond (2016, 2017), Licht (2019), Arnold & Guzmán (2021), Ern, Guzmán, Potu, & Vohralík (2026), ...

De Rham diagram

Commuting de Rham diagram

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

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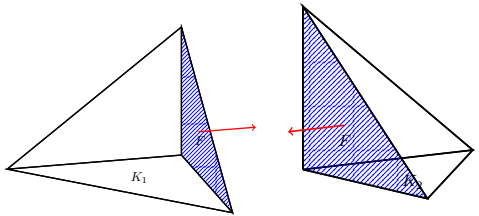
1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (minimal regularity, partial BCs)

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (minimal regularity, partial BCs)

Canonical **elementwise** interpolation excluded



$p=0$ **requests** $(\boldsymbol{v}_h|_{K_1}\cdot\boldsymbol{n}_F)|_F=(\boldsymbol{v}_h|_{K_2}\cdot\boldsymbol{n}_F)|_F=\langle\boldsymbol{v}\cdot\boldsymbol{n}_F,1\rangle_F/|F|$

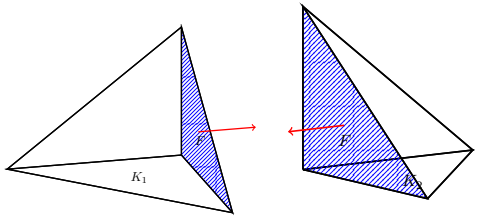
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1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (minimal regularity, partial BCs)

   Canonical **elementwise** interpolation excluded

   Face normal trace integrals $\langle \boldsymbol{v}\cdot\boldsymbol{n}_F, 1\rangle_F$ not available in $\boldsymbol{H}(\mathrm{div})$.



$p = 0$ **requests** $(\boldsymbol{v}_h|_{K_1}\cdot\boldsymbol{n}_F)|_F = (\boldsymbol{v}_h|_{K_2}\cdot\boldsymbol{n}_F)|_F = \langle \boldsymbol{v}\cdot\boldsymbol{n}_F, 1\rangle_F/|F|$

De Rham diagram: **optimal stable local commuting projectors**

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2. is defined **locally** (in neighborhood of each mesh element)

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

2. is defined **locally** (in neighborhood of each mesh element)

Energy constrained minimizer excluded

$$\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h=\Pi_h^p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v}-\boldsymbol{v}_h\|$$

is a global problem, defined over the entire Ω.

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

3 is defined simply, ideally starting from the **elementwise div-unconstrained L^2-orthogonal projection**

$$\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K \qquad \forall K\in\mathcal{T}_h$$

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

4. has optimal *hp* **approximation property**, that of $\boldsymbol{\tau}_h$:

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($\lesssim$: up to a generic constant that only depends on the shape-regularity of the mesh $\mathcal{T}_h$) (global–local equivalence)

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5. is locally p-**stable in L^2** (up to data oscillation)

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$$\|\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K \lesssim \|\boldsymbol{v}\|_{\widetilde{\omega}_K} \quad \text{when } \nabla\cdot\boldsymbol{v} \in \mathcal{P}_p(\mathcal{T}_h)$$

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Averaging-type quasi-interpolation excluded

Uses degrees of freedom (unavailable at the continuous level).

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

6. satisfies the **commuting property** expressed by the arrows

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Averaging-type quasi-interpolation excluded

Does not hit the correct gradient, curl, or divergence.

De Rham diagram: **optimal stable local commuting projectors**

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6. satisfies the **commuting property** expressed by the arrows

Averaging-type quasi-interpolation excluded

Does not hit the correct gradient, curl, or divergence.

"Inheritance" from the left of the de Rham sequence excluded

Π_h^p, the L^2-orthogonal projector onto $\mathcal{P}_p(\mathcal{T}_h)$, is imposed.

De Rham diagram: **optimal stable local commuting projectors**

$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

7. is **projector**, i.e., leaves intact piecewise polynomials,

$$\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$$

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$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

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Clément-type quasi-interpolation excluded

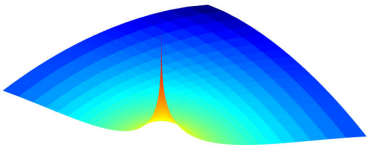
Not a projector.

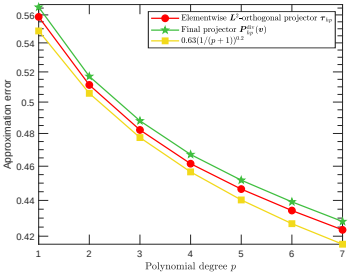
De Rham diagram: **optimal stable local commuting projectors**

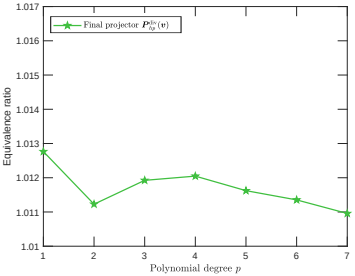
$$\begin{array}{ccc} \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\ \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\ \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega) \end{array}$$

1. is defined over the **entire** $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ (minimal regularity, partial BCs)
2. is defined **locally**
3. is defined simply, by **equilibrated flux reconstruction** from $\boldsymbol{\tau}_h|_K := \arg\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K$
4. has optimal *hp* **approximation property** (global–local equivalence): $\|\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K \lesssim \|\boldsymbol{v}-\boldsymbol{\tau}_h\|_{\widetilde{\omega}_K}$ when $\nabla\cdot\boldsymbol{v}\in\mathcal{P}_p(\mathcal{T}_h)$
5. is locally *p*-**stable in** L^2 (up to data oscillation): $\|\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|_K \lesssim \|\boldsymbol{v}\|_{\widetilde{\omega}_K}$ when $\nabla\cdot\boldsymbol{v}\in\mathcal{P}_p(\mathcal{T}_h)$
6. satisfies the **commuting property** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

A p-stable local commuting projector $P_h^{p,\mathrm{div}}$







Singular function $w(x,y) = \ln(|\ln(\sqrt{x^2+y^2})|)$;
$\boldsymbol{v} := (\partial_y w, -\partial_x w)^t$, $\nabla\cdot\boldsymbol{v} = 0$

Approximation error by elementwise L^2-orthogonal projection $\|\boldsymbol{v} - \boldsymbol{\tau}_h\|$ (red) and by the local commuting projector $\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\|$ (green)

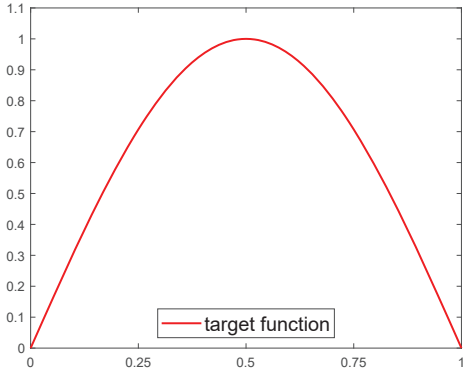
Equivalence ratio $\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\| / \|\boldsymbol{v} - \boldsymbol{\tau}_h\|$

L. Demkowicz, M. Vohralík, Math. Comp. (2026)

Outline

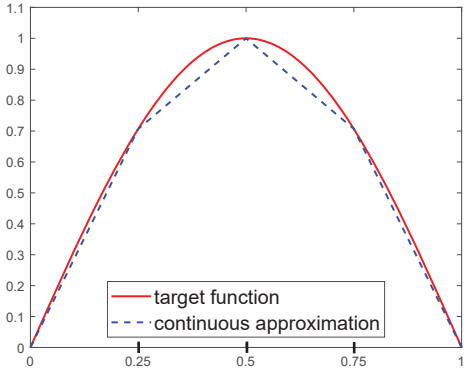
1. Introduction
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   - A priori error estimates
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Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



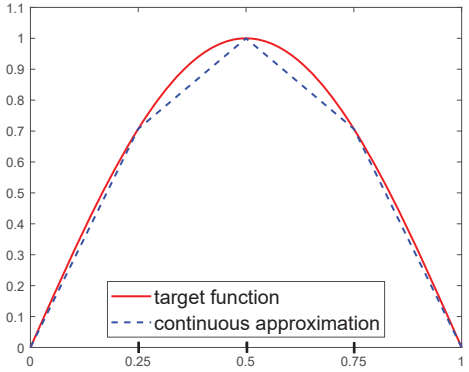
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

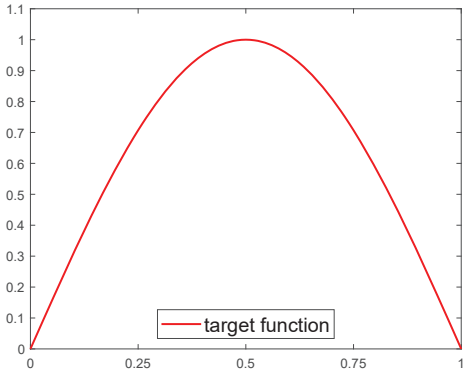


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

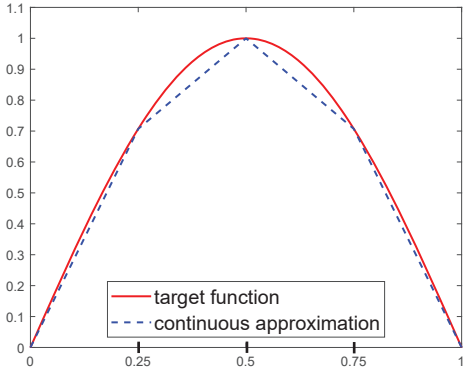


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$

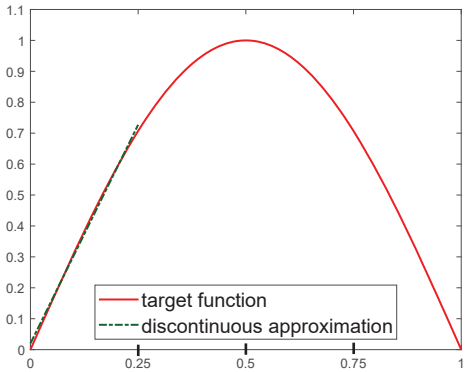


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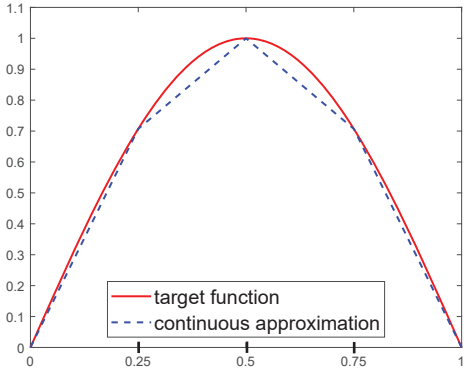


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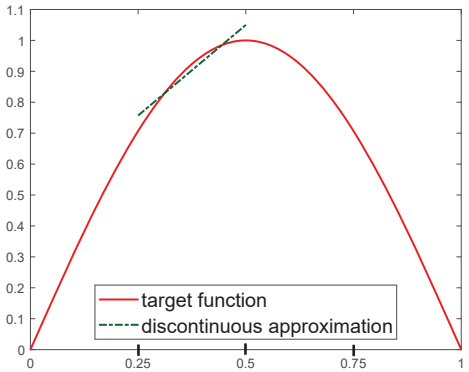


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

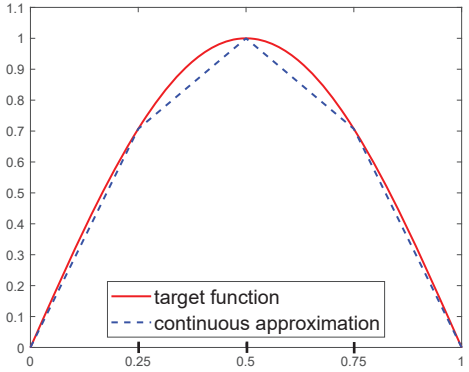


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$

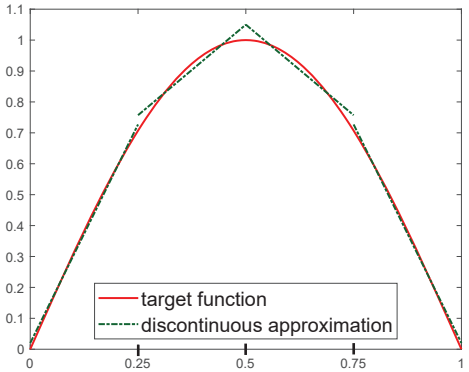


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

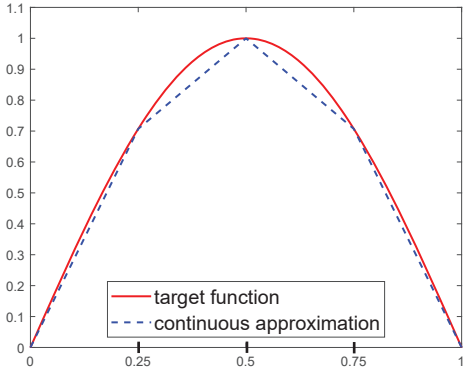


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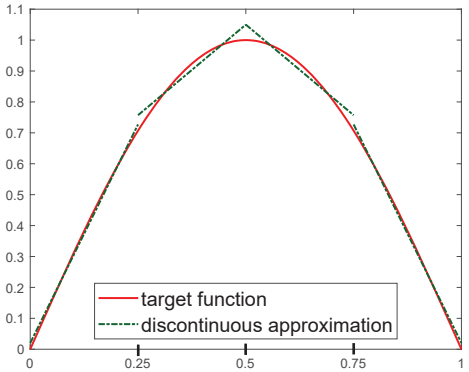


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Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

