Equilibration in $\boldsymbol{H}(\mathrm{curl})$ and applications in a priori and a posteriori magnetostatic analysis

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Outline

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The curl–curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A}) &= \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A} = 0 && \textit{in } \Omega, \\ \boldsymbol{A}\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, && \textit{on } \Gamma_{\mathrm{D}}, \\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0 && \textit{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

The curl–curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla{\cdot}\boldsymbol{j} = 0$)

Weak formulation (consequence)

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Property of the weak solution

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ (primal variable)

Consequence of the weak formulation

$\boldsymbol{h} := \nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$, $\nabla\times\boldsymbol{h} = \boldsymbol{j}$ (dual variable)

Primal Nédélec approximation

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$;

$\boldsymbol{A}_h \in \boldsymbol{V}_h$ *such that*

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h$$

Dual Nédélec approximation

$$\boldsymbol{h}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \Pi_p \boldsymbol{j}}} \|\boldsymbol{v}_h\|^2$$

gives

$$\|\boldsymbol{h} - \boldsymbol{h}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \Pi_p \boldsymbol{j}}} \|\boldsymbol{h} - \boldsymbol{v}_h\|$$

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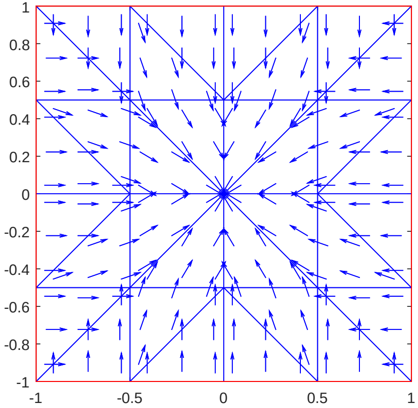
Outline

Equilibration in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)



Flux $\boldsymbol{\iota}_h$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0 \ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

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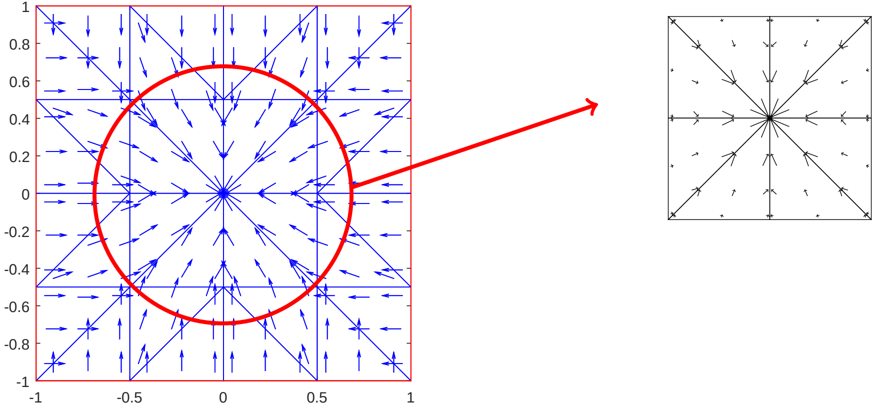

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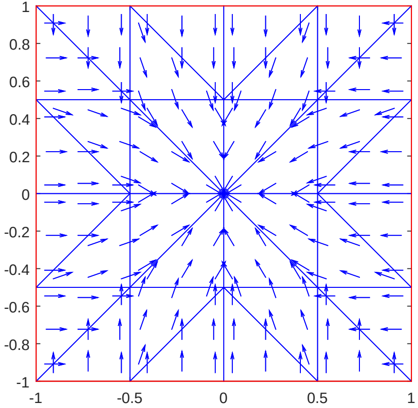

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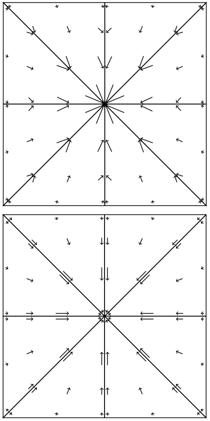

