Local space-time efficiency for the heat and wave equations

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in collaboration with Vít Dolejší, Alexandre Ern, Nicolas Hugot, and Alexandre Imperiale

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Outline

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- provide sharp **computable bounds** on the (unknown) error between the (unavailable) exact solution u and its (computed) numerical approximation $u_{h\tau}$

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This talk

Guaranteed a posteriori error estimate

$$|||u - u_{h\tau}|||^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$$

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Guaranteed a posteriori error estimate **efficient**

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Guaranteed a posteriori error estimate **efficient** and **robust** with respect to the **final time**.

$$|||u - u_{h\tau}|||^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 \leq C_{\text{eff}}^2 |||u - u_{h\tau}|||^2_{\Omega\times(0,T)}, \quad C_{\text{eff}} \text{ indep. of } T$$

A posteriori error estimates

- provide sharp **computable bounds** on the (unknown) error between the (unavailable) exact solution u and its (computed) numerical approximation $u_{h\tau}$ $\Longrightarrow$ **error certification**
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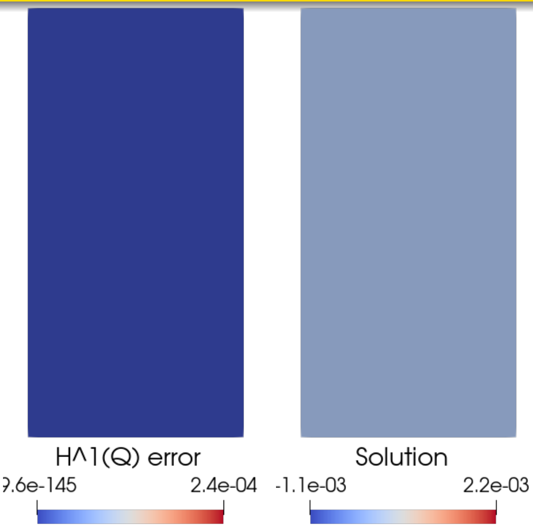
This talk

Guaranteed a posteriori error estimate **locally space-time efficient** and **robust** with respect to the **final time**.

$$|||u - u_{h\tau}|||^2_{\Omega\times(0,T)} \le \sum_{n=1}^{N} \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$$

$$\eta_K^n(u_{h\tau})^2 \le C_{\rm eff}^2 |||u - u_{h\tau}|||^2_{\omega_K\times I_n} \quad \text{for all } 1 \le n \le N \text{ and } K \in \mathcal{T}_h^n$$

Goal



N. Hugot, A. Imperiale, M. Vohralík, to be submitted (2025).

Outline

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The heat & wave equations

The heat equation

Find $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \Omega \times (0, T), \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u &= 0 \quad \text{on } \Omega \times 0. \end{aligned}$$

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Setting

- T: final time
- Ω: space domain
- $Q := \Omega \times (0, T)$: space-time domain
- f piecewise polynomial for simplicity

Function spaces and their associated norms

Spaces and norms

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$$\|v\|_X^2 := \|\nabla v\|_Q^2 = \int_0^T \|\nabla v(\cdot, t)\|_\Omega^2 \,\mathrm{d}t = \int_0^T \int_\Omega |\nabla v(\boldsymbol{x}, t)|^2 \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t,$$

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\|v\|_X^2 &:= \|\nabla v\|_Q^2,\\
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\begin{aligned}
Y_T &:= \{v \in X \cap H^1(0, T; H^{-1}(\Omega));\ v = 0 \text{ on } \Omega \times T\},\\
\|v\|_{Y_T}^2 &:= \int_0^T \{\|\partial_t v\|^2_{H^{-1}(\Omega)} + \|\nabla v\|^2\}\,\mathrm{d}t + \|v(\cdot, 0)\|^2
\end{aligned}
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Weak formulations

The heat equation

Definition (Weak solution)

$u \in X$ such that, for all $v \in H^1_{,T}(Q)$,

$$-(u, \partial_t v)_Q + (\nabla u, \nabla v)_Q = (f, v)_Q.$$

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Nonsymmetry

Trial space X or $H^1_{0,}(Q)$, test space $H^1_{,T}(Q)$.

Residuals and their dual norms

The heat equation

Definition (Residual)

For $u_{h\tau} \in X$, $\mathcal{R}(u_{h\tau}) \in (H^1_{,T}(Q))'$,

$$\langle \mathcal{R}(u_{h\tau}), v \rangle := (f, v)_Q + (u_{h\tau}, \partial_t v)_Q - (\nabla u_{h\tau}, \nabla v)_Q, \quad v \in H^1_{,T}(Q).$$

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$\mathcal{R}(u_{h\tau}) = 0$ if and only if $u_{h\tau} = u$.

