Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors

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Outline

1. Introduction
2. Laplace eigenvalue problem equivalences
   - Generic equivalences
   - Dual norm of the residual equivalences
   - Representation of the residual and eigenvalue bounds
3. A posteriori estimates
   - Eigenvalues
   - Eigenvectors
   - Improvements under elliptic regularity
4. Application to conforming finite elements
5. Numerical experiments
6. Extension to nonconforming discretizations
7. Conclusions and future directions

Setting

Energy minimization ($\Omega \subset \mathbb{R}^d$, $d = 2, 3$, polygon/polyhedron)
Find $u_1 \in V := H_0^1(\Omega)$ such that $(u_1, 1) > 0$ and

$$u_1 := \arg \inf_{v \in V, \|v\|=1} \left\{ \frac{1}{2} \|\nabla v\|^2 \right\}.$$

Laplace eigenvalue problem
Find eigenvector & eigenvalue pair (u_1, λ_1) such that

$$\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 && \text{in } \Omega, \\ u_1 &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Full problem, weak formulation
Find $(u_k, \lambda_k) \in V \times \mathbb{R}^+$, $k \geq 1$, with $\|u_k\| = 1$, such that

$$(\nabla u_k, \nabla v) = \lambda_k (u_k, v) \qquad \forall v \in V \Rightarrow \|\nabla u_k\|^2 = \lambda_k.$$

- $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \to \infty$
- u_k, $k \geq 1$, form an orthonormal basis of $L^2(\Omega)$

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Previous results, Laplace eigenvalue bounds

- Plum (1997), Goerisch and He (1989), Still (1988), Kuttler and Sigillito (1978), Moler and Payne (1968), Fox and Rheinboldt (1966), Bazley and Fox (1961), Weinberger (1956), Forsythe (1955), Kato (1949)
- ...

Previous results, guaranteed lower bounds on λ_1

- Carstensen and Gedicke (2014): $\oplus$ guaranteed bound, arbitrarily coarse mesh; $\ominus$ a priori arguments (largest mesh element diameter), only lowest-order nonconforming FEs
- Hu, Huang, Lin (2014): $\oplus$ bounds in nonconforming FEs; $\ominus$ saturation assumption may be necessary
- Armentano and Durán (2004): $\oplus$ bounds in nonconforming FEs; $\ominus$ only asymptotic
- Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013): $\oplus$ general guaranteed bounds; $\ominus$ condition on applicability, suboptimal convergence speed
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Previous results, Laplace eigenvector bounds

- Rannacher, Westenberger, Wollner (2010), Grubišić and Ovall (2009), Durán, Padra, Rodríguez (2003), Heuveline and Rannacher (2002), Larson (2000), Maday and Patera (2000), Verfürth (1994) ...
- ...typically contain uncomputable terms, higher-order on fine enough meshes

The game

Assumption A (Conforming variational solution)

There holds

- $(u_h, \lambda_h) \in V \times \mathbb{R}^+$
- $\|u_h\| = 1$
- $(u_h, 1) > 0$
- $\|\nabla u_h\|^2 = \lambda_h$ $(\Rightarrow \lambda_h \geq \lambda_1)$

We want to estimate

1. first eigenvalue error

$$\tilde{\eta}(u_h, \lambda_h) \leq \sqrt{\lambda_h - \lambda_1} \leq \eta(u_h, \lambda_h)$$

2. first eigenvector energy error

$$\|\nabla(u_1 - u_h)\| \leq \eta(u_h, \lambda_h)$$

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$$\|\nabla(u_1 - u_h)\| \leq \eta(u_h, \lambda_h) \leq C_{\text{eff}} \|\nabla(u_1 - u_h)\|$$

- C_{eff} only depends on the shape regularity of the mesh
- we give computable upper bounds on C_{eff}

The pathway

1. estimate the $L^2(\Omega)$ error:

$$\|u_1 - u_h\| \leq \alpha_h$$

2. prove equivalence of the eigenvalue & eigenvector errors:

$$C\|\nabla(u_1 - u_h)\|^2 \leq \lambda_h - \lambda_1 \leq \|\nabla(u_1 - u_h)\|^2$$