$\psi^{\boldsymbol{a}} \boldsymbol{\iota}_h$

$\boldsymbol{h}_h^{\boldsymbol{a}}$

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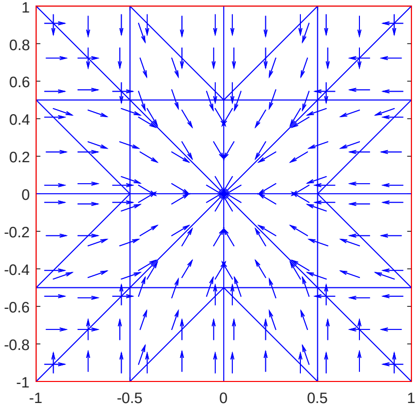
Equilibration in $\boldsymbol{H}(\mathrm{div})$

Destuynder and Métivet (1998), Braess & Schöberl (2008)



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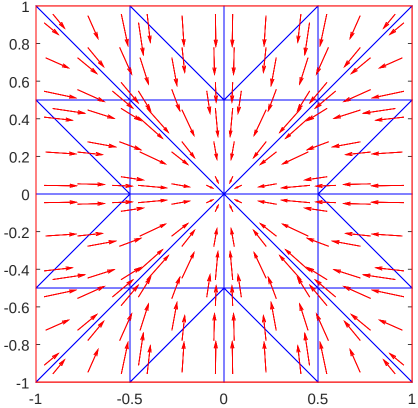

Equilibrated flux reconstruction $\boldsymbol{h}_h$

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Equilibration in $\boldsymbol{H}(\mathrm{curl})$

Previous contributions

- Braess & Schöberl (2008): lowest-order case $p = 0$
- Licht (2019): a conceptual discussion
- Gedicke, Geevers, & Perugia (2020): equilibrated-residual-style construction
- Gedicke, Geevers, Perugia, & Schöberl (2021): p-robust modification

Our construction

$$\underbrace{\iota_h \in \mathcal{N}_p(\mathcal{T}_h), \boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)}_{???=0\ \forall ???}$$

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Equilibration in $\boldsymbol{H}$(curl)

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Outline

1. The curl–curl problem and its Nédélec approximation
2. Equilibration in $\boldsymbol{H}(\mathrm{curl})$
3. A posteriori error estimates in $\boldsymbol{H}(\mathrm{curl})$
4. A stable local commuting projector in $\boldsymbol{H}(\mathrm{curl})$
5. Local-best–global-best equivalence in $\boldsymbol{H}(\mathrm{curl})$
6. Approximation error estimates in $\boldsymbol{H}(\mathrm{curl})$
7. Conclusions

A posteriori error estimates

$\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ s.t. $\nabla\times\boldsymbol{h}_h = \boldsymbol{j}$: local equilibrated flux reconstruction

Theorem (Guaranteed upper bound, efficiency, and p-robustness)

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla\times\boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\text{computable estimator}} \lesssim \underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}}$$