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- induced by the weak formulation (problem-dependent $\times$ fixed space & norm)
- sometimes the only choice (sign-changing coefficients, implicit const. laws

Inf-sup equalities, norms of the difference $u - u_{h\tau}$

The heat equation

- For the slightly bigger test space $Y_T \supset H^1_{,T}(Q)$,

$$\|u - u_{h\tau}\|_X = \|\mathcal{R}(u_{h\tau})\|_{(Y_T)'}$$

by standard inf-sup theory.

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The wave equation

- For the slightly bigger trial space $Y_0 \supset H^1_{0,}(Q)$,

$$\|u - u_{h\tau}\|_{Y_0} = \|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$$

by inf-sup of Steinbach & Zank (2022).

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- There holds

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$$= \sup_{\substack{v \in H^1_{,T}(Q) \\ |v|_{H^1(Q)}=1}} \left\{(u_{h\tau} - u, \partial_t v)_Q - (\nabla(u_{h\tau} - u), \nabla v)_Q\right\}$$
$$\leq \left(\|u - u_{h\tau}\|_Q^2 + \|\nabla(u - u_{h\tau})\|_Q^2\right)^{1/2}.$$

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\end{aligned}$$

Previous results: heat equation

- Picasso / Verfürth (1998), work with the energy norm of X:
  - ✓ upper bound $\|u - u_{h\tau}\|_X^2 \leq C^2 \sum_{n=1}^N \sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2$
  - ✗ constrained lower bound (number of elements $|\mathcal{T}_h^n|$ and time step τ linked)
- Repin (2002), guaranteed upper bound
- Verfürth (2003) (cf. also Bergam, Bernardi, & Mghazli (2005)), Y norm:
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  - ✓ efficiency $\sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 \leq C^2 \|u - \mathcal{I}u_{h\tau}\|_{Y(I_n)}^2$
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- Eriksson & Johnson (1991), duality techniques & Makridakis & Nochetto (2003), elliptic reconstruction: $L^2(L^2)$ / $L^\infty(L^2)$ / $L^\infty(L^\infty)$ / higher-order norms
- Makridakis & Nochetto (2006): Radau reconstruction $\mathcal{I}u_{h\tau}$ for any order
- Schötzau & Wihler (2010), τq adaptivity
- Ern, Smears, & Vohralík (2017): local space-time efficiency in the Y norm
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- Repin (2002), guaranteed upper bound
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- Eriksson & Johnson (1991), duality techniques & Makridakis & Nochetto (2003), elliptic reconstruction: $L^2(L^2)$ / $L^\infty(L^2)$ / $L^\infty(L^\infty)$ / higher-order norms
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Previous results: heat equation

- Picasso / Verfürth (1998), work with the energy norm of X:
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Outline

Context

Recall

- $Q = \Omega \times (0, T)$
- $H^1_{,T}(Q) = \{v \in H^1(Q);\ v = 0 \text{ on } \partial\Omega \times (0, T) \text{ and } \Omega \times T\}$
- $|v|^2_{H^1(Q)} = \|\partial_t v\|^2_Q + \|\nabla v\|^2_Q$
- $\mathcal{R}(u_{h\tau}) \in (H^1_{,T}(Q))'$
- $\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} = \sup_{\substack{v \in H^1_{,T}(Q) \\ |v|_{H^1(Q)}=1}} \langle \mathcal{R}(u_{h\tau}), v \rangle$

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Observations, questions

- $\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$: dual, a priori global norm
- a priori no localization
- local space-time efficiency in a nonlocal norm?

Localization of dual norms

Theorem (Localization of dual norms)

Let $\mathcal{R}(u_{h\tau}) \in (H^1_{,T}(Q))'$ *be arbitrary. Let, for an index set* $\mathcal{V}$,

$$\psi^{\boldsymbol{a}} \in W^{1,\infty}(Q) \subset H^1_{,T}(Q) \quad \begin{cases} \text{have local space–time supports, } \overline{\omega_{\boldsymbol{a}}} \\ \text{form a partition of unity } \sum_{\boldsymbol{a}\in\mathcal{V}} \psi^{\boldsymbol{a}} = 1. \end{cases}$$

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- known from elliptic a posteriori error analysis
- generalizes to $(W_0^{1,\alpha}(Q))'$, $1 \leq \alpha \leq \infty$, see Blechta, Málek, & Vohralík (2020) and the references therein

Localization of the space-time dual norm $\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$