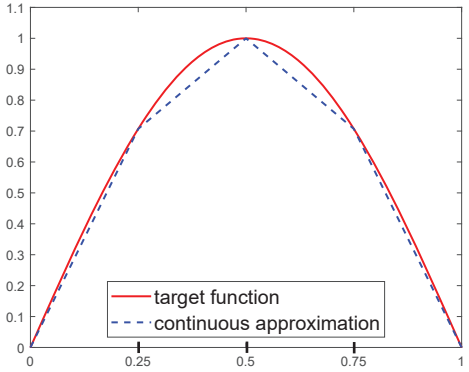


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

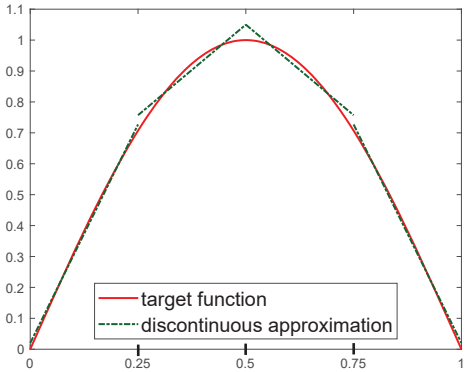


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

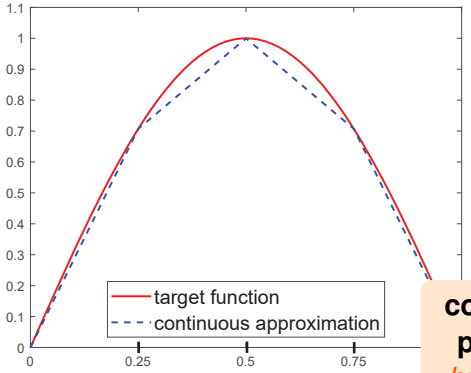


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

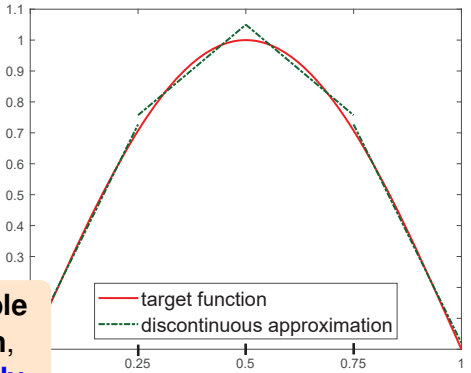


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



comparable precision, *hp*-robustly



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, **smaller** space

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_p(\mathcal{T}_h)$, **bigger** space

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeser (2016), Canuto, Nochetto, Stevenson, & Verani (2017)**)**

$$\min_{\text{CG space}} \approx_p \min_{\text{DG space}}$$

- $\approx_p$: up to a generic constant that only depends on the shape-regularity of the mesh $\mathcal{T}_h$ and the polynomial degree p

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Global–local equivalence in H_0^1, Vohralík (2026))

$$\min_{\text{CG space}} \approx \min_{\text{DG space}}$$

- $\approx$: up to a generic constant that only depends on the shape-regularity of the mesh $\mathcal{T}_h$

Equivalence of **local-** and **global-best** approximations in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & Vohralík (2022)**)**

$$\min_{\text{MFE space with constraints}} \approx_p \min_{\text{broken MFE space without constraints}}$$

Equivalence of **local-** and **global-best** approximations in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Demkowicz & Vohralík (2026))

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Optimal *hp* approximation (a priori estimates)

Theorem (*hp*-optimal approximation, minimal elementwise Sobolev regularity, Demkowicz & Vohralík (2026))

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *with*

$$\boldsymbol{v}|_L \in \boldsymbol{H}^{s_L}(L), \quad (\nabla\cdot\boldsymbol{v})|_L \in [\mathcal{P}_p(L)]^3 \qquad \forall L \in \mathcal{T}_h$$

for $s_L \geq 0$.

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for $s_L \geq 0$. *Then, for all* $K \in \mathcal{T}_h$,

$$\left\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\right\|_K^2 \leq C(\kappa_{\widetilde{\mathcal{T}}_K}, s)^2 \sum_{L\in\widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min\{p+1,s_L\}}}{(p+1)^{s_L}} \|\boldsymbol{v}\|_{\boldsymbol{H}^{s_L}(L)} \right)^2.$$

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- minimal $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$-regularity
- additional $\boldsymbol{H}^{s_L}(L)$-regularity requested **separately on each mesh element**
- optimal $h_L^{\min\{p+1,s_L\}}/(p+1)^{s_L}$ bound, **separately on each mesh element**

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- minimal $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$-regularity
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- optimal $h_L^{\min\{p+1,s_L\}}/(p+1)^{s_L}$ bound, **separately on each mesh element**

Nédélec (1980), Monk (1994), Schwab (1998), Hiptmair (2002), Demkowicz & Buffa (2005), Boffi, Brezzi, Fortin (2013), Melenk & Rojik (2020) ...

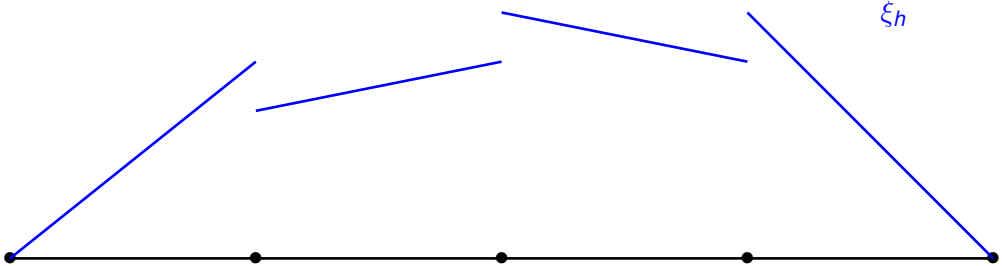
Outline

1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

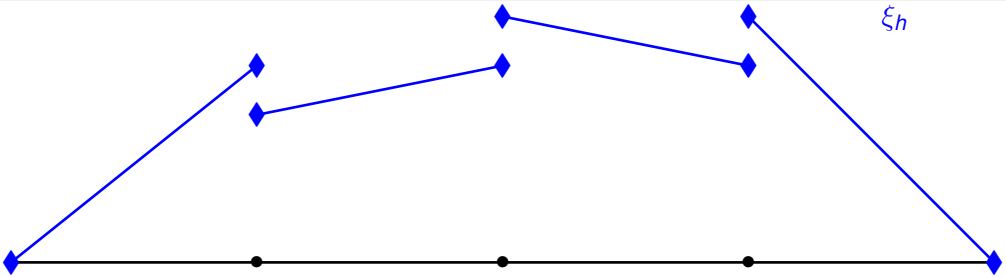
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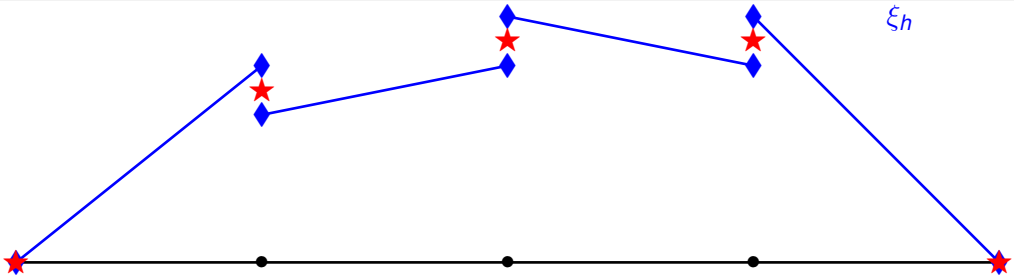
Potential reconstruction by **averaging**



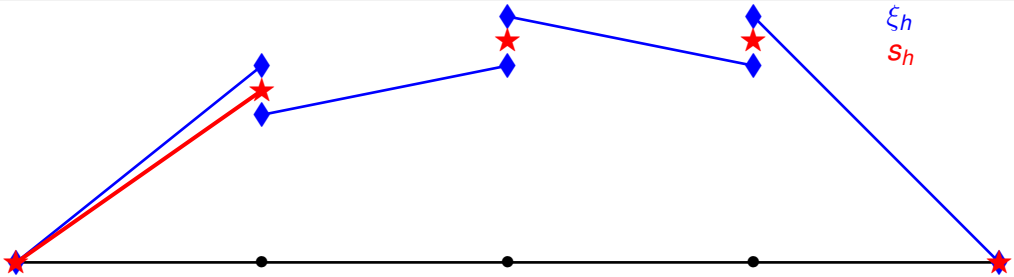
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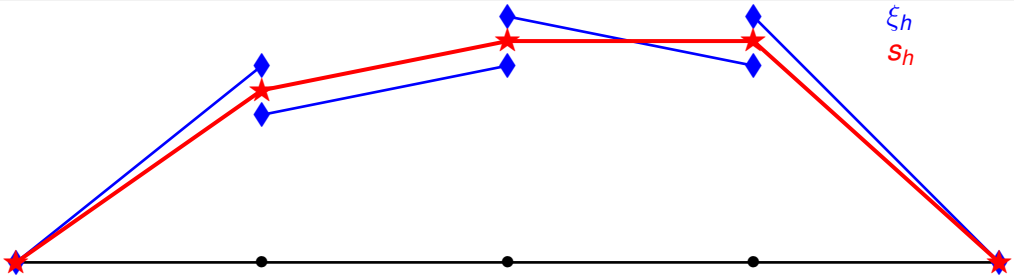
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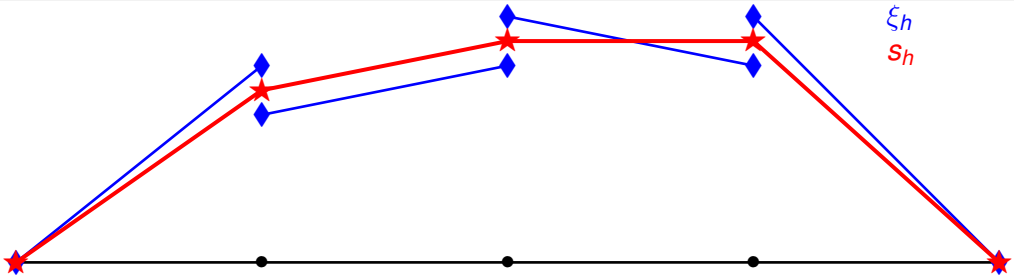


Potential reconstruction by **averaging**



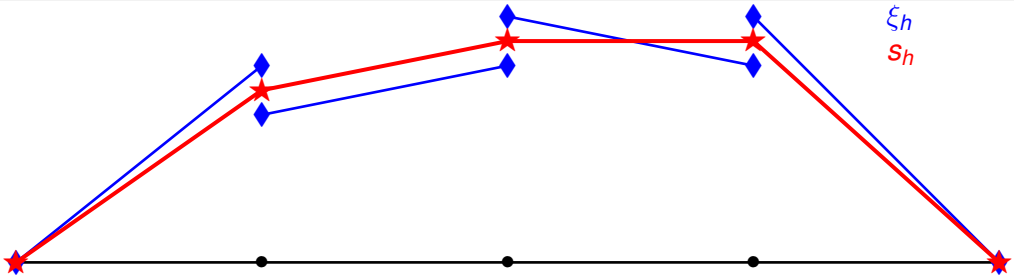
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Potential reconstruction by **averaging**



- the **easiest** potential reconstruction
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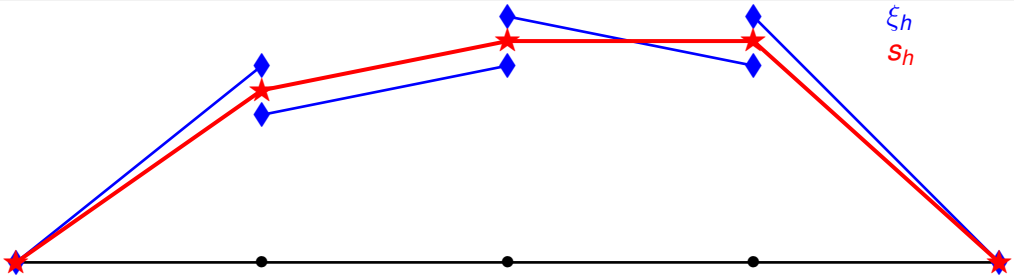
Potential reconstruction by **averaging**



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not used here

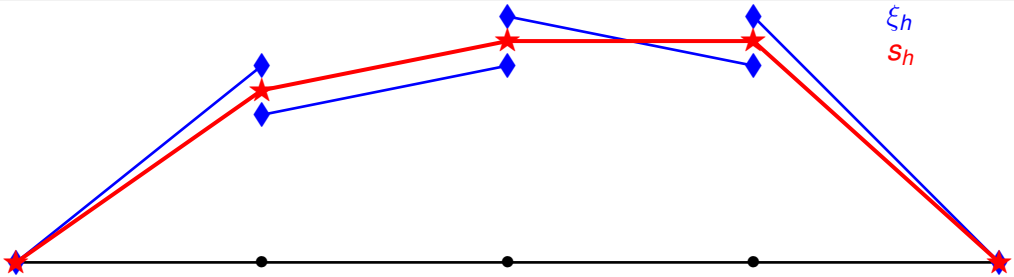
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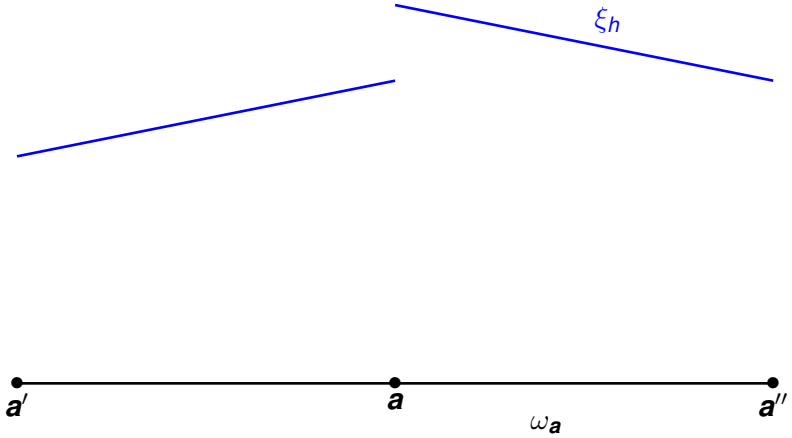
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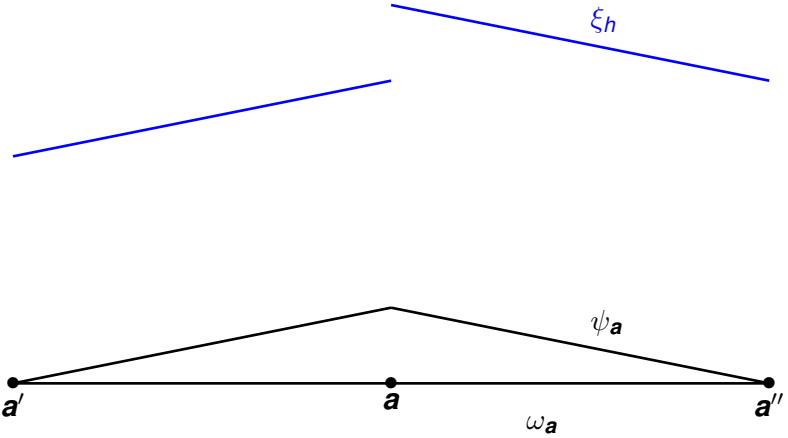
- the **easiest** potential reconstruction
- employs $p' = p$
- based on **degrees of freedom** $\Longrightarrow$ will not be p-robust
- **cumbersome/impossible** extension to $\boldsymbol{H}(\mathrm{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$ (not only make conforming but also impose the constraint)

not used here

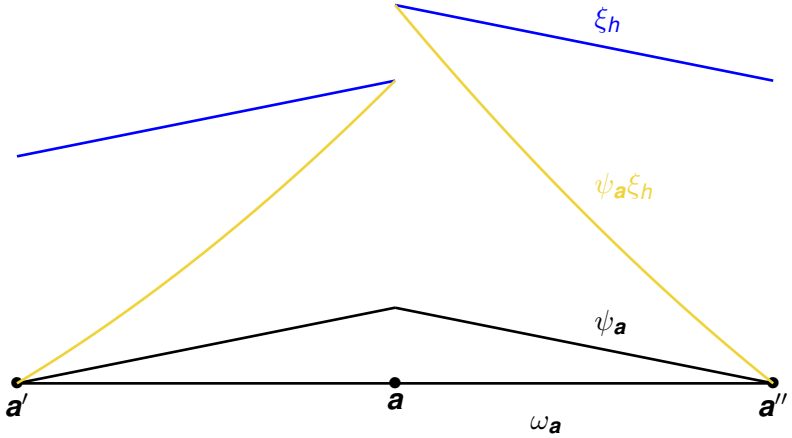
Potential reconstruction by **local energy minimization**, $p' = p + 1$