- $\lesssim$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$

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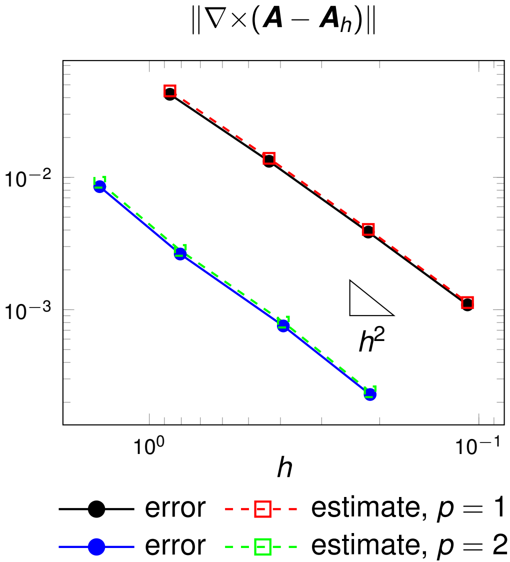
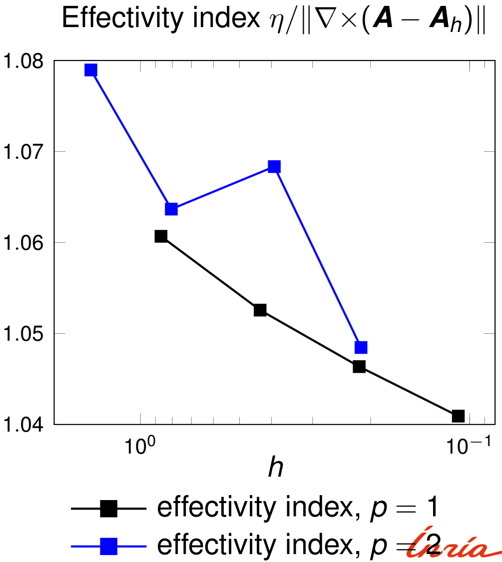
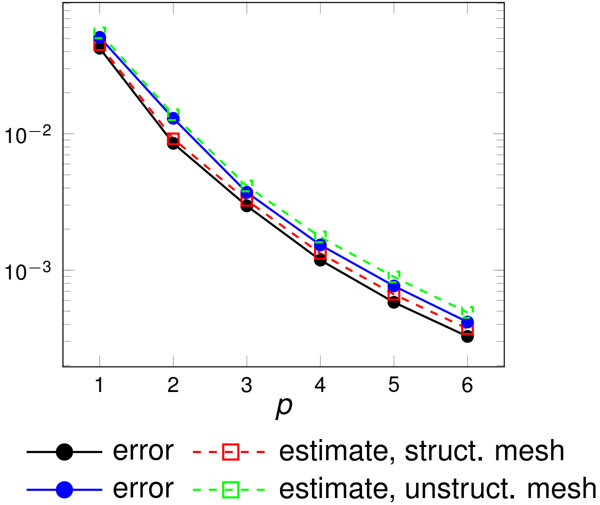
H^3 solution, uniform h-refinement



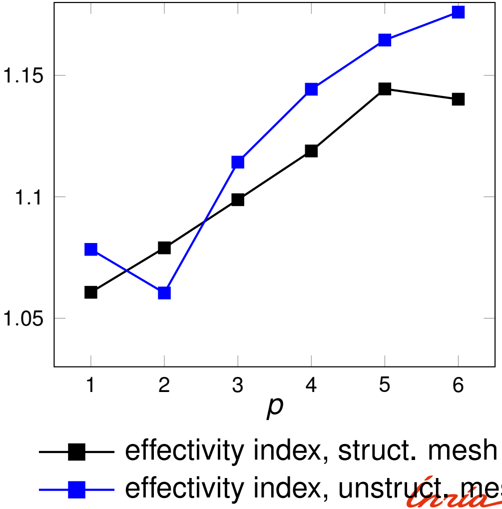


H^3 solution, uniform p-refinement

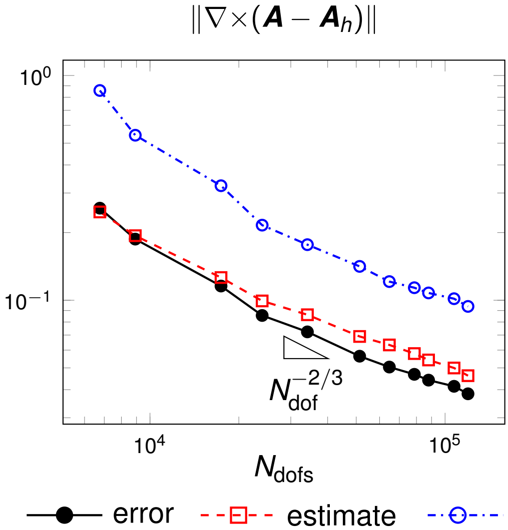
$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$