Descriptions

- inspired by the preceding theorem, we succeed to localize $\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$
- in general, localization entails overlapping
- we overlap in space but not in time (localization is per time interval)
- $\psi^{\mathbf{a}}$ above is a space-time function; **orthogonality** to lowest-order **space finite element basis functions** on **each time interval** turns out to be sufficient
- orthogonality not for the residual: time-derivative term is integrated by parts
- this will request to increase by one the time regularity of $u_{h\tau}$

Structural observations

- in the elliptic case, since the problem is boundary value, to obtain orthogonality of the residual wrt (lowest-order) finite element basis functions, one needs to solve global a linear system
- in **parabolic/hyperbolic cases**, no condition is prescribed on the final time $\Longrightarrow$ **implicit and even explicit** time stepping is sufficient (no global linear system)

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Discrete setting

- discrete times $\{t^n\}_{0\le n\le N}$, $t^0 = 0$ and $t^N = T$
- time intervals $I_n := (t^{n-1}, t^n]$, time steps $\tau^n := t^n - t^{n-1}$
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Theorem (Localization of the intrinsic dual residual norm)

Let $u_{h\tau} \in H^1_{0,}(Q)$ *(heat) or* $u_{h\tau} \in H^1_{0,}(Q) \cap H^2(0, T; L^2(\Omega))$ *(wave).*

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where the hidden constants only depend on the space dimension d and shape-regularity of the space and time meshes.

Outline

Crank–Nicolson method for the heat equation

Definition (Crank–Nicolson)

Set $u_h^0 := 0$. Find u_h^n, $1 \leq n \leq N$, such that

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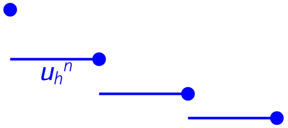

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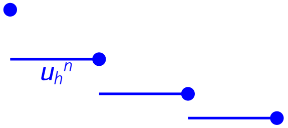

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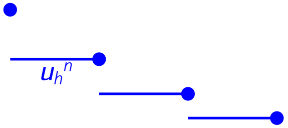

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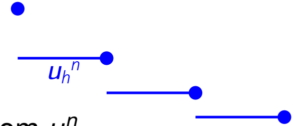

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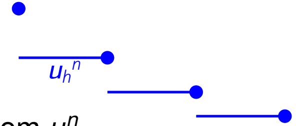

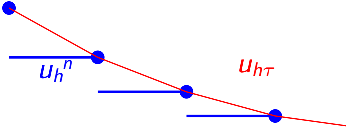

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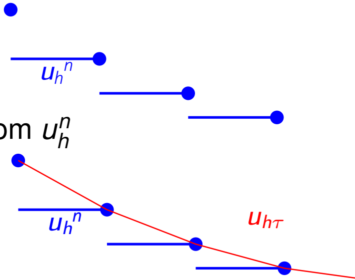

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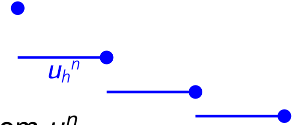

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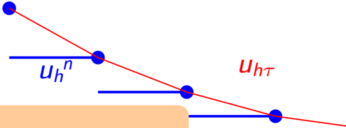

Orthogonality condition satisfied

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Reliable and locally space-time efficient a posteriori error estimates

Theorem (A posteriori error estimates)

Let $u_{h\tau} \in H^1_{0,}(Q)$ (heat) or $u_{h\tau} \in H^1_{0,}(Q) \cap H^2(0, T; L^2(\Omega))$ (wave) be piecewise polynomials (order p in space, order q in time).

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$$\|\mathcal{R}(u_{h\tau})\|^2_{(H^1_{,T}(Q))'} \lesssim \sum_{n=1}^{N}\sum_{K\in\mathcal{T}_h^n} \eta_K^n(u_{h\tau})^2 \lesssim \|\mathcal{R}(u_{h\tau})\|^2_{(H^1_{,T}(Q))'}$$

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where the hidden constants only depend on the space dimension d, shape-regularity of the space and time meshes, and p and q.