3. prove equivalence of the eigenvector error & of the dual norm of the residual:

$$\underline{C}\|\mathrm{Res}(u_h, \lambda_h)\|_{-1} \leq \|\nabla(u_1 - u_h)\| \leq \bar{C}\|\mathrm{Res}(u_h, \lambda_h)\|_{-1},$$

where

$$\langle \mathrm{Res}(u_h, \lambda_h), v\rangle_{V',V} := \lambda_h(u_h, v) - (\nabla u_h, \nabla v) \qquad v \in V$$

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4. prove equivalence of the dual residual norm & its estimate:

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$L^2(\Omega)$ bound

Lemma ($L^2(\Omega)$ bound via a quadratic residual inequality)

Let Assumption A hold and let

$$\lambda_h < \lambda_2$$

and

$$\beta_h := \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \|\mathfrak{r}_{(h)}\| < 1,$$

$$\alpha_h^2 := 2\left(1 - \sqrt{1 - \beta_h^2}\right) \leq |\Omega|^{-1}(u_h, 1)^2.$$

Then

$$\|u_1 - u_h\| \leq \alpha_h.$$

Riesz representation of the residual $\mathfrak{r}_{(h)} \in V$

$$(\nabla \mathfrak{r}_{(h)}, \nabla v) = \langle \mathrm{Res}(u_h, \lambda_h), v \rangle_{V',V} \qquad \forall v \in V$$

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$(\mathscr{r}_{(h)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla \mathscr{r}_{(h)}) = \frac{1}{\lambda_k}\left(\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)\right)$$
$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $\mathscr{r}_{(h)}$

$$\|\mathscr{r}_{(h)}\|^2 =$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

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assumption $\lambda_h < \lambda_2$:

$$\min_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

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dropping the first term above, $e_h := \|u_1 - u_h\|^2$:

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|r_{(h)}\|^2 \geq 0$$

quadratic residual inequality in e_h, under assumption on β_h:

$$e_h \leq 2\left(1 - \sqrt{1 - \beta_h^2}\right) \quad \text{or} \quad e_h \geq 2(1 + \sqrt{1 - \beta_h^2})$$

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First eigenvalue error equivalences

Theorem (Eigenvalue error – eigenvector error equivalence)

Under the above assumptions, there holds

$$\frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\left(1-\frac{\alpha_h^2}{4}\right)\|\nabla(u_1-u_h)\|^2 \leq \lambda_h-\lambda_1 \leq \|\nabla(u_1-u_h)\|^2,$$

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Key arguments of the proof

- there holds
$$\lambda_h-\lambda_1=\|\nabla(u_h-u_1)\|^2-\lambda_1\|u_1-u_h\|^2$$
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First eigenvector error equivalences

Theorem (Eigenvector error – dual norm of the residual equivalence)

Under the above assumptions, there holds

$$\left(\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1\right)^{-1} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}^2$$

$$\leq \|\nabla(u_1 - u_h)\|^2 \leq \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-2} \left(1 - \frac{\alpha_h^2}{4}\right)^{-1} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}^2,$$

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First eigenvector error equivalences

Key arguments of the proof I.

- use $\|\nabla v\|^2 = \sum_{k\geq 1} \lambda_k (v, u_k)^2$ for $v = \mathscr{r}_{(h)}$:

$$\|\nabla \mathscr{r}_{(h)}\|^2 = \sum_{k\geq 1} \lambda_k (\mathscr{r}_{(h)}, u_k)^2 = \sum_{k\geq 1} \lambda_k \left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2$$

- obtain as in the $L^2(\Omega)$ bound lemma:

$$\|\nabla \mathscr{r}_{(h)}\|^2 \geq C_h \|\nabla(u_1 - u_h)\|^2 - \frac{C_h}{4} \|\nabla(u_1 - u_h)\|^2 \alpha_h^2$$

- estimate in the other direction:

$$\|\nabla \mathscr{r}_{(h)}\|^2 \leq \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k\geq 2} \lambda_k (u_h - u_1, u_k)^2$$

$$\leq \lambda_1 \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 + \|\nabla(u_1 - u_h)\|^2$$

First eigenvector error equivalences

Key arguments of the proof II.