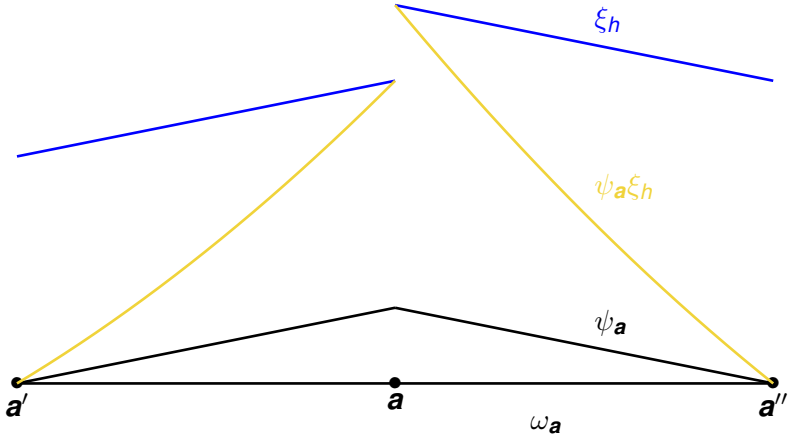
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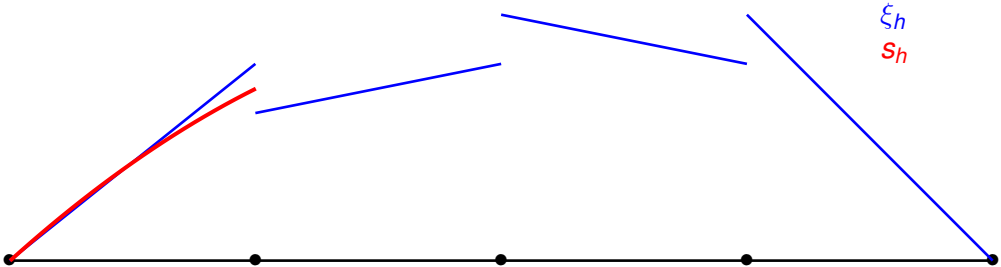


Potential reconstruction by **local energy minimization**, $p' = p + 1$



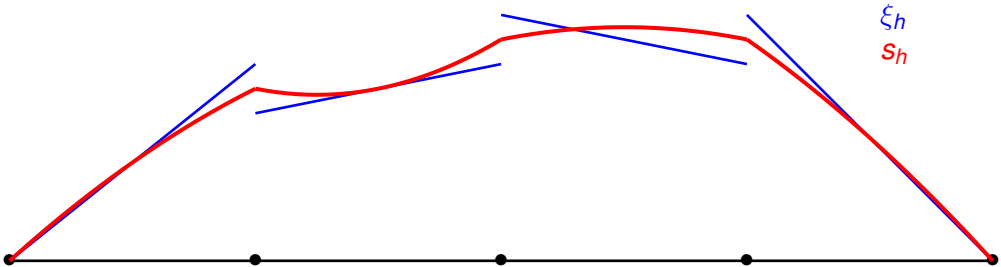
- local conforming projection of $\psi_{\boldsymbol{a}}\xi_h$: $s_h^{\boldsymbol{a}} := \arg \min_{v_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(\underbrace{\psi_{\boldsymbol{a}}\xi_h}_{\in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})} - v_h)\|_{\omega_{\boldsymbol{a}}}$

Potential reconstruction by **local energy minimization**, $p' = p + 1$



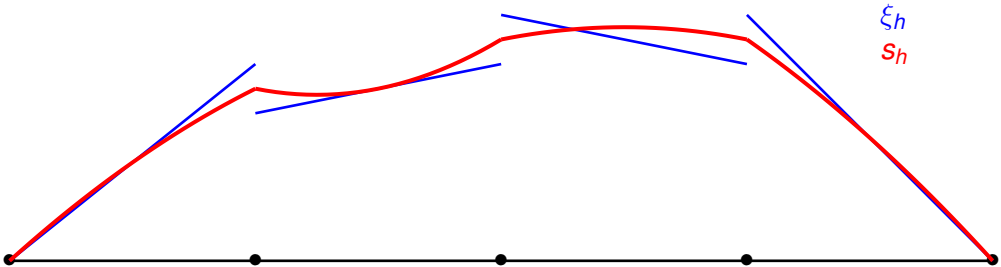
- $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_{p+1}(\mathcal{T}_h)$

Potential reconstruction by **local energy minimization**, $p' = p + 1$



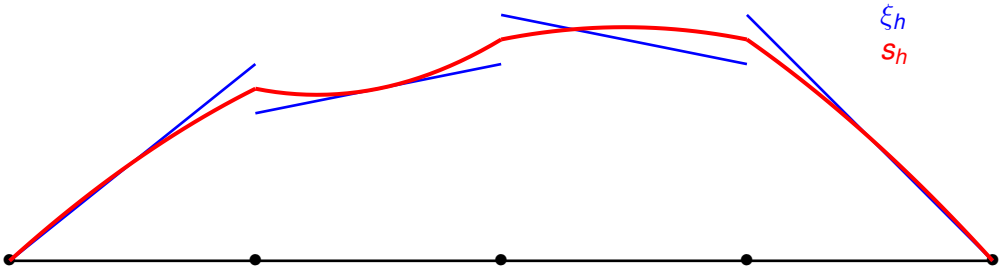
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- extends to infinite dimension (no degrees of freedom) $\Longrightarrow$ p-**robustness**

Potential reconstruction by **local energy minimization**, $p' = p + 1$



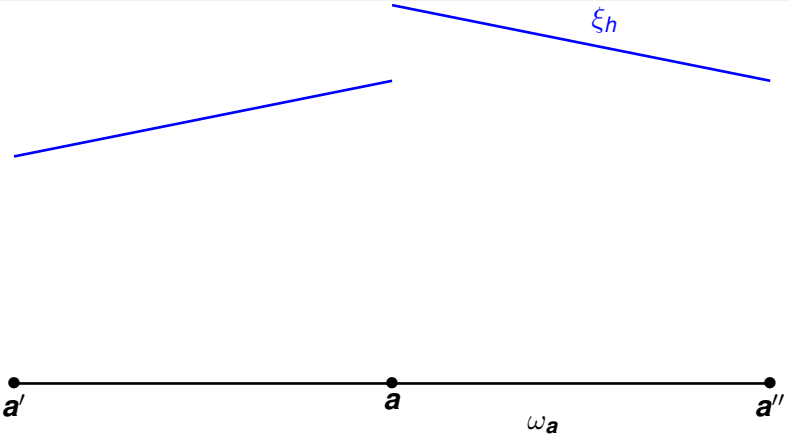
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Potential reconstruction by **local energy minimization**, $p' = p + 1$

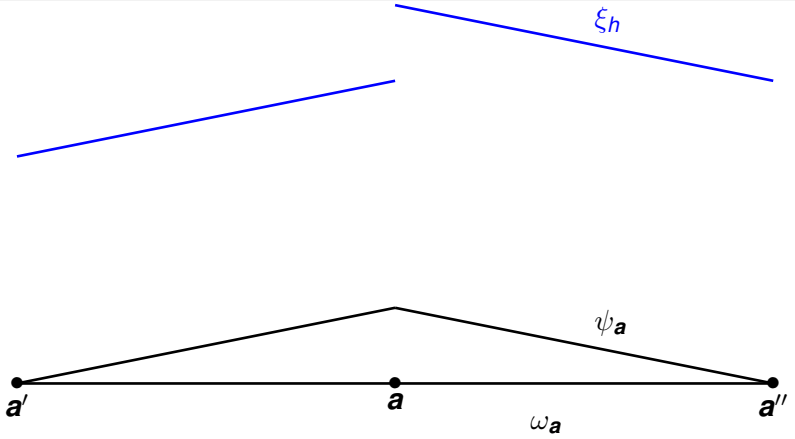


- $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_{p+1}(\mathcal{T}_h)$
- extends to infinite dimension (no degrees of freedom) $\Longrightarrow$ p-**robustness** using the **broken polynomial extension operators** as a theoretical tool
- used in a posteriori error analysis, where $p' = p + 1$ is allowed

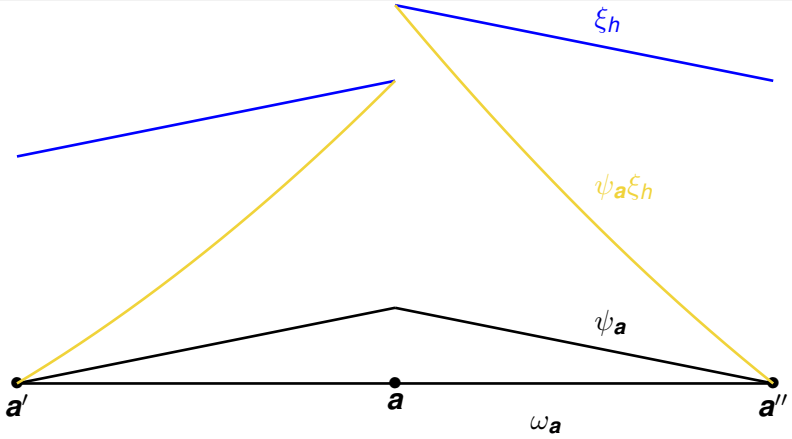
Potential reconstruction by **local energy minimization**, $p' = p$



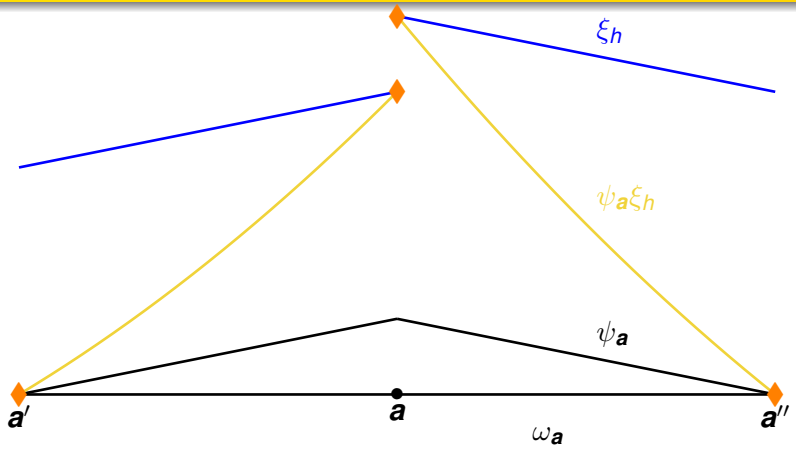
Potential reconstruction by **local energy minimization**, $p' = p$



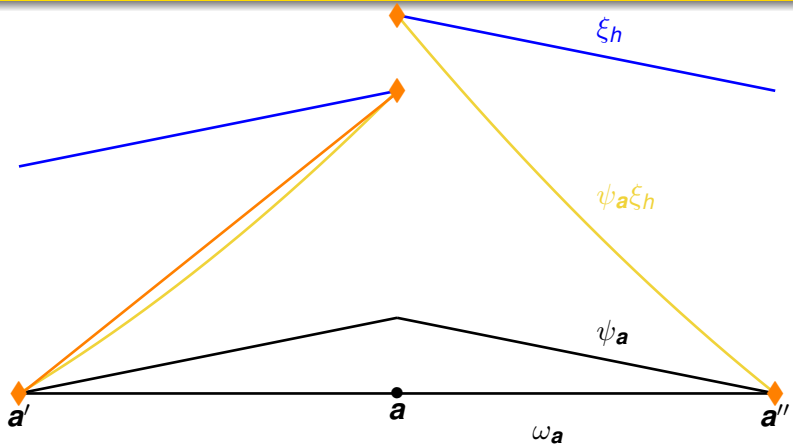
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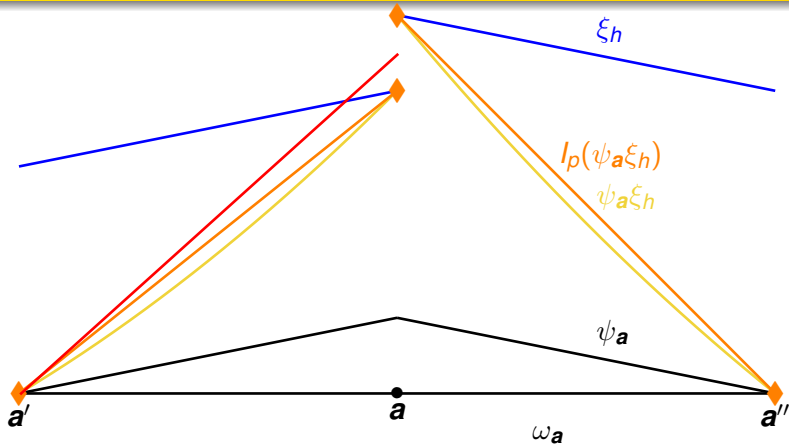