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Estimators

Residual-based estimators

$$\eta_K^n(u_{h\tau}) := \underbrace{h_{K\times I_n}\|f - \partial_{tt} u_{h\tau} + \Delta u_{h\tau}\|_{K\times I_n}}_{\text{volume residual}} + \left\{ \sum_{F\in\mathcal{F}_K^{\mathrm{int}}} \underbrace{h_{F\times I_n}\|[\![\nabla u_{h\tau}]\!]\cdot\boldsymbol{n}_F\|_{F\times I_n}^2}_{\text{face normal component jump}} \right\}^{1/2}$$

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Equilibrated fluxes estimators (reliability constant becomes 1)

$$\boldsymbol{\sigma}_{h\tau} \in \boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathrm{div},\Omega)) \text{ with } (f - \partial_{tt}u_{h\tau} - \nabla\cdot\boldsymbol{\sigma}_{h\tau}, 1)_{K\times I_n} \quad \forall 1 \le n \le N,\ \forall K \in \mathcal{T}_h,$$

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Outline

Inspection of the local efficiency proof (element residual)

- $v_{K,n} := (f - \partial_{tt} u_{h\tau} + \Delta u_{h\tau})|_{K \times I_n}$
- space-time bubble $\psi_{K,n}$, product of the barycentric coordinates on K and of the barycentric coordinates on I_n
- norm equivalence in finite-dimensional spaces:

$$(v_{K,n}, v_{K,n})_{K\times I_n} \lesssim (v_{K,n}, \psi_{K,n} v_{K,n})_{K\times I_n}$$

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- congruently, in $|v|^2_{H^1(Q)} = \|\partial_t v\|^2_Q + \|\nabla v\|^2_Q$, the physical units are different
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- space-time bubble $\psi_{K,n}$, product of the barycentric coordinates on K and of the barycentric coordinates on I_n
- norm equivalence in finite-dimensional spaces:

$$(v_{K,n}, v_{K,n})_{K \times I_n} \lesssim (v_{K,n}, \psi_{K,n} v_{K,n})_{K \times I_n}$$

- inverse inequality separately in space and in time:

$$h_K \|\nabla(\psi_{K,n} v_{K,n})\|_{K \times I_n} \lesssim \|\psi_{K,n} v_{K,n}\|_{K \times I_n},$$
$$\tau^n \|\partial_t(\psi_{K,n} v_{K,n})\|_{K \times I_n} \lesssim \|\psi_{K,n} v_{K,n}\|_{K \times I_n}$$

- congruently, in $|v|^2_{H^1(Q)} = \|\partial_t v\|^2_Q + \|\nabla v\|^2_Q$, the physical units are different
- $\Longrightarrow$ space-time weighted mesh-dependent norm imposed on the test space

$$|v|^2_{H^1(Q)} := \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h^n} \left\{ (\tau^n)^2 \|\partial_t v\|^2_{K \times I_n} + h_K^2 \|\nabla v\|^2_{K \times I_n} \right\}$$

- $\Longrightarrow$ units ✓, space-time anisotropy ✓, time-evolving meshes ✓

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Outline

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Setting

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- incomplete interior penalty discontinuous Galerkin space discretization with polynomial degrees $p = 1, 2, 3$
- Crank–Nicolson in time
- space and time meshes both uniformly refined: $m = 1, 2, 3$

Effectivity indices

- dual norm

$$i_e := \frac{\eta}{\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'} + \text{jumps}} \geq 1$$

- (weighted) L^2 norm:

$$i_{e,H^1} := \frac{\eta}{\left(\tau^{-2}\|u - u_{h\tau}\|_Q^2 + h^{-2}\|\nabla(u - u_{h\tau})\|_Q^2\right)^{1/2} + \text{jumps}} (< 1 \text{ possible})$$

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Viscous Burgers equation

Viscous Burgers equation

$$\partial_t u - \nabla\cdot(\varepsilon\nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$ or $\varepsilon = 10^{-4}$
- $\phi(u) = (u^2/2, u^2/2)^{\mathrm{T}}$
- $\Omega = (-1, 1) \times (-1, 1)$
- $T = 1$

Exact solution

- $$u(x, y, t) = \left(1 + \exp\left(\frac{x + y + 1 - t}{2\varepsilon}\right)\right)^{-1}$$

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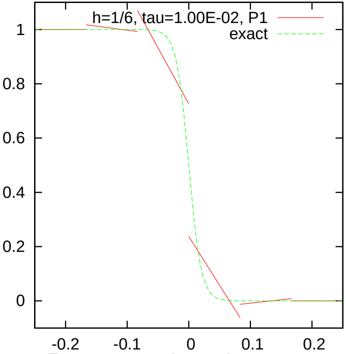
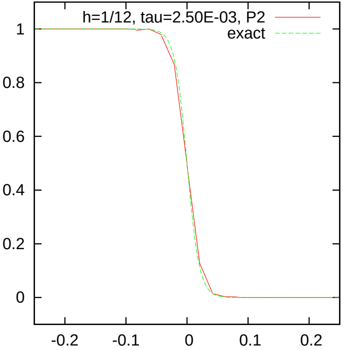
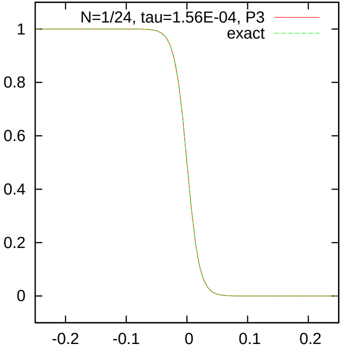
Exact and approximate solutions, $\varepsilon = 10^{-2}$