- for the last estimate:

$$\begin{aligned} &\|\nabla(u_1 - u_h)\|^2 \\ &= (\nabla(u_1 - u_h), \nabla(u_1 - u_h)) \\ &= \lambda_1(u_1, u_1 - u_h) + \langle \mathrm{Res}(u_h, \lambda_h), u_1 - u_h\rangle_{V',V} - \lambda_h(u_h, u_1 - u_h) \\ &= \langle \mathrm{Res}(u_h, \lambda_h), u_1 - u_h\rangle_{V',V} + \frac{\lambda_1 + \lambda_h}{2}\|u_1 - u_h\|^2 \end{aligned}$$

- Young inequality:

$$\|\nabla(u_1 - u_h)\|^2 \leq \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}^2 + (\lambda_1 + \lambda_h)\|u_1 - u_h\|^2$$

- finish by $\lambda_h \geq \lambda_1$ & $L^2(\Omega)$ bound $\|u_1 - u_h\| \leq \alpha_h$

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Motivation

Weak solution

$$(\nabla u_1, \nabla v) = \lambda_1 (u_1, v) \ \forall v \in V \Rightarrow -\nabla u_1 \in \mathbf{H}(\mathrm{div}, \Omega), \nabla\cdot(-\nabla u_1) = \lambda_1 u_1$$

Ideal discrete imitation $(-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega))$

$$\sigma_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \lambda_h u_h} \|\nabla u_h + \mathbf{v}_h\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega) \Rightarrow$ **global** minimization, too expensive

Local flux reconstruction (partition of unity cut-off)

$$\sigma_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\overbrace{\psi_{\mathbf{a}}}^{\text{hat function}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

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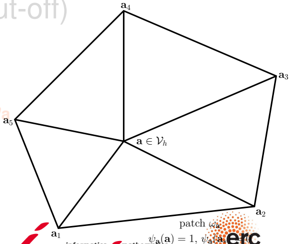

Destuynder & Métivet (1999), Braess & Schöberl (2008)

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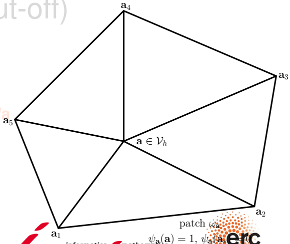

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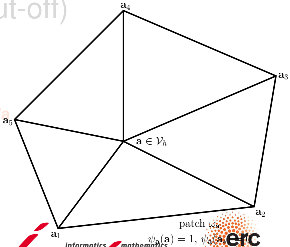

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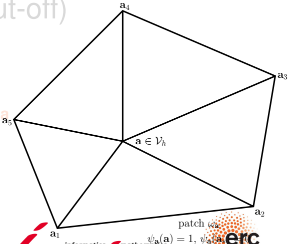

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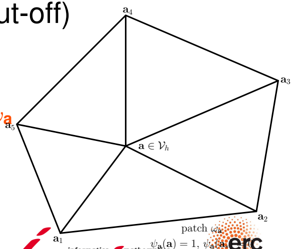

Destuynder & Métivet (1999), Braess & Schöberl (2008)

$H_0^1(\Omega)$- and $\mathbf{H}(\mathrm{div},\Omega)$-conforming local residual liftings

Definition (Mixed local Neumann problems: equilibrated flux)

For all $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ by solving

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\\ \nabla\cdot\mathbf{v}_h = \Pi_{Q_h}(\psi_{\mathbf{a}}\lambda_h u_h - \nabla u_h\cdot\nabla\psi_{\mathbf{a}})}} \|\psi_{\mathbf{a}}\nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h,$$

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Definition (Conforming local Neumann problems: lifted residual)

For each $\mathbf{a} \in \mathcal{V}_h$, define $r_h^{\mathbf{a}} \in X_h^{\mathbf{a}} \subset H^1(\omega_{\mathbf{a}})$ by

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \mathrm{Res}(u_h,\lambda_h), \psi_{\mathbf{a}} v_h\rangle_{V',V} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

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Babuška & Strouboulis (2001), Repin (2008)

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For each $\mathbf{a} \in \mathcal{V}_h$, define $r_h^{\mathbf{a}} \in X_h^{\mathbf{a}} \subset H^1(\omega_{\mathbf{a}})$ by