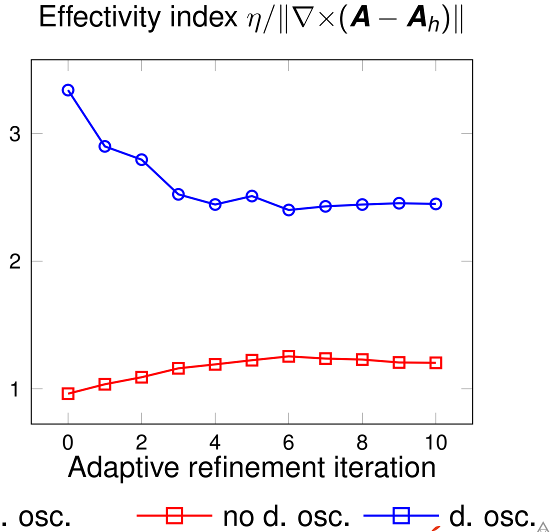


Effectivity index $\eta/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$



Singular solution, adaptive mesh refinement ($p = 2$)





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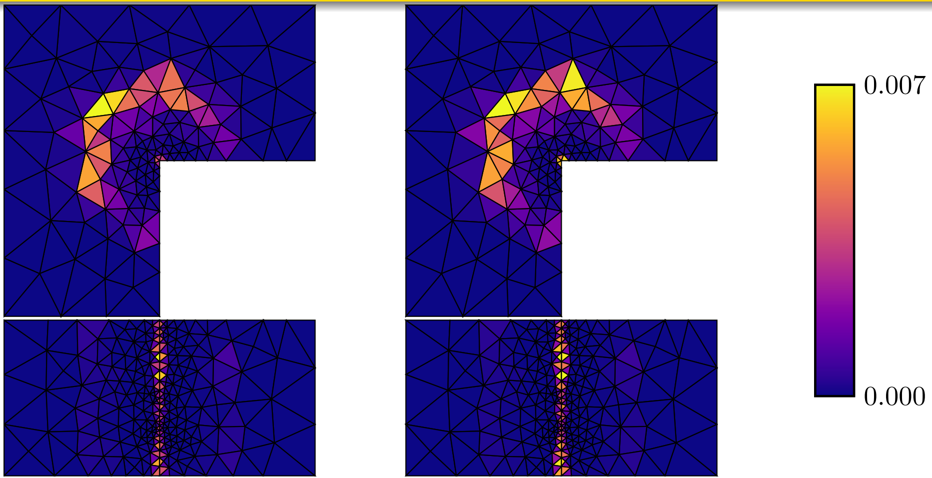

Estimators (left) and actual error (right), adaptive mesh refinement iteration #10.
Top view (top) and side view (bottom)

Outline

Commuting de Rham diagram with operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\Big\downarrow P_h^{p+1,\mathrm{grad}} & & \Big\downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \Big\downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \Big\downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Commuting de Rham diagram with operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathcal{RT}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

Properties of $\boldsymbol{P}_h^{p,\mathrm{curl}}$

1. is defined over the **entire** infinite-dimensional space $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$
2. is defined **locally** (in neighborhood of mesh elements)
3. is defined simply (starting from elementwise polynomial projections)
4. has optimal approximation properties, that of **elementwise curl-unconstrained L^2-orthogonal projector** (local–global equivalence)
5. is **stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting properties** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable
- Ern, Gudi, Smears, and Vohralík (2022): all the above properties in $\boldsymbol{H}(\mathrm{div})$