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Effectivity indices for varying ε and (h_0, τ_0)

ε		10^{-2}		10^{-2}		10^{-2}		10^{-4}	
(h_0, τ_0)		$(1/6, 0.05)$		$(1/6, 0.2)$		$(1/6, 0.0125)$		$(1/6, 0.05)$	
m	p	i_e	i_{e,H^1}	i_e	i_{e,H^1}	i_e	i_{e,H^1}	i_e	i_{e,H^1}
1	1	1.85	1.15	2.21	1.28	3.00	0.81	1.45	0.71
2	1	1.71	1.35	2.38	1.12	2.45	1.03	1.68	1.06
3	1	1.25	1.36	2.15	0.90	1.33	1.03	1.82	1.34
1	2	2.15	1.01	3.13	1.71	3.69	0.67	1.38	0.62
2	2	1.65	0.94	2.74	1.58	2.16	0.49	1.41	0.62
3	2	1.53	1.08	2.38	1.52	1.83	0.58	1.54	0.69
1	3	1.71	0.59	2.74	1.47	3.00	0.34	1.26	0.31
2	3	1.75	0.73	2.63	1.67	3.15	0.46	1.13	0.21
3	3	2.54	0.97	2.77	1.73	—	0.69	1.03	0.15

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Degenerate advection-diffusion equation

Degenerate advection-diffusion problem (Kačur 2001)

$$\partial_t u - \nabla\cdot(2\varepsilon u \nabla u - \phi(u)) = 0 \quad \text{in } Q$$

- $\varepsilon = 10^{-2}$
- $\phi(u) = 0.5(u^2, 0)^{\mathrm{T}}$
- $\Omega = (0,1) \times (0,1)$
- $T = 1$

Exact solution

- $$u(x,y,t) = \begin{cases} 1 - \exp\left(\frac{v(x-vt-x_0)}{2\varepsilon}\right) & \text{for } x \le vt + x_0, \\ 0 & \text{for } x > vt + x_0 \end{cases}$$
- $x_0 = 1/4$ is the initial position of the front

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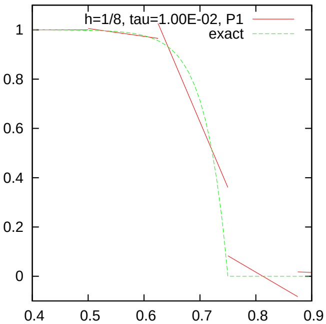
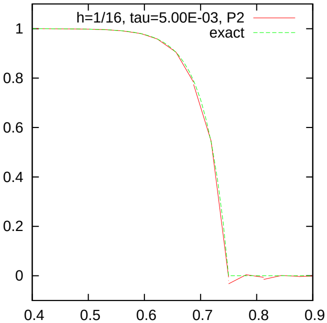
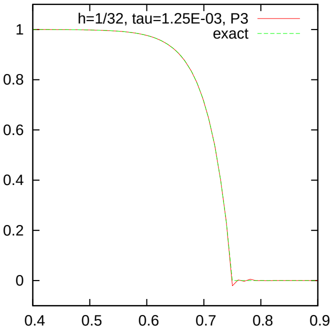
Exact and approximate solutions