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = \langle \mathrm{Res}(u_h,\lambda_h), \psi_{\mathbf{a}} v_h\rangle_{V',V} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

Then set

$$r_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \psi_{\mathbf{a}} r_h^{\mathbf{a}} \in V.$$

Babuška & Strouboulis (2001), Repin (2008)

$H_0^1(\Omega)$- and $\mathbf{H}(\mathrm{div},\Omega)$-conforming local residual liftings

Definition (Mixed local Neumann problems: equilibrated flux)

For all $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ by solving

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\ \nabla\cdot\mathbf{v}_h = \Pi_{Q_h}(\psi_{\mathbf{a}}\lambda_h u_h - \nabla u_h \cdot \nabla \psi_{\mathbf{a}})}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h,$$

and set $\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$, $\nabla\cdot\boldsymbol{\sigma}_h = \lambda_h u_h$.

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Numerical assumptions

Assumption B (Galerkin orthogonality of the residual to $\psi_{\mathbf{a}}$)

There holds, for all $\mathbf{a} \in \mathcal{V}_h^{\rm int}$,

$$\lambda_h(u_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathrm{Res}(u_h, \lambda_h), \psi_{\mathbf{a}} \rangle_{V', V} = 0.$$

Assumption C (Shape regularity & piecewise polynomial form)

The meshes $\mathcal{T}_h$ *are shape regular. There holds*

$u_h \in \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, *and spaces* $\mathbf{V}_h \times Q_h$ *are of degree* $p+1$.

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Dual norm of the residual equivalences

Theorem (Dual norm of the residual equivalences)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ verifying Assumption B be arbitrary. Then

$$\frac{\langle \mathrm{Res}(u_h, \lambda_h), r_h \rangle_{V',V}}{\|\nabla r_h\|} \le \|\mathrm{Res}(u_h, \lambda_h)\|_{-1} \le \|\nabla u_h + \boldsymbol{\sigma}_h\|.$$

Moreover, under Assumption C, there holds

$$\|\nabla u_h + \boldsymbol{\sigma}_h\| \le (d+1) C_{\mathrm{st}} C_{\mathrm{cont,PF}} \|\mathrm{Res}(u_h, \lambda_h)\|_{-1}.$$

- C_{st} and $C_{\mathrm{cont,PF}}$ independent of the polynomial degree p
- we can compute upper bounds on C_{st} and $C_{\mathrm{cont,PF}}$

Ern & Vohralík (2015)

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Dual norm of the residual bounds

Sketch of the proof.

equilibrated flux $\boldsymbol{\sigma}_h$ definition, Green's theorem, CS inequality:

$$\langle \mathrm{Res}(u_h,\lambda_h),v\rangle_{V',V} = \lambda_h(u_h,v)-(\nabla u_h,\nabla v) = (\nabla\cdot\boldsymbol{\sigma}_h,v)-(\nabla u_h,\nabla v)$$
$$= -(\nabla u_h+\boldsymbol{\sigma}_h,\nabla v) \le \|\nabla u_h+\boldsymbol{\sigma}_h\|\|\nabla v\|$$

dual norm and residual lifting r_h definitions:

$$\sup_{v\in V,\,\|\nabla v\|=1} \langle \mathrm{Res}(u_h,\lambda_h),v\rangle_{V',V}$$
$$\ge \frac{\langle \mathrm{Res}(u_h,\lambda_h),r_h\rangle_{V',V}}{\|\nabla r_h\|} = \frac{\sum_{\mathbf{a}\in\mathcal{V}_h}\langle \mathrm{Res}(u_h,\lambda_h),\psi_{\mathbf{a}} r_h^{\mathbf{a}}\rangle_{V',V}}{\|\nabla r_h\|}$$
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Bounds on the Riesz representation of the residual