Outline

Global-best approximation $\approx$ local-best approximation

Previous contributions

- Carstensen, Peterseim, Schedensack (2012): H^1 (lowest-order case $p = 1$)
- Aurada, Feischl, Kemetmüller, Page, Praetorius (2013): H^1 (boundary approximation context)
- Veeser (2016): H^1 (any p)
- Canuto, Nochetto, Stevenson, and Verani (2017): H^1 (improvement of the dependence of the equivalence constant in 2D)
- Ern, Gudi, Smears, and Vohralík (2022): $\boldsymbol{H}(\mathrm{div})$
- Chaumont-Frelet & Vohralík (2021): $\boldsymbol{H}(\mathrm{curl})$ without data oscillation

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

$$\textit{bigger} \approx_p \textit{smaller}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

$$\min_{\textit{smaller space } \textbf{with curl constraints}} \approx_p \min_{\textit{bigger space } \textbf{without curl constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}$(curl)

Theorem (Constrained equivalence in $\boldsymbol{H}$(curl))

$$\min_{\textit{conforming}\text{ Nédélec space }\textbf{\textit{with curl constraints}}} \approx_p \min_{\textit{broken}\text{ Nédélec space }\textbf{\textit{without curl constraints}}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2}_{\text{global-best on } \Omega;\ \text{tangential-trace-continuity constraint};\ \text{curl constraint}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2\right]}_{\text{local-best on each } K\in\mathcal{T}_h;\ \text{no tangential-trace-continuity constraint};\ \text{no curl constraint}}.$$

global-best on Ω
tangential-trace-continuity constraint
curl constraint

local-best on each $K \in \mathcal{T}_h$
no tangential-trace-continuity constraint
no curl constraint

- $\approx_p$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \textbf{curl constraint}}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h} \Bigg[\underbrace{\min_{\boldsymbol{v}_h \in \mathcal{N}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2}_{\substack{\text{local-best on each } K\in\mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \textbf{no curl constraint}}}\Bigg].$$

- $\approx_p$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

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global-best on Ω
tangential-trace-continuity constraint
curl constraint

$$\approx_p \sum_{K\in\mathcal{T}_h} \Bigg[\underbrace{\min_{\boldsymbol{v}_h \in \mathcal{N}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \Pi_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_K \right)^2}_{\text{local-best on each } K\in\mathcal{T}_h} \Bigg].$$

local-best on each $K \in \mathcal{T}_h$
no tangential-trace-continuity constraint
no curl constraint

- $\approx_p$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

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$$\approx_p \sum_{K\in\mathcal{T}_h}\Bigg[\underbrace{\min_{\boldsymbol{v}_h\in\mathcal{N}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \Pi^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_K\right)^2}_{\substack{\text{local-best on each } K\in\mathcal{T}_h\\ \text{no tangential-trace-continuity constraint}\\ \text{no curl constraint}}}\Bigg].$$

- $\approx_p$: only depends on the shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Outline

Approximation error estimates: context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1/2$. *Then*

$$\min_{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl},\Omega)} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

- Nédélec (1980), Hiptmair (2002), Boffi, Brezzi, Fortin (2013)
- Monk (1994, rectangular meshes)

Approximation error estimates: context

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Approximation error estimates: context

hp approximation estimate

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- Melenk and Rojik (2020)
- Ciarlet Jr. (2016)

Approximation error estimates: context

hp approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}^s(\Omega)$, $s > 1$. *Then*

$$\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl},\Omega)} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, s, \cancel{p}) \cancel{\ln(p)} \frac{h^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

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Approximation error estimates

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^s(K), \quad (\nabla \times \boldsymbol{v})|_K \in \boldsymbol{H}^t(K) \qquad \forall K \in \mathcal{T}_h$$

for $s \geq 0$ *and* $s \geq t \geq \max\{0, s-1\}$. *Then*

$$\min_{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)} \left[\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times (\boldsymbol{v} - \boldsymbol{v}_h)\|_K \right)^2 \right]$$

$$\leq C(\kappa_{\mathcal{T}_h}, s, t) \sum_{K \in \mathcal{T}_h} \left[\left(\frac{h_K^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(K)} \right)^2 + \left(\frac{h_K}{p+1} \frac{h_K^{\min\{p+1,t\}}}{(p+1)^t} \|\nabla \times \boldsymbol{v}\|_{\boldsymbol{H}^t(K)} \right)^2 \right].$$