P_1 approximation on $\{h_1, \tau_1\}$



P_2 approximation on $\{h_2, \tau_2\}$



P_3 approximation on $\{h_3, \tau_3\}$

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (1/8, 0.05)$

m	p	$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$	$\eta_{\rm F}$	$\eta_{\rm R}$	$\eta_{\rm NC}$	$\eta_{\rm IC}$	$\eta_{\rm qd}$	η	$i_{\rm e}$	$i_{{\rm e},H^1}$
1	1	9.91E-03	1.00E-02	6.02E-03	2.77E-02	2.31E-02	2.17E-03	6.62E-02	1.76	0.97
2	1	7.39E-03	7.71E-03	5.68E-03	1.62E-02	7.71E-03	1.23E-03	3.66E-02	1.55	1.02
		(0.42)	(0.37)	(0.08)	(0.78)	(1.59)	(0.82)	(0.86)		
3	1	4.58E-03	4.52E-03	4.95E-03	8.33E-03	1.86E-03	5.22E-04	1.89E-02	1.47	1.16
		(0.69)	(0.77)	(0.20)	(0.96)	(2.05)	(1.23)	(0.95)		
1	2	2.62E-03	3.30E-03	5.40E-03	9.33E-03	6.27E-03	6.74E-04	2.35E-02	1.97	0.73
2	2	1.11E-03	1.43E-03	1.93E-03	4.22E-03	1.09E-03	2.67E-04	8.34E-03	1.56	0.62
		(1.23)	(1.21)	(1.48)	(1.14)	(2.52)	(1.34)	(1.50)		
3	2	4.26E-04	5.63E-04	6.13E-04	1.84E-03	1.51E-04	1.00E-04	3.06E-03	1.35	0.57
		(1.38)	(1.34)	(1.65)	(1.20)	(2.85)	(1.42)	(1.45)		
1	3	6.48E-04	8.83E-04	1.03E-03	3.57E-03	1.19E-03	2.31E-04	6.47E-03	1.53	0.36
2	3	1.94E-04	2.63E-04	1.45E-04	1.21E-03	1.07E-04	6.39E-05	1.69E-03	1.21	0.25
		(1.74)	(1.74)	(2.84)	(1.56)	(3.48)	(1.85)	(1.93)		
3	3	4.42E-05	7.58E-05	2.58E-05	4.04E-04	7.47E-06	1.67E-05	5.07E-04	1.13	0.21
		(2.13)	(1.80)	(2.49)	(1.58)	(3.84)	(1.94)	(1.74)		

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Porous medium equation

Porous medium equation

$$\partial_t u - \nabla\cdot(\boldsymbol{K}(u)\nabla u) = 0 \quad \text{in } Q$$

- $\boldsymbol{K}(u) = \kappa |u|^{\kappa-1} \underline{\boldsymbol{I}}$,
- $\kappa = 2$ or $\kappa = 4$
- $\Omega = (-6, 6) \times (-6, 6)$
- $T = 1$

Barenblatt solution

- $$u(x, y, t) = \left\{ \frac{1}{t+1} \left[1 - \frac{\kappa - 1}{4\kappa^2} \frac{x^2 + y^2}{(t+1)^{1/\kappa}} \right]_+^{\frac{\kappa}{\kappa-1}} \right\}^{\frac{1}{\kappa}}$$

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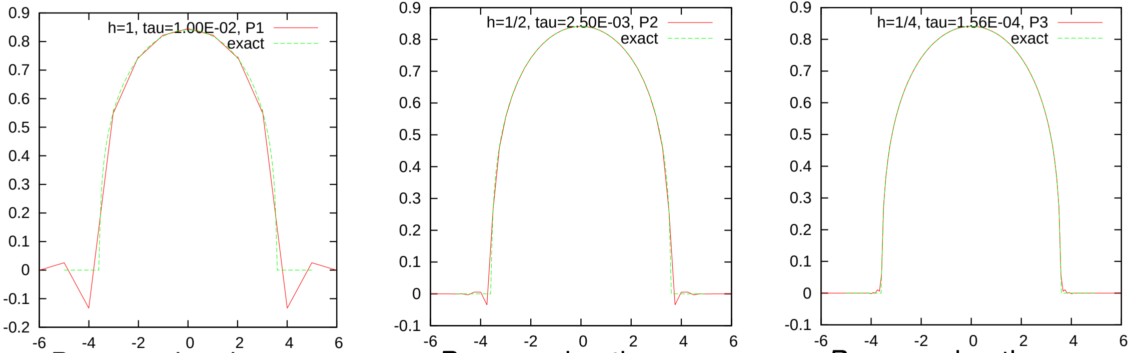
Exact and approximate solutions, $\kappa = 4$



P_1 approximation on $\{h_1, \tau_1\}$

P_2 approximation on $\{h_2, \tau_2\}$

P_3 approximation on $\{h_3, \tau_3\}$

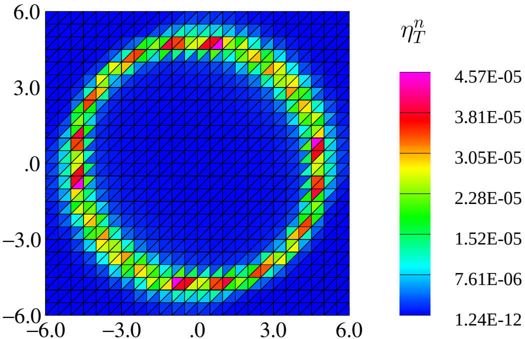
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Errors, estimators, and effectivity indices, $(h_0, \tau_0) = (0.5, 0.02)$