Lemma (Poincaré–Friedrichs bound on $\|r_{(h)}\|$)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ be arbitrary. There holds

$$\|r_{(h)}\| \leq \frac{1}{\sqrt{\lambda_1}} \|\nabla r_{(h)}\|.$$

Lemma (Elliptic regularity bound on $\|r_{(h)}\|$)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ satisfy Assumption B and let the solution $\zeta_{(h)}$ of

$$(\nabla \zeta_{(h)}, \nabla v) = (r_{(h)}, v) \qquad \forall v \in V$$

belong to $H^{1+\delta}(\Omega)$, $0 < \delta \leq 1$, with

$$\inf_{v_h \in V_h} \|\nabla(\zeta_{(h)} - v_h)\| \leq C_{\mathrm{I}} h^{\delta} |\zeta_{(h)}|_{H^{1+\delta}(\Omega)},$$

$$|\zeta_{(h)}|_{H^{1+\delta}(\Omega)} \leq C_{\mathrm{S}} \|r_{(h)}\|.$$

Then

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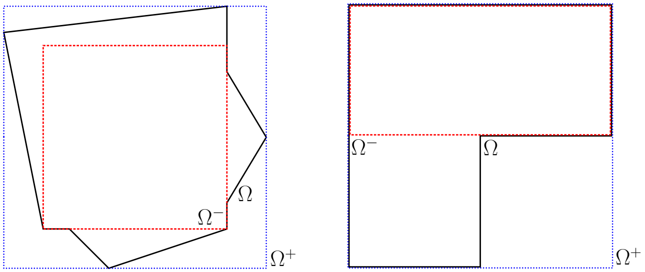
Estimates of eigenvalues via domain inclusion



$$\Omega \subset \Omega^+ \quad \Rightarrow \quad \lambda_k \geq \lambda_k(\Omega^+), \qquad \forall k \geq 1$$
$$\Omega \supset \Omega^- \quad \Rightarrow \quad \lambda_k \leq \lambda_k(\Omega^-),$$

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Guaranteed bounds for the first eigenvalue

Theorem (Eigenvalue bounds)

Let $0 < \underline{\lambda_2} \le \lambda_2$ and $0 < \underline{\lambda_1} \le \lambda_1$. Let $\lambda_h < \underline{\lambda_2}$ and let Assumptions A and B hold. With $\boldsymbol{\sigma}_h$ and r_h from above, let

$$\underbrace{\beta_h}_{\searrow 0} := \frac{1}{\sqrt{\underline{\lambda_1}}} \left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-1} \|\nabla u_h + \boldsymbol{\sigma}_h\| < 1,$$

$$\underbrace{\alpha_h^2}_{\searrow 0} := 2\left(1 - \sqrt{1 - \beta_h^2}\right) \le |\Omega|^{-1}(u_h, 1)^2.$$

Then

$$\lambda_1 \ge \lambda_h - \overbrace{\underbrace{\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-2}}_{\text{no if elliptic reg.}} \underbrace{\left(1 - \frac{\alpha_h^2}{4}\right)^{-1}}_{\searrow 1} \|\nabla u_h + \boldsymbol{\sigma}_h\|^2}^{\eta^2},$$

$$\lambda_1 \le \lambda_h - \frac{1}{2}\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)\left(1 - \frac{\alpha_h^2}{4}\right)\frac{\underline{\lambda_1}}{2}\left(\sqrt{1 + \frac{4}{\underline{\lambda_1}} \frac{\langle \mathrm{Res}(u_h, \lambda_h), r_h\rangle^2_{V', V}}{\|\nabla r_h\|^2}} - 1\right).$$

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Guaranteed bounds for the first eigenvector

Theorem (Eigenvector bounds)

Under the assumptions of the eigenvalue theorem,

$$\|\nabla(u_1 - u_h)\| \leq \eta.$$

Moreover, under Assumption C,

$$\eta \leq (d+1) C_{\mathrm{cont,PF}} C_{\mathrm{st}} \underbrace{\left(\frac{\|\nabla(u_1-u_h)\|^2}{\lambda_1} + 1\right)^{\frac{1}{2}}}_{\searrow 1}$$

$$\underbrace{\left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1}}_{\searrow \left(1-\frac{\lambda_1}{\lambda_2}\right)^{-1}} \underbrace{\left(1 - \frac{\alpha_h^2}{4}\right)^{-\frac{1}{2}}}_{\searrow 1} \|\nabla(u_1 - u_h)\|.$$