Potential reconstruction by **local energy minimization**, $p' = p$



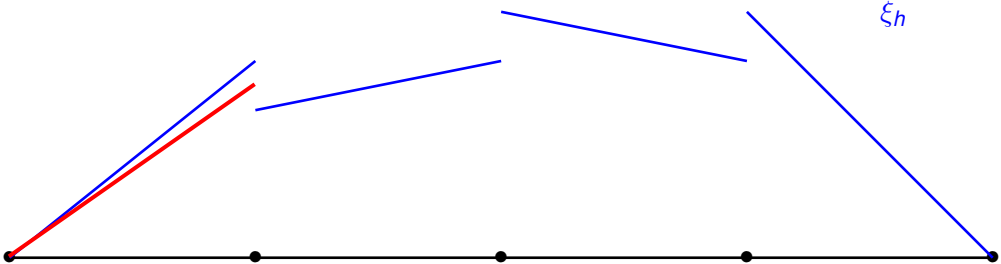
- $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$: downsampling of $\psi_{\boldsymbol{a}}\xi_h$ by the Lagrange elementwise projector

Potential reconstruction by **local energy minimization**, $p' = p$



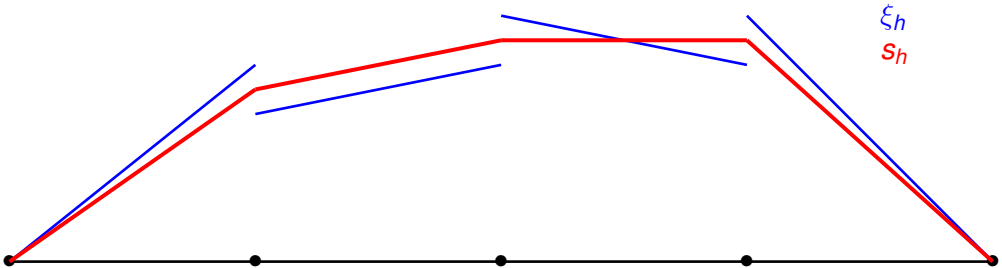
- $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$: downsampling of $\psi_{\boldsymbol{a}}\xi_h$ by the Lagrange elementwise projector
- local conforming projection of $I_p(\psi_{\boldsymbol{a}}\xi_h)$: $s_h^{\boldsymbol{a}} := \arg \min_{v_h \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \| \nabla_h (\underbrace{I_p(\psi_{\boldsymbol{a}}\xi_h)}_{\in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} - v_h) \|_{\omega_{\boldsymbol{a}}}$

Potential reconstruction by **local energy minimization**, $p' = p$



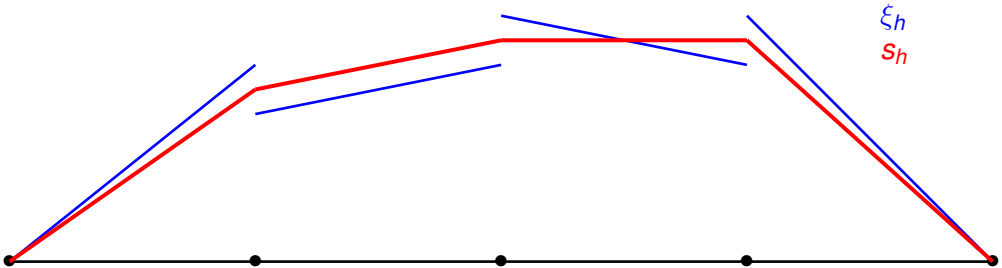
- $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_p(\mathcal{T}_h)$

Potential reconstruction by **local energy minimization**, $p' = p$



- $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_p(\mathcal{T}_h)$
- based on **degrees of freedom** $\Longrightarrow$ will not be p-robust
- used in a priori error analysis, where $p' = p$ is needed
- extends to $\boldsymbol{H}(\mathrm{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$, imposing the constraint

Potential reconstruction by **local energy minimization**, $p' = p$



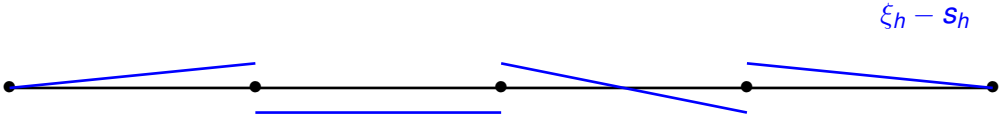
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consolidated, available

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$

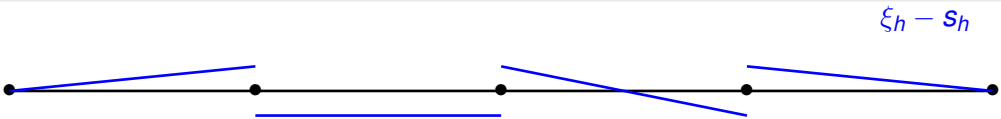
- downsample $\psi_{\boldsymbol{a}}\xi_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ to $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$
- employ $I_p(\psi_{\boldsymbol{a}}\xi_h)$ in local energy minimization for $s_h^{\boldsymbol{a}}$, $s_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_p(\mathcal{T}_h)$

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$



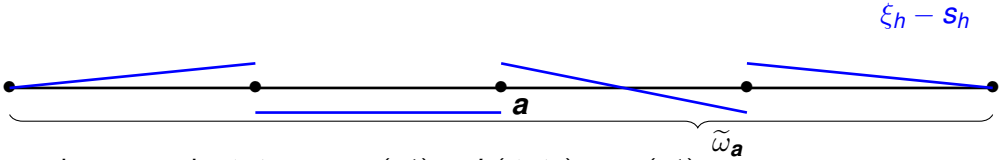
- downsample $\psi_{\boldsymbol{a}}\xi_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ to $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$
- employ $I_p(\psi_{\boldsymbol{a}}\xi_h)$ in local energy minimization for $s_h^{\boldsymbol{a}}$, $s_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_p(\mathcal{T}_h)$
- consider the remainder $\xi_h - s_h$

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$



- downsample $\psi_{\boldsymbol{a}}\xi_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ to $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$
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- consider the remainder $\xi_h - s_h$
- this remainder is zero when $\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, can be made elementwise mean-value-free for $p \geq d+1$

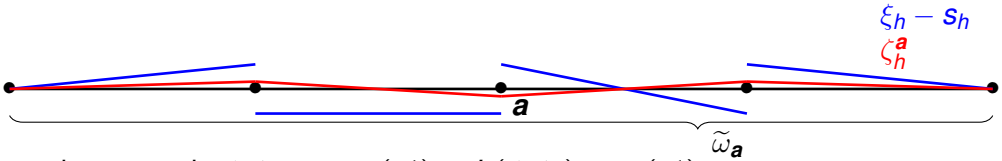
Potential reconstruction by **local energy minimization** and *p*-stable **decomposition**, $p' = p$



- downsample $\psi_{\boldsymbol{a}}\xi_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ to $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$
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- this remainder is zero when $\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, can be made elementwise mean-value-free for $p \geq d+1$

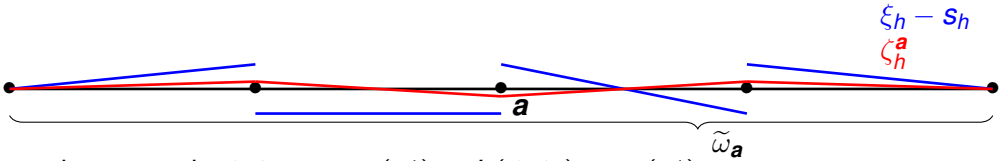
extended patches $\widetilde{\omega}_{\boldsymbol{a}}$

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$



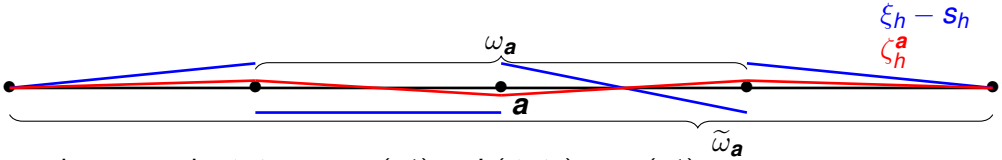
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- consider the remainder $\xi_h - s_h$
- this remainder is zero when $\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, can be made elementwise mean-value-free for $p \geq d+1$
- conforming projection $\zeta_h^{\boldsymbol{a}}$ of $\xi_h - s_h$ by local energy minimization on extended patches $\widetilde{\omega}_{\boldsymbol{a}}$ (no hat functions $\psi_{\boldsymbol{a}}$, polynomial degree p)

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$



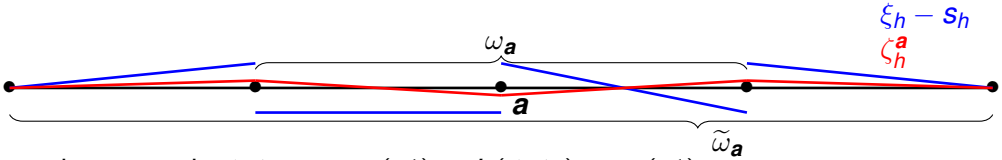
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- conforming projection $\zeta_h^{\boldsymbol{a}}$ of $\xi_h - s_h$ by local energy minimization on extended patches $\widetilde{\omega}_{\boldsymbol{a}}$ (no hat functions $\psi_{\boldsymbol{a}}$, polynomial degree p)
- *p*-**stable decomposition** of $\zeta_h^{\boldsymbol{a}}$, $\zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}'\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'}$, $\zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}'}) \cap H_0^1(\omega_{\boldsymbol{a}'})$

Potential reconstruction by **local energy minimization** and *p*-**stable decomposition**, $p' = p$



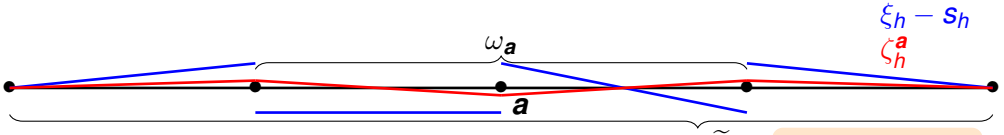
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- ***p*-stable decomposition** of $\zeta_h^{\boldsymbol{a}}$, $\zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}'\in\widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'}$, $\zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}'}) \cap H_0^1(\omega_{\boldsymbol{a}'})$
- **correction** $\zeta_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \zeta_h^{\boldsymbol{a},\boldsymbol{a}}$, $s_h \to s_h + \zeta_h$

Potential reconstruction by **local energy minimization** and p-**stable decomposition**, $p' = p$



- downsample $\psi_{\boldsymbol{a}}\xi_h \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ to $I_p(\psi_{\boldsymbol{a}}\xi_h) \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$
- employ $I_p(\psi_{\boldsymbol{a}}\xi_h)$ in local energy minimization for $s_h^{\boldsymbol{a}}$, $s_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathcal{P}_p(\mathcal{T}_h)$
- consider the remainder $\xi_h - s_h$
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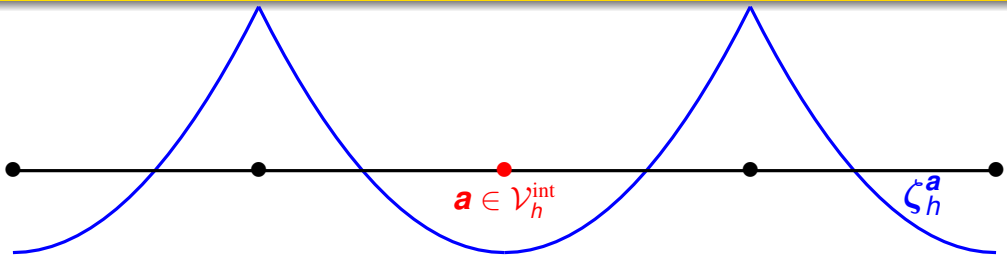
Potential reconstruction by **local energy minimization** and p-**stable decomposition**, $p' = p$



sharpest, recent

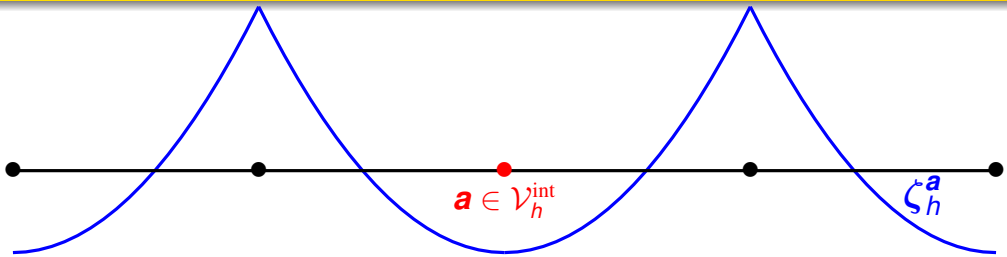
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p-stable decomposition (no global component) ~ Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