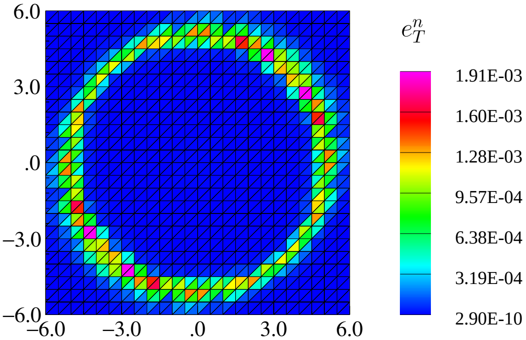
			$\kappa = 2$								$\kappa = 4$	
m	p	$\|\mathcal{R}(u_{h\tau})\|_{(H^1_{,T}(Q))'}$	η_{F}	η_{R}	η_{NC}	η_{IC}	η_{qd}	η	i_{e}	i_{e,H^1}	i_{e}	i_{e,H^1}
1	1	7.90E-03	5.90E-03	1.32E-02	9.10E-03	3.23E-02	7.08E-05	5.88E-02	3.46	0.92	4.68	0.98
2	1	8.36E-03	4.64E-03	1.71E-02	8.46E-03	1.11E-02	3.99E-05	4.03E-02	2.40	1.46	3.72	1.62
		(-0.08)	(0.35)	(-0.38)	(0.10)	(1.54)	(0.83)	(0.54)				
3	1	8.91E-03	4.38E-03	2.18E-02	9.56E-03	3.44E-03	1.83E-05	3.87E-02	2.09	2.49	3.38	2.68
		(-0.09)	(0.08)	(-0.35)	(-0.18)	(1.69)	(1.13)	(0.06)				
1	2	1.09E-03	1.06E-02	1.06E-01	3.12E-02	1.35E-02	1.74E-04	1.61E-01	4.99	3.22	5.13	3.18
2	2	4.02E-04	8.04E-03	8.12E-02	2.37E-02	5.16E-03	6.40E-05	1.18E-01	4.90	3.89	5.05	3.84
		(1.43)	(0.40)	(0.39)	(0.40)	(1.39)	(1.45)	(0.45)				
3	2	1.28E-04	5.22E-03	5.33E-02	1.55E-02	1.69E-03	2.23E-05	7.57E-02	4.84	4.26	4.97	4.30
		(1.65)	(0.62)	(0.61)	(0.61)	(1.61)	(1.52)	(0.64)				
1	3	6.53E-04	2.26E-02	3.27E-01	7.58E-02	8.39E-03	1.36E-04	4.33E-01	5.67	5.01	5.67	4.88
2	3	1.78E-04	9.26E-03	1.38E-01	3.13E-02	3.14E-03	3.51E-05	1.82E-01	5.76	5.17	5.78	5.03
		(1.87)	(1.29)	(1.24)	(1.27)	(1.42)	(1.95)	(1.25)				
3	3	3.83E-05	3.41E-03	5.08E-02	1.15E-02	1.14E-03	8.89E-06	6.68E-02	5.80	5.21	5.85	5.10
		(2.22)	(1.44)	(1.44)	(1.45)	(1.46)	(1.98)	(1.44)				

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

Exact and approximate error, $\kappa = 4$, $t = T$, $p = 2$, $m = 2$



Estimators η_K^n



Local error upper bounds

V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2013)