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Comments

Eigenvalue bounds

- **guaranteed**
- **optimally convergent**
- **improvement of the upper bound**
- valid under explicit, a posteriori verifiable conditions

Eigenvector bounds

- **efficient** and **polynomial-degree robust**
- $\|\nabla u_h + \sigma_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \sigma_h\|_K^2 \Rightarrow$ **adaptivity-ready**
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Improved bounds for the first eigenvalue

Theorem (Elliptic regularity eigenvalue bounds)

Let the elliptic regularity bound on $\|r_{(h)}\|$ *hold. Let*

$$\overbrace{\left(1-\frac{\lambda_h}{\underline{\lambda_2}}\right)^{-1} C_{\rm I} C_{\rm S} h^{\delta}}^{\gamma_h \searrow 0} \|\nabla u_h + \boldsymbol{\sigma}_h\| =: \beta_h < 1,$$

$$\alpha_h^2 := 2\left(1-\sqrt{1-\beta_h^2}\right) \le |\Omega|^{-1}(u_h, 1)^2.$$

Then

$$\lambda_1 \ge \lambda_h - \overbrace{(1+4\lambda_h\gamma_h^2)}^{\searrow 1}\|\nabla u_h + \boldsymbol{\sigma}_h\|^2,$$

$$\lambda_1 \le \lambda_h + 2\lambda_h\gamma_h^2\|\nabla u_h+\boldsymbol{\sigma}_h\|^2 - \frac{\underline{\lambda_1}}{2}\left(\sqrt{1+\frac{4}{\underline{\lambda_1}}\frac{\langle \mathrm{Res}(u_h,\lambda_h), r_h\rangle_{V',V}^2}{\|\nabla r_h\|^2}}-1\right).$$

Improved bounds for the first eigenvalue

Theorem (Elliptic regularity eigenvalue bounds)

Let the elliptic regularity bound on $\| r_{(h)} \|$ *hold. Let*

$$\overbrace{\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-1} C_{\mathrm{I}} C_{\mathrm{S}} h^{\delta}}^{\gamma_h \searrow 0} \|\nabla u_h + \boldsymbol{\sigma}_h\| =: \beta_h < 1,$$

$$\alpha_h^2 := 2\left(1 - \sqrt{1 - \beta_h^2}\right) \leq |\Omega|^{-1} (u_h, 1)^2.$$

Then

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Improved bounds for the first eigenvector

Theorem (Elliptic regularity eigenvector bounds)

Let the assumptions of the elliptic regularity eigenvalue theorem be verified. Then

$$\|\nabla(u_1 - u_h)\|^2 \leq (1 + 4\lambda_h \gamma_h^2)\|\nabla u_h + \boldsymbol{\sigma}_h\|^2.$$

Moreover, under Assumption C, this estimator is efficient as above.

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Application to conforming finite elements

Finite element method

Find $(u_h, \lambda_h) \in V_h \times \mathbb{R}^+$ with $\|u_h\| = 1$ and $(u_h, 1) > 0$, where $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap V$, $p \geq 1$, such that,

$$(\nabla u_h, \nabla v_h) = \lambda_h(u_h, v_h) \qquad \forall v_h \in V_h.$$

Assumptions verification

- $V_h \subset V$
- $\|u_h\| = 1$ and $(u_h, 1) > 0$ by definition
- $\|\nabla u_h\|^2 = \lambda_h$ follows upon taking $v_h = u_h$ ($\Rightarrow$ Assumption A)
- Assumption B follows upon taking $v_h = \psi_{\mathbf{a}} \in V_h$
- Assumption C is technical

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Unit square

Setting

- $\Omega = (0,1)^2$
- $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$ known explicitly
- $u_1(x,y) = \sin(\pi x)\sin(\pi y)$ known explicitly

Parameters

- convex domain: $C_{\mathrm{S}} = 1$, $\delta = 1$, $C_{\mathrm{I}} \approx 1/\sqrt{8}$
- $\underline{\lambda_1} = 1.5\pi^2$, $\underline{\lambda_2} = 4.5\pi^2$