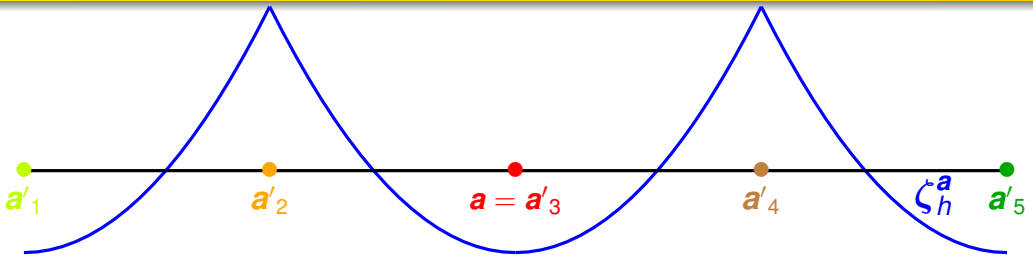
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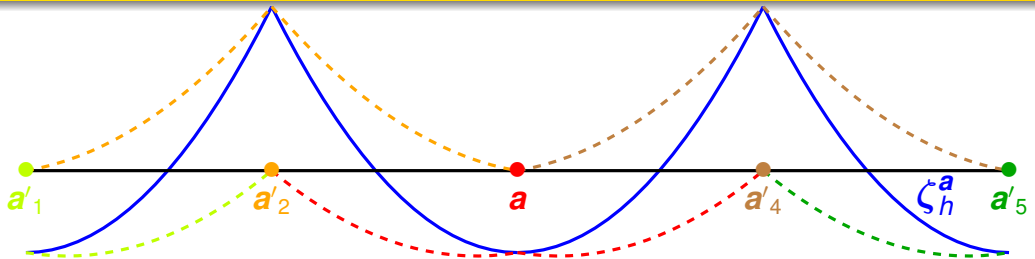
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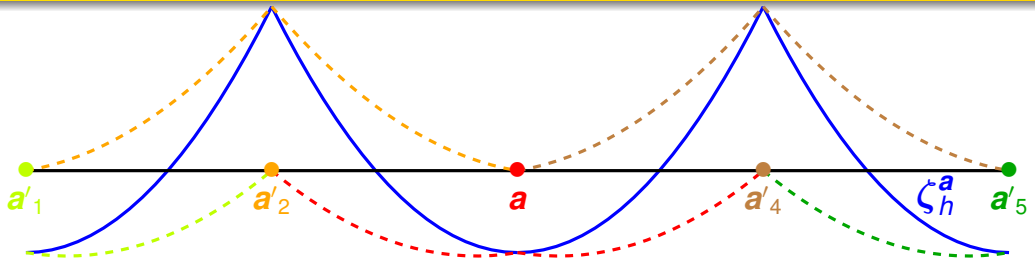
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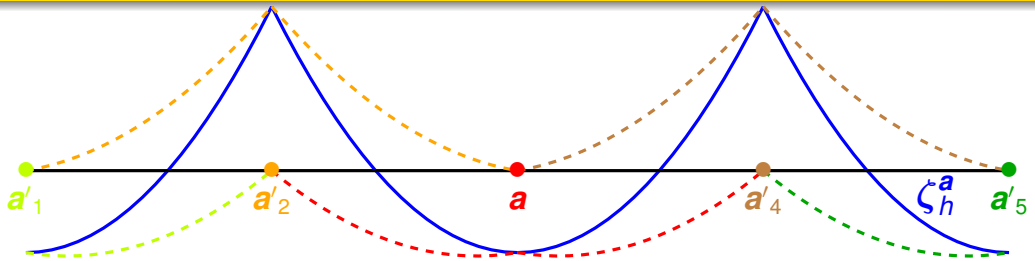
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- p-**stable** $\|\nabla \zeta_h^{\boldsymbol{a},\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla \zeta_h^{\boldsymbol{a}}\|_{\widetilde{\omega}_{\boldsymbol{a}}}$
- no global (low-order) component since $\zeta_h^{\boldsymbol{a}}$ is elementwise mean-value free

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

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Definition (Construction of s_h Ern & Vohralík (2015), $\approx$ Carstensen and Merdon (2013))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, perform **local energy minimization**

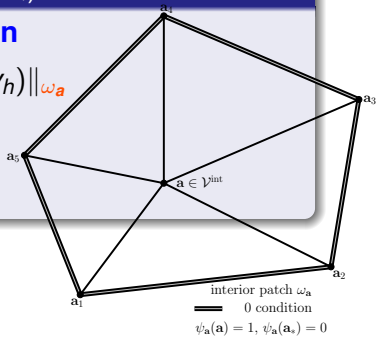
$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}}} \|\nabla_h(\psi_{\boldsymbol{a}} \xi_h - v_h)\|_{\omega_{\boldsymbol{a}}}$$

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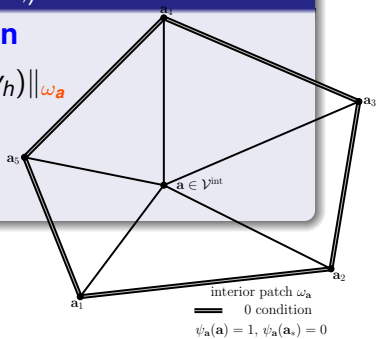


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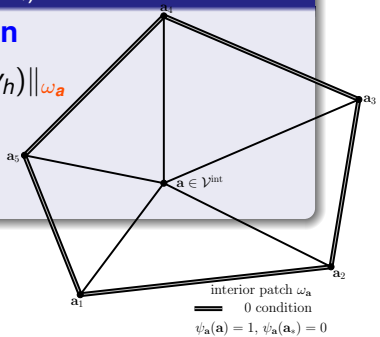
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and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}.$$



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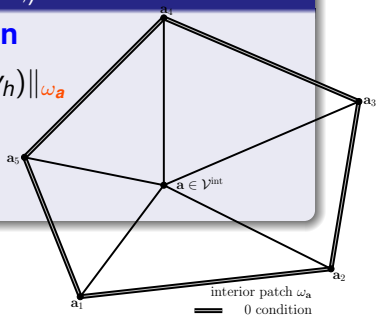
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Euler–Lagrange conditions (orthogonal projection):
Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h \quad (\psi_{\boldsymbol{a}} \xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

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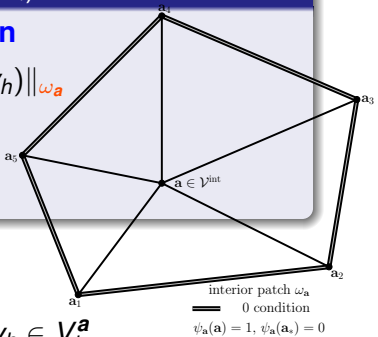
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3 choices

- $p' = p + 1$

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

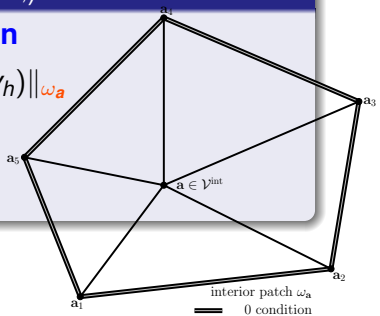
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and combine

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Euler–Lagrange conditions (orthogonal projection):
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3 choices

- $p' = p + 1$
- $p' = p$

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

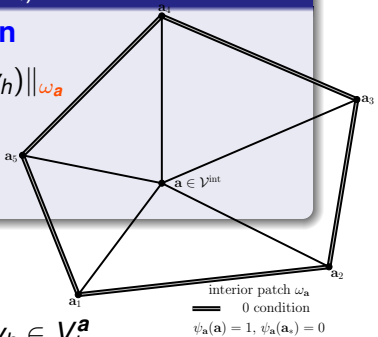
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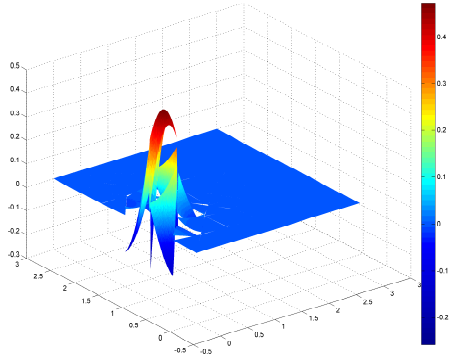
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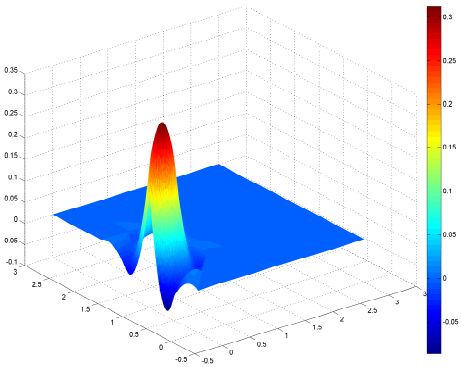
3 choices

- $p' = p + 1$
- $p' = p$
- $p' = p$ plus a correction by p-**stable decomposition** of conforming projections of the reminder $\xi_h - s_h$

Potential reconstruction



Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to \boldsymbol{s}_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H^1(\Omega)$$

Approximation of the potential reconstruction

Theorem (Local approximation Ern & Vohralík (2015, 2020), using ▸ Tools **)**

There holds

$$\min_{v_h \in \mathcal{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(I_{p'}(\psi_{\boldsymbol{a}}\xi_h) - v_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{v \in H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(I_{p'}(\psi_{\boldsymbol{a}}\xi_h) - v)\|_{\omega_{\boldsymbol{a}}}.$$

Approximation of the potential reconstruction

Corollary (Global approximation; $p' = p+1$, Ern & Vohralík (2015, 2020))

Up to a jump term, s_h is closer to ξ_h than any $u \in H^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F\in\mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

s_h so good that no $u \in H^1(\Omega)$ can do better

Approximation of the potential reconstruction

Corollary (Global approximation; $p' = p$, Karakashian & Pascal (2003), Chaumont-Frelet & Vohralík (2026))

Up to a jump term, s_h is closer to ξ_h than any $u \in H^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim_p \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

s_h **so good that no** $u \in H^1(\Omega)$ **can do better**

Approximation of the potential reconstruction

Corollary (Global approximation; $p' = p$ after a p-stable decomposition correction, Vohralík (2026))

Up to a jump term, s_h is closer to ξ_h than any $u \in H^1(\Omega)$:

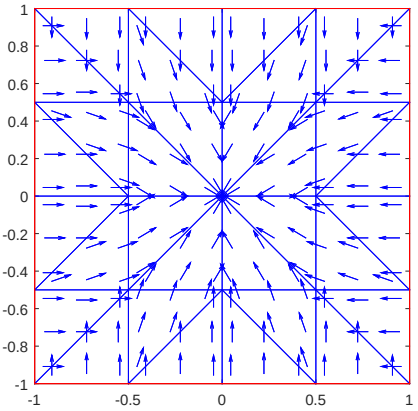
$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

s_h so good that no $u \in H^1(\Omega)$ can do better

Outline

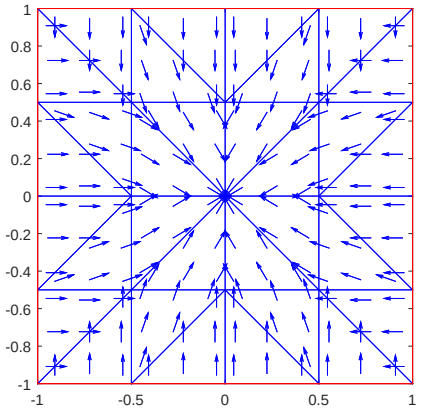
1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

flux

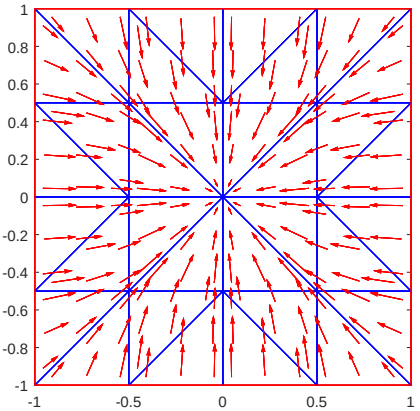


Flux ξ_h

flux reconstruction

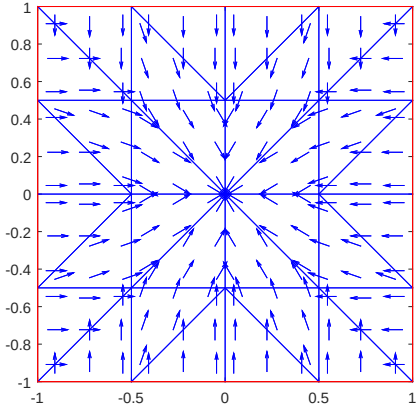


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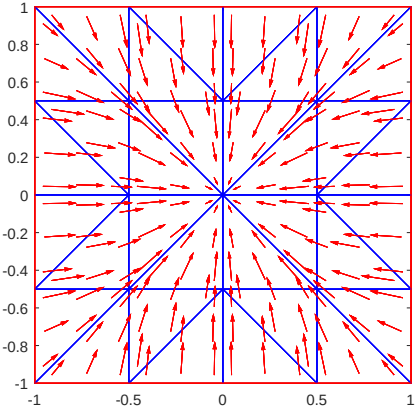


flux reconstruction σ_h

flux reconstruction



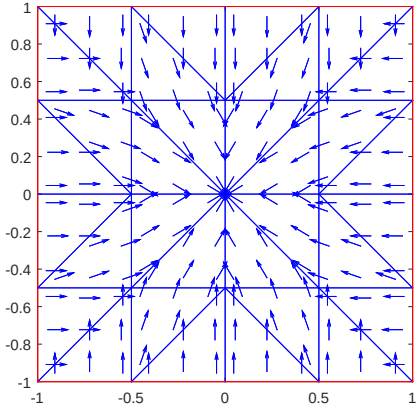
Flux $\boldsymbol{\xi}_h$



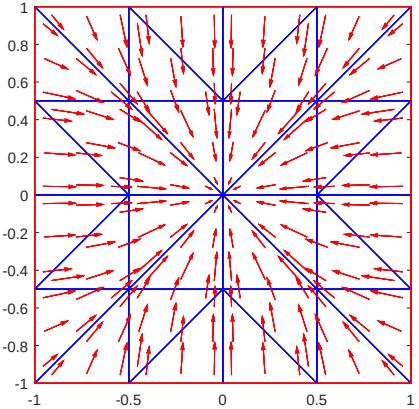
flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h)} \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div}, \Omega)$$

flux reconstruction



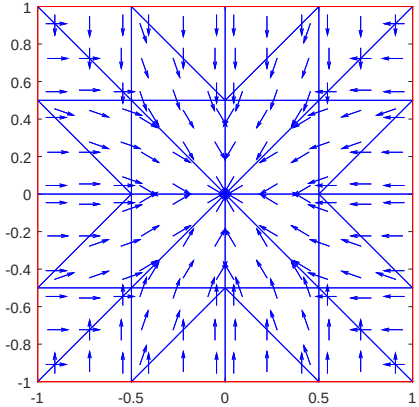
Flux $\boldsymbol{\xi}_h$



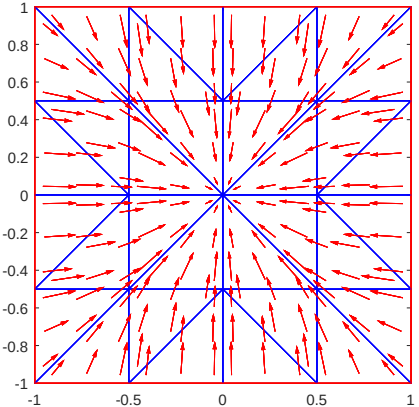
flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})\omega_{\boldsymbol{a}}+(\boldsymbol{\xi}_h,\nabla\psi_{\boldsymbol{a}})\omega_{\boldsymbol{a}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega)$$

Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\xi}_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_h^{p'} f$$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

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Assumption (Orthogonality wrt hat functions)

There holds $(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\xi_h, \nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \qquad \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}.$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

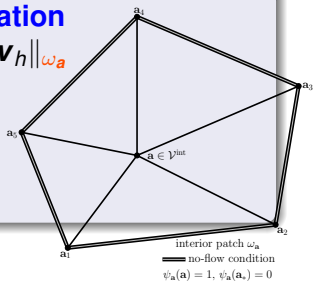
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Definition (Constr. of σ_h, Destuynder & Métivet (1999) & Braess & Schöberl (2008), Ern & Vohralík (2013))

For each $\boldsymbol{a} \in \mathcal{V}_h$, perform **local constrained energy minimization**

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}} \\ \nabla\cdot\boldsymbol{v}_h =}} \| \psi_{\boldsymbol{a}}\boldsymbol{\xi}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$



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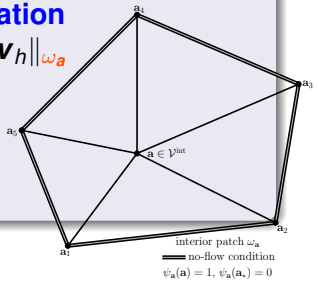
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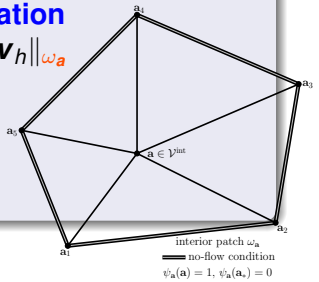
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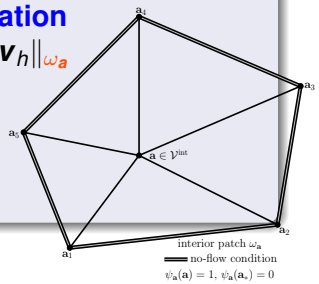
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and combine $\sigma_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \sigma_h^{\boldsymbol{a}}.$



Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

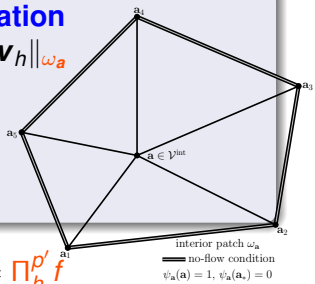
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Equilibrium

$$\nabla\cdot\boldsymbol{\sigma}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \Pi_h^{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}}) = \Pi_h^{p'} f$$

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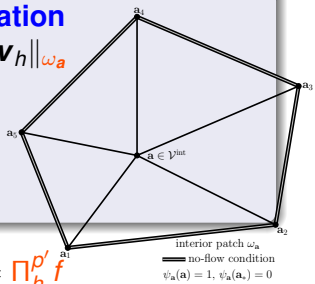
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3 choices

- $p' = p + 1$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

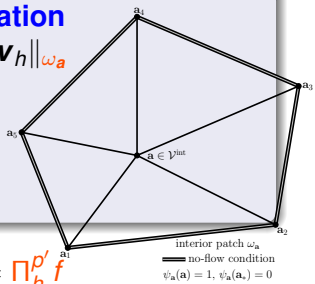
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and combine $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$



Equilibrium

$$\nabla\cdot\boldsymbol{\sigma}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \Pi_h^{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}}) = \Pi_h^{p'} f$$

3 choices

- $p' = p+1$
- $p' = p$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

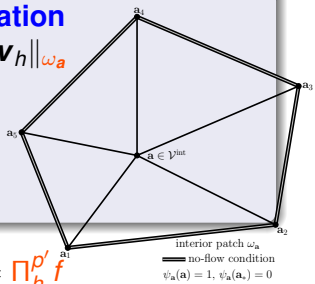
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and combine $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}.$



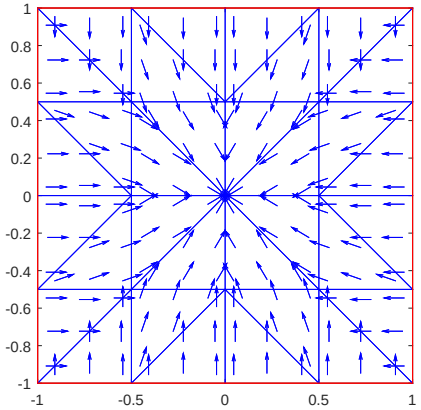
Equilibrium

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3 choices

- $p' = p+1$
- $p' = p$
- $p' = p$ plus a correction by p-**stable decomposition**

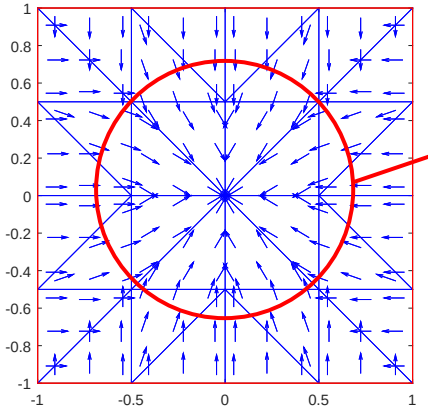
Equilibrated flux reconstruction



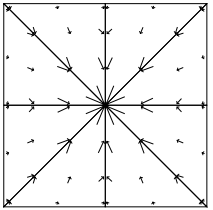
Flux $\boldsymbol{\xi}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}$$
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Equilibrated flux reconstruction



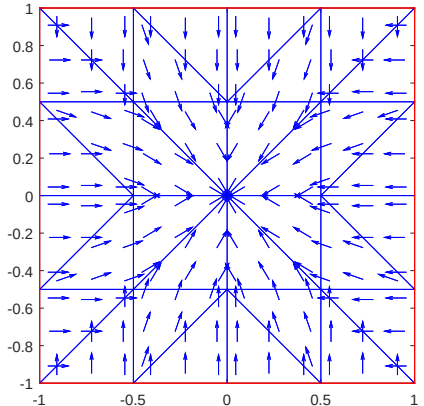
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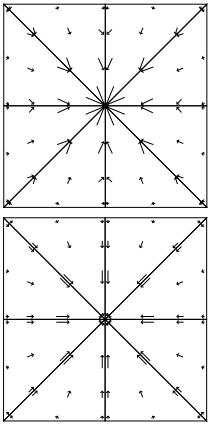
$\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}$$
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Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



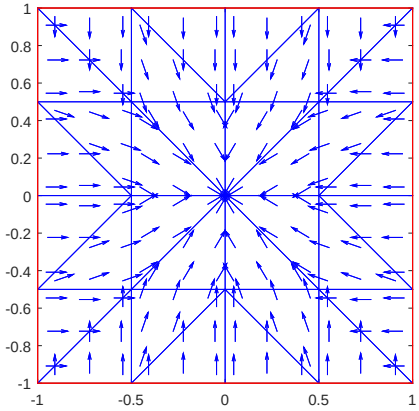
$\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h$

$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

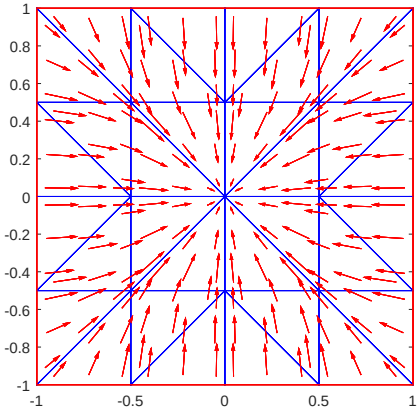
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Equilibrated flux reconstruction



Flux $\boldsymbol{\xi}_h$



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\boldsymbol{\xi}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\xi}_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \rightarrow \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_h^{p'} f$$

Approximation of the flux reconstruction

Theorem (Local approximation Braess, Pillwein, Schöberl (2009; 2D), Ern & Vohralík (2020; 3D), using ▸ Tools **)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v} = \Pi_h^{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h\cdot\nabla\psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}}\boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Approximation of the flux reconstruction

Corollary (Global approximation; $p' = p+1$, Braess, Pillwein, Schöberl (2009; 2D), Ern & Vohralík (2020; 3D)**)**

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ such that $\nabla\cdot\boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_h^p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$ can do better

Approximation of the flux reconstruction

Corollary (Global approximation; $p' = p$, Ern, Gudi, Smears, & Vohralík (2022))

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla\cdot\boldsymbol{\sigma} = f$:

$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim_p \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_h^p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$ can do better

Approximation of the flux reconstruction

Corollary (Global approximation; $p' = p$ after a p-stable decomposition correction, Demkowicz & Vohralík (2026))

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Outline

1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

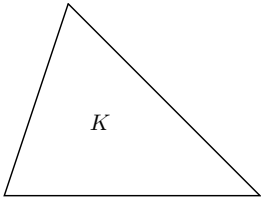
Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ be given.

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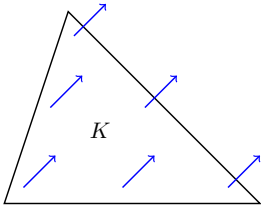
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$$\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K \qquad \forall K \in \mathcal{T}_h$$



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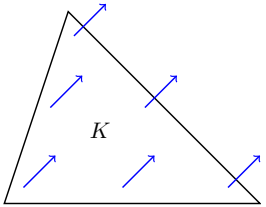
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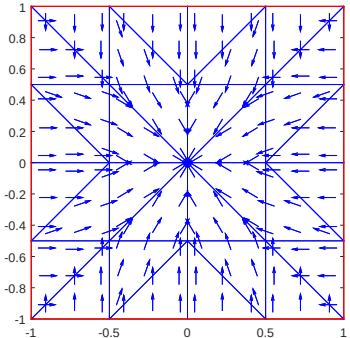
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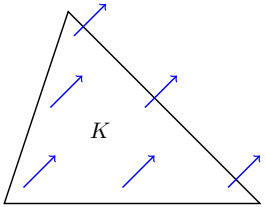
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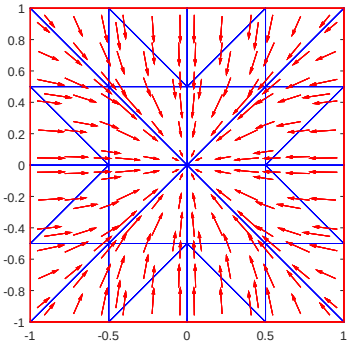
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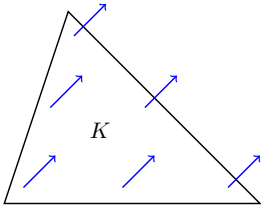


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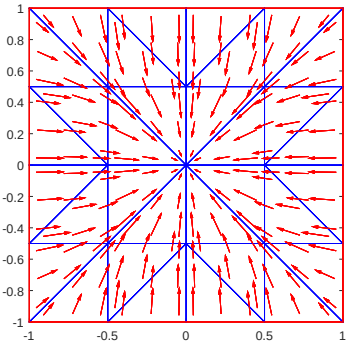
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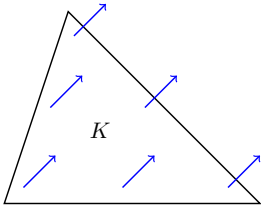


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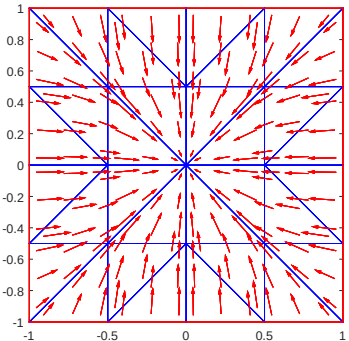
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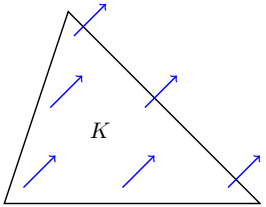


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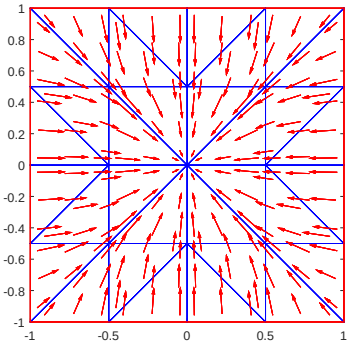
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- henceforth: manipulation of piecewise polynomials



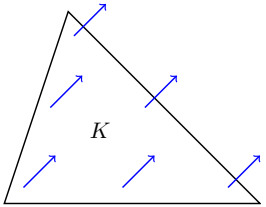
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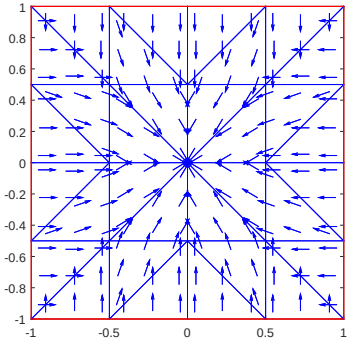
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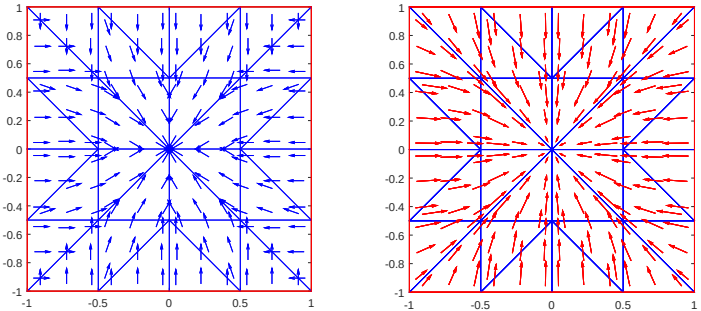
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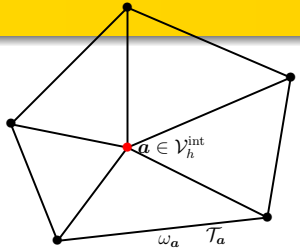
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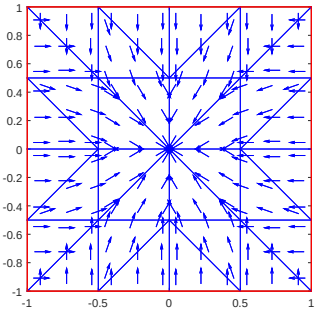