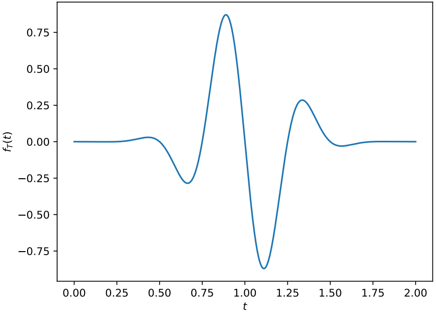
Outline

Data and solution

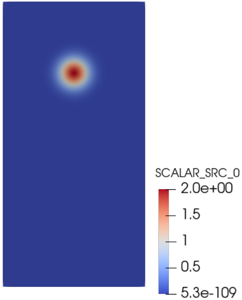
$$\Omega = (0,1) \times (0,2)$$

$$f_T(t) = -\sin(4\pi(t-1)) \times e^{-\left(\frac{t-1}{0.1}\right)^2}$$

$$f_X(x,y) = \exp\left(-\frac{(x-0.5)^2 + (y-1.5)^2}{0.1^2}\right)$$

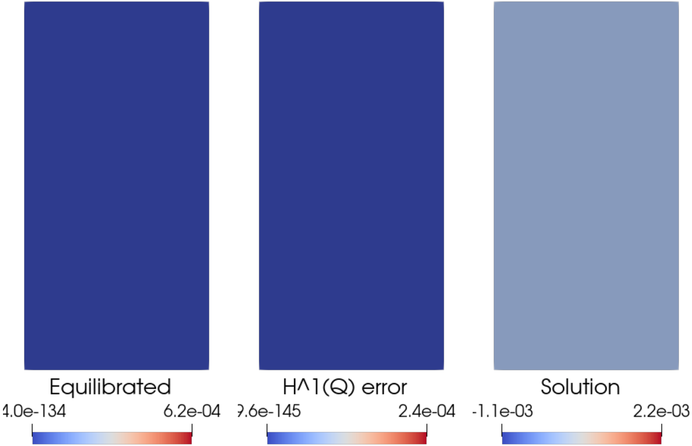


f_T



f_X

Error and estimators



Remarks:

- cumulated local errors, for $K \in \mathcal{T}_h$ and $1 \leq n \leq N$:

$$\left\{ \sum_{i=0}^{n} (\eta_K^i(u_{h\tau}))^2 \right\}^{1/2}$$

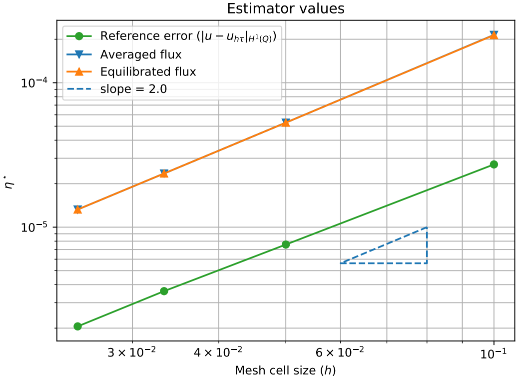
- local H^1 norms in place of the dual norm:

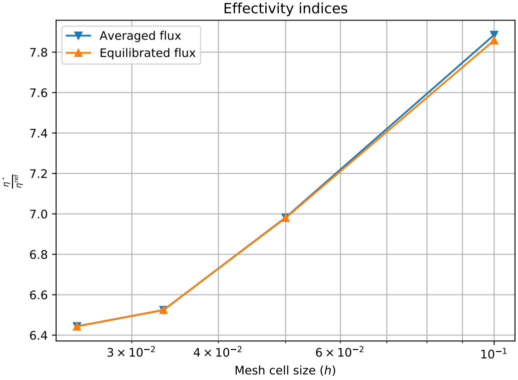
$$|||\mathcal{R}(u_{h\tau})\|_{(H_0^1(\omega_K \times I_n))'} \leq |v|^2_{H^1(\omega_K \times I_n)}$$

- quadrature err. ignored

N. Hugot, A. Imperiale, M. Vohralík, to be submitted (2025).

Convergence rates and effectivity indices





N. Hugot, A. Imperiale, M. Vohralík, to be submitted (2025).

Outline

Conclusions

Take-away messages

- **inexpensive** (explicit) **estimators** of the global error ($\times$ estimates of local quantities of interest for a price of a solution of a backward problem)
- same methodology for both **parabolic** & **hyperbolic problems**
- **localization** of **global space-time norms** with an orthogonality property
- covers **implicit & explicit time-stepping** as well as time-evolving meshes

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References

DOLEJŠÍ V., ERN A., VOHRALÍK M. A framework for robust a posteriori error control in unsteady nonlinear advection-diffusion problems, *SIAM J. Numer. Anal.* **51** (2013), 773–793.

HUGOT N., IMPERIALE A., VOHRALÍK M. Reliable and locally space-time efficient a posteriori estimates for the transient acoustic wave equation. To be submitted, 2025.

Conclusions

Take-away messages

- **inexpensive** (explicit) **estimators** of the global error ($\times$ estimates of local quantities of interest for a price of a solution of a backward problem)
- same methodology for both **parabolic** & **hyperbolic problems**
- **localization** of **global space-time norms** with an orthogonality property
- covers **implicit & explicit time-stepping** as well as time-evolving meshes

Outlook

- local space-time efficiency $\Longrightarrow$ use local-space-time mesh refinement?
- convergence, optimality?

References

DOLEJŠÍ V., ERN A., VOHRALÍK M. A framework for robust a posteriori error control in unsteady nonlinear advection-diffusion problems, *SIAM J. Numer. Anal.* **51** (2013), 773–793.

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Thank you for your attention!