Comments

- *hp* case: $\Gamma_{\mathrm{D}} = \emptyset$ and convex patch subdomains $\omega_{\boldsymbol{a}}$ for all vertices

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Comments

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Outline

Conclusions

Equilibration in $\boldsymbol{H}(\mathrm{curl})$:

- guaranteed, locally efficient, and p-robust a posteriori error estimates
- a stable local commuting projector
- local-best–global-best equivalence
- local hp approximation (a priori) error estimates under min. Sobolev regularity

Conclusions

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CHAUMONT-FRELET T., VOHRALÍK M. Equivalence of local-best and global-best approximations in ***H***(curl). *Calcolo* **58** (2021), 53.

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Thank you for your attention!

Conclusions

Equilibration in *H*(curl):

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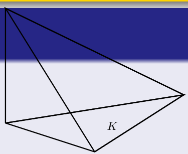
Outline

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Definition (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)



Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ be given (minimal regularity).

1. For each $K \in \mathcal{T}_h$, prepare the datum $\boldsymbol{\tau}_h|_K$

$$\boldsymbol{\tau}_h|_K := \arg \min_{\substack{\boldsymbol{w}_h \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{w}_h = 0}} \|\nabla\times\boldsymbol{v} - \boldsymbol{w}_h\|_K$$

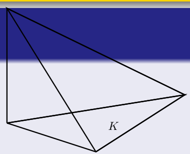
   and define $\boldsymbol{\iota}_h|_K$ by the **elementwise (constrained) projection**

$$\boldsymbol{\iota}_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(K) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

   (discrete, tangential-trace discontinuous).

2. Obtain $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ by applying the **flux equilibration procedure** to $\boldsymbol{\iota}_h$; in particular, $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) := \boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}$, where $\boldsymbol{h}_h^{\boldsymbol{a}}$ are obtained by local energy minimizations on the patch subdomains $\omega_{\boldsymbol{a}}$.

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$



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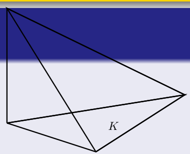
$$\boldsymbol{\iota}_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(K) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

   (discrete, tangential-trace discontinuous).

2. Obtain $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ by applying the **flux equilibration procedure** to $\boldsymbol{\iota}_h$; in particular, $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) := \boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}$, where $\boldsymbol{h}_h^{\boldsymbol{a}}$ are obtained by local energy minimizations on the patch subdomains $\omega_{\boldsymbol{a}}$.

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Definition (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)

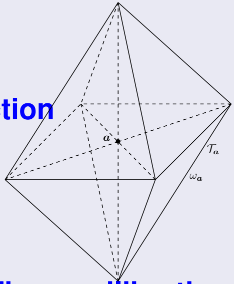


Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ be given (minimal regularity).

1. For each $K \in \mathcal{T}_h$, prepare the datum $\boldsymbol{\tau}_h|_K$

$$\boldsymbol{\tau}_h|_K := \arg \min_{\substack{\boldsymbol{w}_h \in \mathcal{RT}_p(K) \\ \nabla \cdot \boldsymbol{w}_h = 0}} \|\nabla \times \boldsymbol{v} - \boldsymbol{w}_h\|_K$$

and define $\boldsymbol{\iota}_h|_K$ by the **elementwise (constrained) projection**



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A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Theorem (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)

$\boldsymbol{P}_h^{p,\mathrm{curl}}$ *is a* **commuting projector** *since*

$$\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega),$$
$$\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v}\in\mathcal{N}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega).$$

Moreover, it has **local-best approximation properties** *and is* L^2 **stable** *up to data oscillation, since, for all* $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ *and* $K\in\mathcal{T}_h$,

$$\|\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times(\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}))\|_K\right)^2$$
$$\lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{N}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\},$$

$$\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 \lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\|\boldsymbol{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}.$$