$\boldsymbol{\tau}_h \quad \rightarrow \quad \boldsymbol{\sigma}_h$

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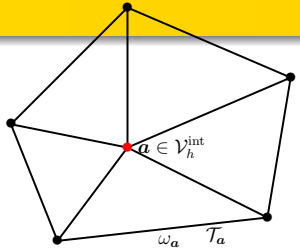
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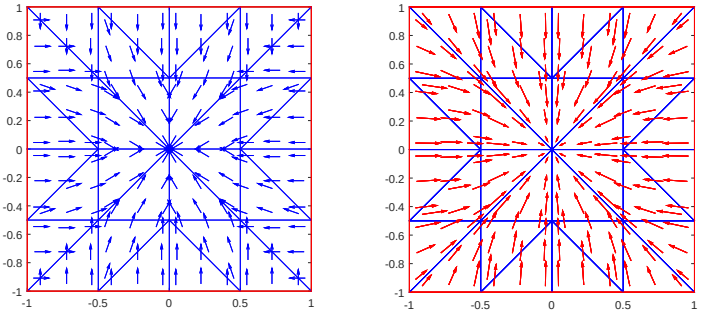


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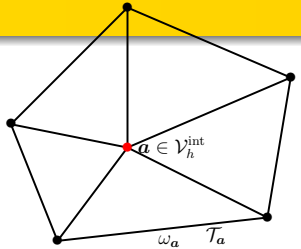


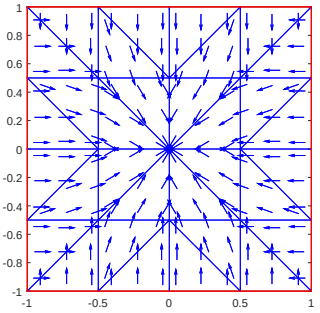
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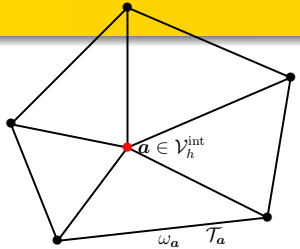




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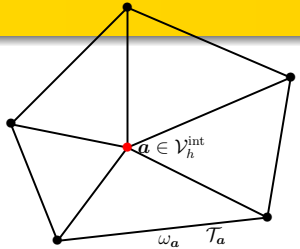
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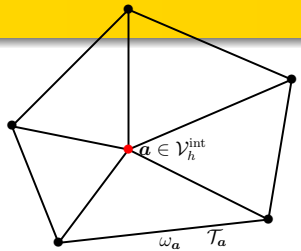
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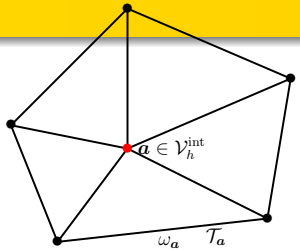
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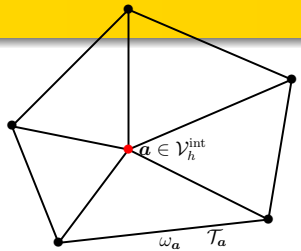
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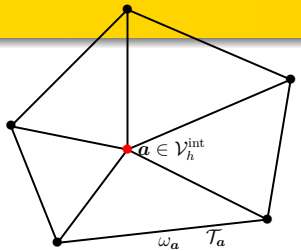


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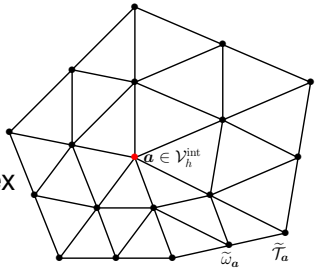
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- the last step will only work on the (conforming projection of the) reminder $\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h$ ("nonpolynomial part"): p-robust correction, no $\psi_{\boldsymbol{a}}$ and $\boldsymbol{I}_{\mathcal{RT}}^{h,p}$

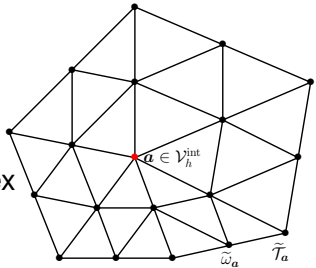
A p-stable local commuting projector $P_h^{p,\mathrm{div}}$

3. Apply to conforming projections $\zeta_h^{\boldsymbol{a}}$ of the reminder $\tau_h - \sigma_h$ a p-**stable decomposition** on extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$



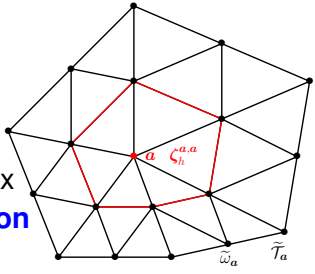
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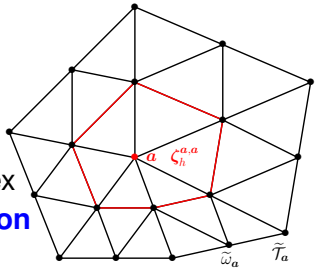


A p-stable local commuting projector $P_h^{p,\mathrm{div}}$

3. Apply to conforming projections $\zeta_h^{\boldsymbol{a}}$ of the reminder $\tau_h - \sigma_h$ a p-**stable decomposition** on extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}} \Rightarrow \zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \Rightarrow$ **correction** $\zeta_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \zeta_h^{\boldsymbol{a},\boldsymbol{a}}$.



A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$



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4. $\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) := \boldsymbol{\sigma}_h + \zeta_h$.

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)

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(patchwise **flux equilibration**)

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Ern, Gudi, Smears, & Vohralík (2022)

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(patchwise **flux equilibration** with an **additional orthogonality constraint**)

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$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

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(patchwise div-free conforming projection of $\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h$ with an additional constraint)

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$$\boldsymbol{\zeta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}'} \text{ with in particular } \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

(patchwise **p-stable** remainder **decomposition**)

A p-stable local commuting projector $P_h^{p,\mathrm{div}}$

1. $\boldsymbol{\tau}_h|_K := \arg \min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$ (**elementwise L^2-orthogonal projection**)
2. $$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^p(\psi_{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{v}) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{I}_p(\psi_{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \left\|\boldsymbol{I}_p(\psi_{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}} \quad \text{if } p \geq 1$$

(patchwise **flux equilibration** with an **additional orthogonality constraint**)

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$$ (gluing patchwise contributions)

3. $$\boldsymbol{\zeta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\widetilde{\omega}_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h,\boldsymbol{r}_h)_K = 0 \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}}} \|\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h - \boldsymbol{v}_h\|_{\widetilde{\omega}_{\boldsymbol{a}}}$$

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(patchwise p-**stable** remainder **decomposition**)

$$\boldsymbol{\zeta}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\zeta}_h^{\boldsymbol{a},\boldsymbol{a}}$$ (gluing patchwise correction contributions)

A p-stable local commuting projector $P_h^{p,\mathrm{div}} := \sigma_h + \zeta_h$

1. $\tau_h|_K := \arg \min_{v_h \in \mathcal{RT}_p(K)} \|v - v_h\|_K$ (**elementwise L^2-orthogonal projection**)
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(patchwise **flux equilibration** with an **additional orthogonality constraint**)

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$$\zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \text{ with in particular } \zeta_h^{\boldsymbol{a},\boldsymbol{a}} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0(\mathrm{div},\omega_{\boldsymbol{a}}),\ \nabla\cdot\zeta_h^{\boldsymbol{a},\boldsymbol{a}} = 0$$

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A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is* ***commuting*** *since*

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$



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$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v}) = \Pi_h^p(\nabla\cdot\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),$$

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It has *hp*-***robust local-best approximation property***

since, for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *and* $K \in \mathcal{T}_h$,

$$\left\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\right\|_K^2 + \left(\frac{h_K}{p+1}\left\|\nabla\cdot\left(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\right)\right\|_K\right)^2$$
$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2\right\},$$



A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$

Theorem ($\boldsymbol{P}_h^{p,\mathrm{div}} : \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \to \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$)

$\boldsymbol{P}_h^{p,\mathrm{div}}$ *is* ***commuting*** *and* ***projector*** *since*

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It has *hp*-***robust local-best approximation property*** *and is* *p*-***robustly*** L^2 ***stable*** *up to data oscillation, since, for all* $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ *and* $K \in \mathcal{T}_h$,

$$\left\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\right\|_K^2 + \left(\frac{h_K}{p+1}\left\|\nabla\cdot\left(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{v})\right)\right\|_K\right)^2$$
$$\lesssim \sum_{L\in\widetilde{\mathcal{T}}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{RT}_p(L)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_L^2 + \left(\frac{h_L}{p+1}\|\nabla\cdot\boldsymbol{v} - \Pi_h^p(\nabla\cdot\boldsymbol{v})\|_L\right)^2\right\},$$

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Outline

1. Introduction
   - Potential reconstruction
   - Flux reconstruction
   - A posteriori error estimates
   - Local commuting projectors
   - Local–global equivalence
   - A priori error estimates
2. Potential reconstruction
3. Equilibrated flux reconstruction
4. A p-stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{div}}$
5. Tools (hp-optimality, p-robustness)
   - Broken polynomial extension operators
   - p-stable decompositions
6. Conclusions and outlook

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1 Introduction
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5 Tools (hp-optimality, p-robustness)
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6 Conclusions and outlook

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}_h$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} .$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}_h$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

Potentials: vertex patch

Theorem (Broken H^1 polynomial extension on a vertex patch Ern & Vohralík (2015, 2020)**)**

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}})$. Suppose the compatibility

$$r_F|_{F\cap\partial\omega_{\boldsymbol{a}}} = 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}},$$
$$\sum_{F\in\mathcal{F}_e} \iota_{F,e}\, r_F|_e = 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.$$

Then

$$\min_{\substack{v_h\in\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})\\ v_h=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v_h]\!]=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v\in H^1(\mathcal{T}_{\boldsymbol{a}})\\ v=0\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}}\\ [\![v]\!]=r_F\ \forall F\in\mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & Vohralík (2020))

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K}(r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K \quad .$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & Vohralík (2020))

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F\in\mathcal{F}_K}(r_F,1)_F = (r_K,1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_p(K)\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}_h=r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K .$$

Context

$$-\Delta\zeta_K = r_K \quad \text{in } K,$$
$$-\nabla\zeta_K\cdot\boldsymbol{n}_K = r_F \quad \text{on all } F \in \mathcal{F}_K^{\mathrm{N}},$$
$$\zeta_K = 0 \quad \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}.$$

Set $\boldsymbol{\varphi}_K := -\nabla\zeta_K$.

Fluxes: one element

Lemma ($H(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & Vohralík (2020)**)**

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

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Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & Vohralík (2020))

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F\in\mathcal{F}_K}(r_F,1)_F = (r_K,1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\|\varphi_{h,K}\|_K \overset{MFEs}{=} \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_p(K)\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}_h=r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)\\ \boldsymbol{v}\cdot\boldsymbol{n}_K=r_F\ \forall F\in\mathcal{F}_K^{\mathrm{N}}\\ \nabla\cdot\boldsymbol{v}=r_K}} \|\boldsymbol{v}\|_K = \|\varphi_K\|_K.$$

Context

$$\begin{aligned} -\Delta\zeta_K &= r_K && \text{in } K,\\ -\nabla\zeta_K\cdot\boldsymbol{n}_K &= r_F && \text{on all } F\in\mathcal{F}_K^{\mathrm{N}},\\ \zeta_K &= 0 && \text{on all } F\in\mathcal{F}_K\setminus\mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla\zeta_K$.

Fluxes: vertex patch

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a vertex patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & Vohralík (2020; 3D)**)**

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

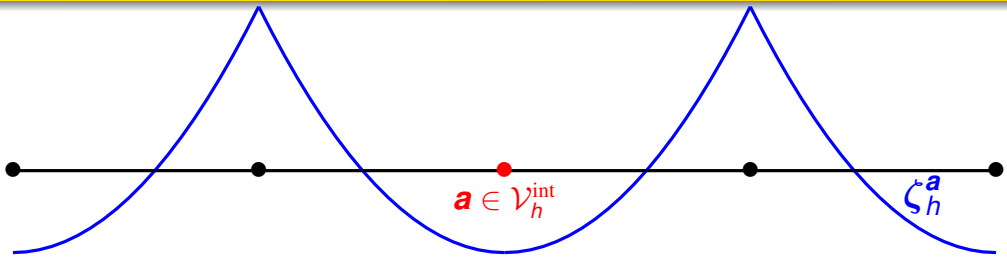
Then

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{ext}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{int}} \\ \nabla_h \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Outline

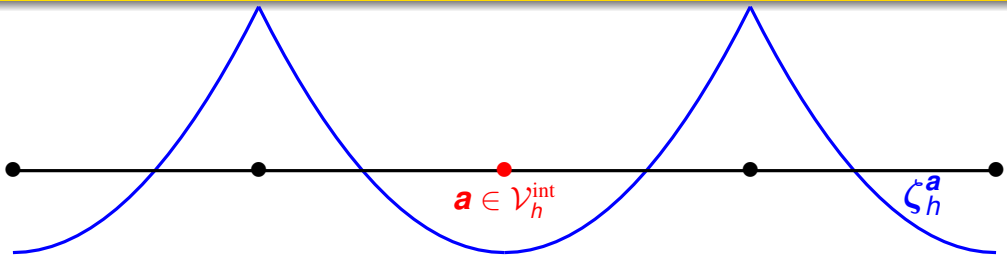
1. Introduction
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p-stable decomposition (no global component) ~ Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



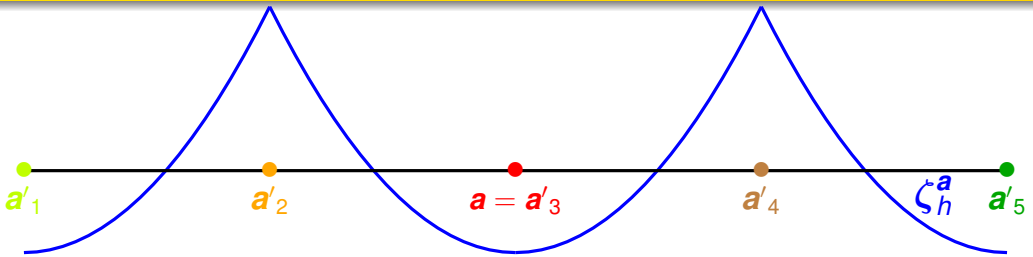
- $\zeta_h^{\boldsymbol{a}} \in \mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap H^1(\widetilde{\omega}_{\boldsymbol{a}})$ on an extended patch domain $\widetilde{\omega}_{\boldsymbol{a}}$ given

p-stable decomposition (no global component) ~ Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



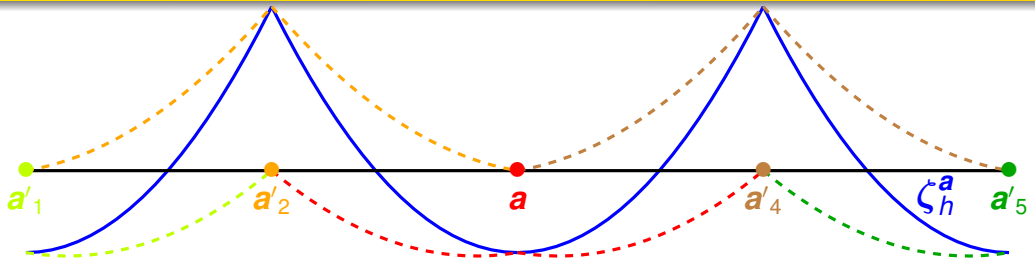
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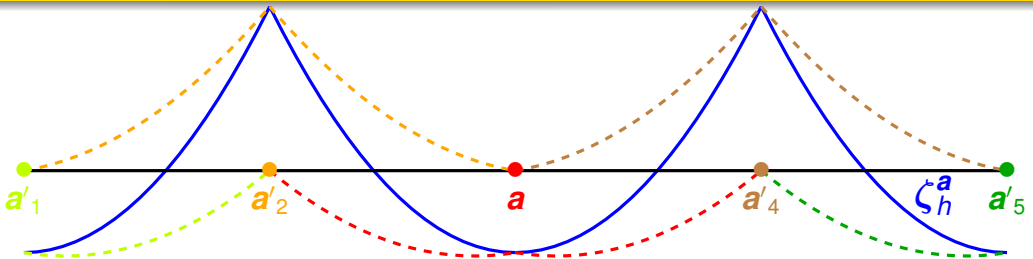
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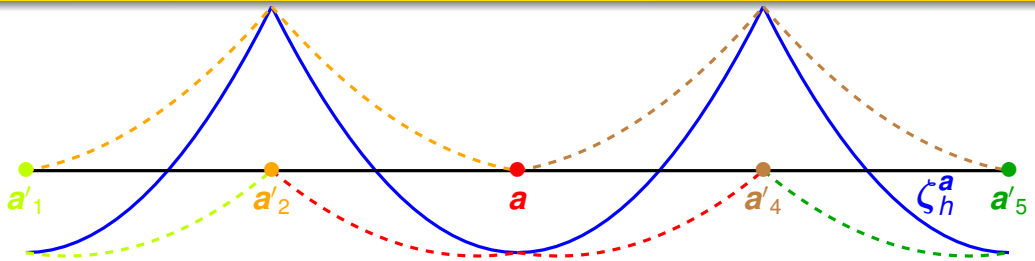
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- **decomposition** $\zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'} = \zeta_h^{\boldsymbol{a},a'_1} + \zeta_h^{\boldsymbol{a},a'_2} + \zeta_h^{\boldsymbol{a},\boldsymbol{a}} + \zeta_h^{\boldsymbol{a},a'_4} + \zeta_h^{\boldsymbol{a},a'_5}$,
  $\zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}'}) \cap H_0^1(\omega_{\boldsymbol{a}'})$

p-stable decomposition (no global component) ~ Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



- $\zeta_h^{\boldsymbol{a}} \in \mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap H^1(\widetilde{\omega}_{\boldsymbol{a}})$ on an extended patch domain $\widetilde{\omega}_{\boldsymbol{a}}$ given
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- p-**stable** $\|\nabla \zeta_h^{\boldsymbol{a},\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla \zeta_h^{\boldsymbol{a}}\|_{\widetilde{\omega}_{\boldsymbol{a}}}$

p-stable decomposition (no global component) ~ Schöberl, Melenk, Pechstein, & Zaglmayr (2008)



- $\zeta_h^{\boldsymbol{a}} \in \mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap H^1(\widetilde{\omega}_{\boldsymbol{a}})$ on an extended patch domain $\widetilde{\omega}_{\boldsymbol{a}}$ given
- $(\zeta_h^{\boldsymbol{a}}, 1)_K = 0$ for all $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, **vanishing mean values**
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- p-**stable** $\|\nabla \zeta_h^{\boldsymbol{a},\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla \zeta_h^{\boldsymbol{a}}\|_{\widetilde{\omega}_{\boldsymbol{a}}}$
- no global (low-order) component since $\zeta_h^{\boldsymbol{a}}$ is elementwise mean-value free

$\boldsymbol{H}(\mathrm{div})$ stable decomposition

Theorem ($\boldsymbol{H}(\mathrm{div})$ stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008))

Let $d = 2$ and let the closure of $\widetilde{\omega}_{\boldsymbol{a}}$ be contractible. Let

$$\zeta_h^{\boldsymbol{a}} \in \mathcal{RT}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}}) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \widetilde{\omega}_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\zeta_h^{\boldsymbol{a}} = 0, \qquad \text{div-free}$$

$$(\zeta_h^{\boldsymbol{a}}, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d,\ \forall K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}. \qquad \text{vanishing means}$$

Then there exists a decomposition of $\zeta_h^{\boldsymbol{a}}$ as

$$\zeta_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_h^{\boldsymbol{a},\boldsymbol{a}'}, \qquad \text{decomposition}$$

where

$\zeta_h^{\boldsymbol{a},\boldsymbol{a}'}$ *are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}'}$, linearly depend on $\zeta_h^{\boldsymbol{a}}$ on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}'}$,*

and satisfy

$$\zeta_h^{\boldsymbol{a},\boldsymbol{a}'} \in \mathcal{RT}_p(\mathcal{T}_{\boldsymbol{a}'}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}'}) \quad \text{with} \quad \nabla\cdot\zeta_h^{\boldsymbol{a},\boldsymbol{a}'} = 0, \qquad \text{div-free}$$

$$\|\zeta_h^{\boldsymbol{a},\boldsymbol{a}'}\|_{\omega_{\boldsymbol{a}'}} \lesssim \|\zeta_h^{\boldsymbol{a}}\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a}' \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}. \qquad p\text{-stable}$$

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Conclusions and outlook

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- piecewise polynomial reconstructions (potential, flux)

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- a posteriori:
  - p-robust error estimates (unified framework for all classical numerical schemes)

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- piecewise polynomial reconstructions (potential, flux)
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- a priori:
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  - p-robust global-best – local-best equivalence (in H^1; in $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$ while removing constraints)
  - optimal hp a priori error estimates under minimal elementwise regularity

Conclusions and outlook

Conclusions

- piecewise polynomial reconstructions (potential, flux)
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  - optimal hp a priori error estimates under minimal elementwise regularity

Ongoing work

- extensions to other complexes

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Thank you for your attention!

Outline

7 Dual roles

8 A priori error estimates

9 A posteriori error estimates

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction

flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace $\to$ continuous normal trace ▸ flux reconstruction

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction

flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\to$ continuous normal trace ▸ flux reconstruction

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\rightarrow$ continuous pw polynomial ▸ potential reconstruction

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\rightarrow$ continuous normal trace & equilibrium ▸ flux reconstruction

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\to$ continuous normal trace & equilibrium ▸ flux reconstruction
- **a posteriori** analysis of **conforming** FEs:

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\rightarrow$ continuous pw polynomial ▸ potential reconstruction

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\rightarrow$ continuous normal trace & equilibrium ▸ flux reconstruction
- **a posteriori** analysis of **conforming** FEs:

  estimate $\approx$ error

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\to$ continuous normal trace & equilibrium ▸ flux reconstruction
- **a posteriori** analysis of **conforming** FEs:

  estimate $\approx$ error: guaranteed & p-robust bounds Braess, Pillwein, Schöberl (2009), Ern, Vohralík (2015, 2020)

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial $\to$ continuous pw polynomial ▸ potential reconstruction
- **a posteriori** analysis of **mixed** and **nonconforming** FEs:

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium $\to$ continuous normal trace & equilibrium ▸ flux reconstruction
- **a posteriori** analysis of **conforming** FEs:

estimate $\approx$ error: guaranteed & p-robust bounds Braess, Pillwein, Schöberl (2009), Ern, Vohralík (2015, 2020)

Potential and flux reconstructions

Potential reconstruction

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- **a priori** analysis of **mixed** (and **nonconforming**) FEs:

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Outline

7 Dual roles

8 A priori error estimates

9 A posteriori error estimates

A model partial differential equation

Laplace equation

Find $u : \Omega \to \mathbb{R}$

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

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Setting

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, Lipschitz polytope (line segment, polygon, polyhedron)
- $f \in L^2(\Omega)$

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Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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Theorem (**Optimal a priori error estimate**)

There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|}$$

global-best

a priori error estimate

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- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$

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a priori error estimate

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- local-best: $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H^1_0(\Omega)$
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- s_h so good that **no** $v \in H^1_0(\Omega)$ **can do better** (up to a constant)

a priori error estimate, **elementwise**

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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global-best

$$= \left\{ \sum_{K\in\mathcal{T}_h} \underbrace{\min_{v_h\in\mathcal{P}_p(K)} \|\nabla(u-v_h)\|^2_K}_{\text{local-best approximation of } u \text{ on each } K} \right\}^{1/2}$$

no interface constraints

regularity only in K counts

- local-best: $\xi_h|_K := \arg\min_{v_h\in\mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H^1_0(\Omega)$
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Optimal a priori error estimate, elementwise, both h and p

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$$

Theorem (Optimal a priori error estimate)

There holds

$$\underbrace{\|\nabla(u-u_h)\|}_{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h)\cap H^1_0(\Omega)} \|\nabla(u-v_h)\|} \leq \|\nabla(u-s_h)\| \leq \|\nabla_h(u-\xi_h)\| + \|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(u-\xi_h)\|$$

$$\text{global-best} = \left\{ \sum_{K\in\mathcal{T}_h} \underbrace{\min_{v_h\in\mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K^2}_{\text{local-best approximation of } u \text{ on each } K} \right\}^{1/2} \lesssim \left(\frac{h}{p}\right)^p.$$

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Optimal a priori error estimate, **elementwise**

Theorem (Optimal a priori error estimate)

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Laplace equation

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Weak formulation

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Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

$$u \in H_0^1(\Omega), \quad -\nabla u \in \boldsymbol{H}(\mathrm{div}, \Omega), \quad \nabla\cdot(-\nabla u) = f$$

Error characterization

Theorem (Error equality)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, be arbitrary. Then

$$\|\nabla_h(u - u_h)\|^2 \qquad .$$

Error characterization

Theorem (Error equality)

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$$\|\nabla_h(u - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla_h(u_h - v)\|^2}_{} + \underbrace{\min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v} = f}} \|\nabla_h u_h + \boldsymbol{v}\|^2}_{} .$$

Error characterization

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Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, be arbitrary. Then

$$\|\nabla_h(u - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla_h(u_h - v)\|^2}_{\text{distance to the correct space}} + \underbrace{\min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v} = f}} \|\nabla_h u_h + \boldsymbol{v}\|^2}_{}.$$

Error characterization

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$$= \underbrace{\max_{v \in H_0^1(\Omega),\, \|\nabla v\|=1} \{(f,v) - (\nabla_h u_h, \nabla v)\}}$$

Error characterization

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Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, be arbitrary. Then

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$$= \underbrace{\max_{v \in H_0^1(\Omega),\, \|\nabla v\| = 1} \{(f, v) - (\nabla_h u_h, \nabla v)\}}_{\text{dual norm of the residual}}.$$

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Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, be arbitrary. Then

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a posteriori error estimate, reliability

Theorem (Optimal a posteriori error estimate)

For any $s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma}_h = f$, there holds

$$\underbrace{\|\nabla_h(u - u_h)\|^2}_{} \leq \boxed{\underbrace{\|\nabla_h(u_h - s_h)\|^2 + \|\nabla_h u_h + \boldsymbol{\sigma}_h\|^2}_{}}$$

a posteriori error estimate, reliability (**guaranteed upper bound**)

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For any $s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma}_h = f$, there holds

$$\underbrace{\|\nabla_h(u - u_h)\|^2}_{\text{unknown error}} \leq \underbrace{\boxed{\|\nabla_h(u_h - s_h)\|^2 + \|\nabla_h u_h + \boldsymbol{\sigma}_h\|^2}}_{\text{computable estimator } \eta(u_h)}$$

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Comments

- **local** construction of **piecewise polynomial** s_h and $\boldsymbol{\sigma}_h$ from u_h

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Theorem (**Optimal a posteriori error estimate**) ($(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla_h u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0$ for all $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

For any $s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma}_h = f$, there holds

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a posteriori error estimate, reliability (**guaranteed upper bound**)

Theorem (**Optimal a posteriori error estimate**) ($(f, \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla_h u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0$ for all $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$)

For **good** $s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma}_h = f$, there holds

$$\underbrace{\|\nabla_h(u - u_h)\|^2}_{\text{unknown error}} \leq \underbrace{\boxed{\|\nabla_h(u_h - s_h)\|^2 + \|\nabla_h u_h + \boldsymbol{\sigma}_h\|^2}}_{\text{computable estimator } \eta(u_h)}$$

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Optimal a posteriori error estimate, reliability (**guaranteed upper bound**), efficiency (p-**robust**)

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For **good** $s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma}_h = f$, there holds

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$$\lesssim \min_{v \in H_0^1(\Omega)} \|\nabla_h(u_h - v)\|^2 + \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega) \\ \nabla \cdot \boldsymbol{v} = f}} \|\nabla_h u_h + \boldsymbol{v}\|^2 = \|\nabla_h(u - u_h)\|^2$$

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Optimal a posteriori error estimate, reliability (**guaranteed upper bound**), efficiency (*p*-**robust**)

Theorem (**Optimal a posteriori error estimate**)

For **good** $s_h \in H^1_0(\Omega)$ and $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div},\Omega)$ such that $\nabla\cdot\boldsymbol{\sigma}_h = f$, there holds

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