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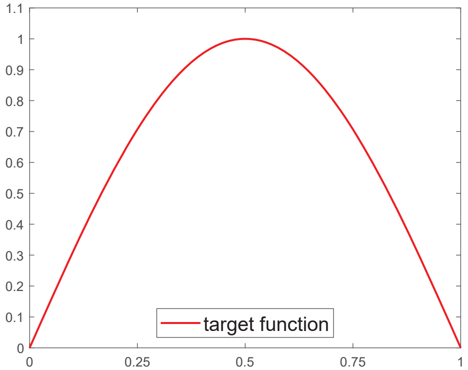
$$\lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\},$$

$$\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 \lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\|\boldsymbol{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}.$$

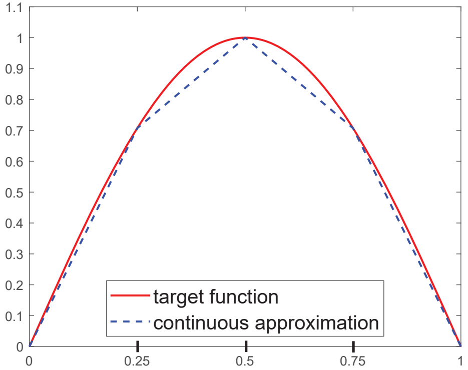
Outline

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**



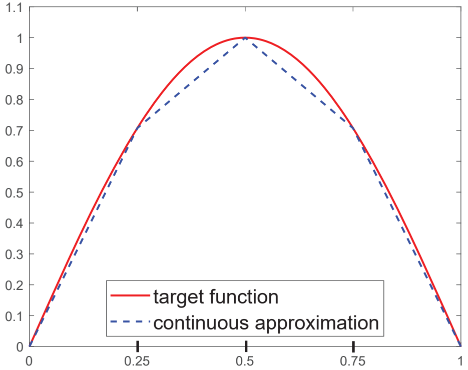
Target function in $H_0^1(\Omega)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

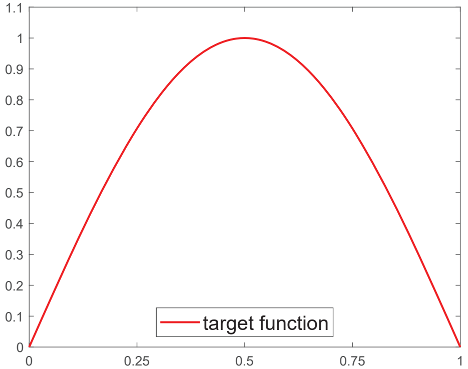


Approximation by **continuous**
piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$: **1D**

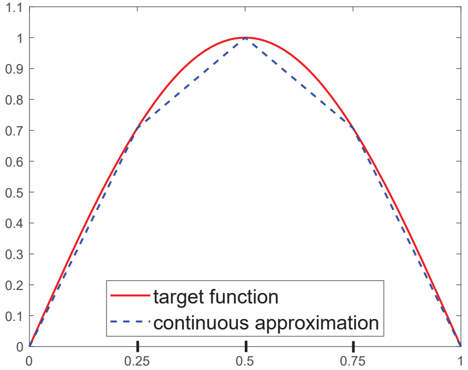


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$

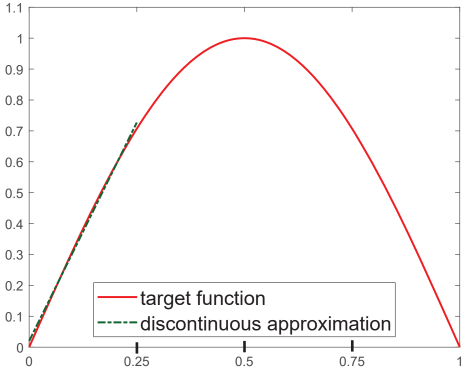


Target function in $H^1_0(\Omega)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

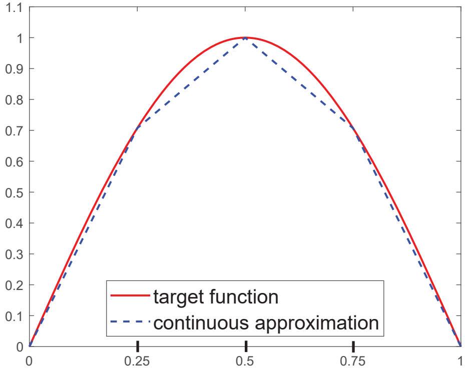


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

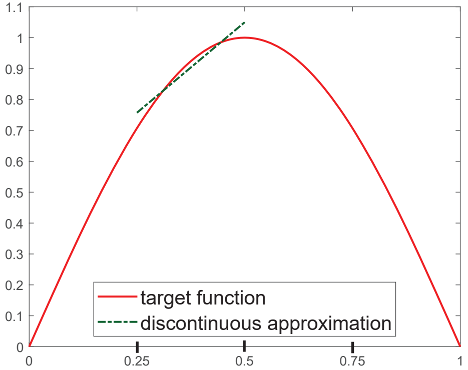


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

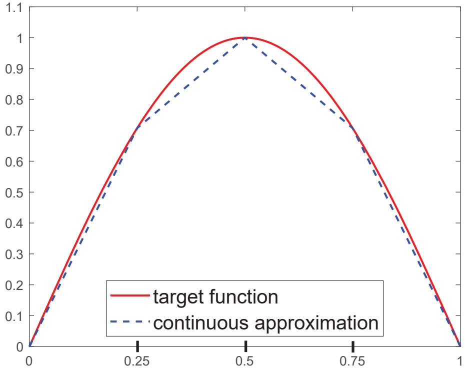


Approximation by **continuous**
piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

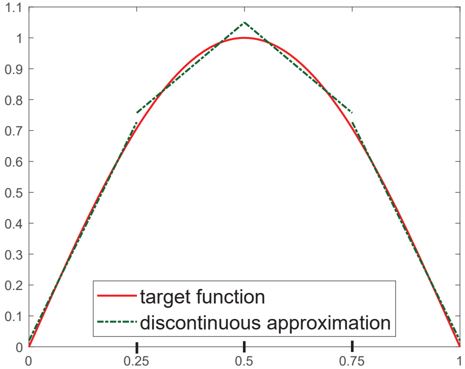


Approximation by **discontinuous**
piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

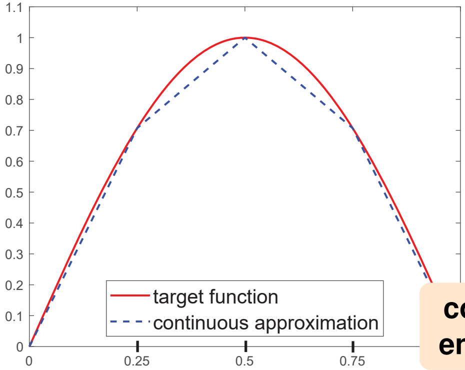


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

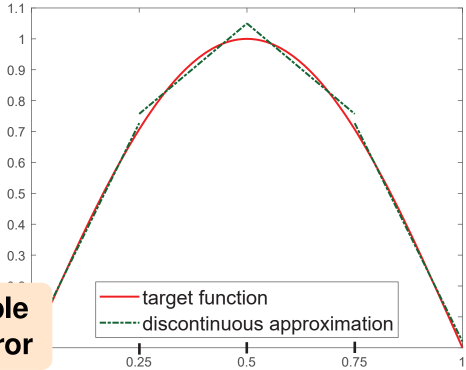


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**



comparable energy error



Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$