Effectivity indices

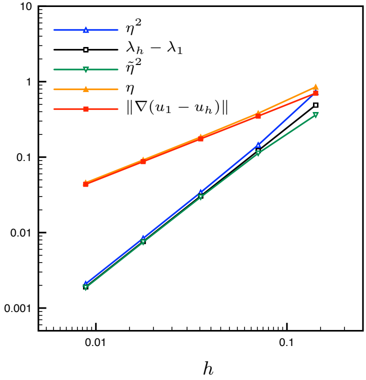
- recall $\tilde{\eta}^2 \le \lambda_h - \lambda_1 \le \eta^2$

$$I_{\lambda,\mathrm{eff}}^{\mathrm{lb}} := \frac{\lambda_h - \lambda_1}{\tilde{\eta}^2}, \qquad I_{\lambda,\mathrm{eff}}^{\mathrm{ub}} := \frac{\eta^2}{\lambda_h - \lambda_1}$$

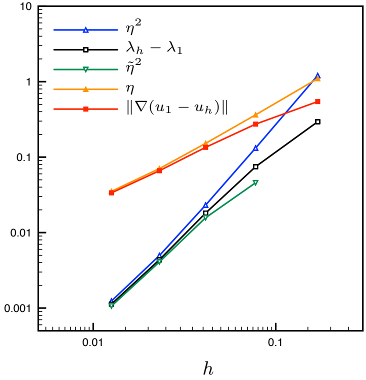
- recall $\|\nabla(u_1 - u_h)\| \le \eta$

$$I_{u,\mathrm{eff}}^{\mathrm{ub}} := \frac{\eta}{\|\nabla(u_1 - u_h)\|}$$

Eigenvalue and eigenvector errors and estimators



Structured meshes



Unstructured meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Structured meshes

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Unstructured meshes

L-shaped domain

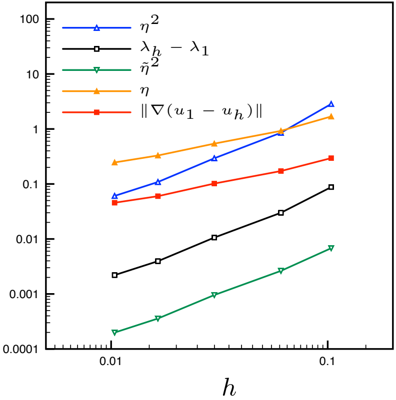
Setting

- $\Omega := (-1,1)^2 \setminus [0,1] \times [-1,0]$
- $\lambda_1 \approx 9.6397238440$

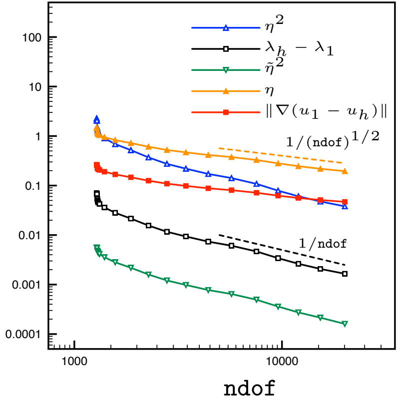
Parameters

- $\underline{\lambda_1} = \pi^2/2$ and $\underline{\lambda_2} = 5\pi^2/4$ by inclusion into the square $(-1,1)^2$

Eigenvalue and eigenvector errors and estimators



Unstructured meshes



Adaptively refined meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
30	0.1038	826	9.63972	9.72744	6.88126	9.72064	12.90	32.45	3.42E-01	5.72
60	0.0608	3,154	9.63972	9.66968	8.81618	9.66705	11.39	28.49	9.21E-02	5.38
120	0.0299	12,747	9.63972	9.65032	9.35716	9.64937	11.08	27.65	3.07E-02	5.32
240	0.0164	49,119	9.63972	9.64367	9.53508	9.64331	11.03	27.51	1.13E-02	5.49
360	0.0104	114,806	9.63972	9.64192	9.58128	9.64173	11.08	27.55	6.29E-03	5.40

Unstructured meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
30	0.1038	826	9.63972	9.72744	6.88126	9.72064	12.90	32.45	3.42E-01	5.72
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Unstructured meshes

Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
2	1,282	9.63972	9.70858	7.56083	9.70303	12.39	31.19	2.48E-01	5.62
6	1,294	9.63972	9.68971	8.35342	9.68509	10.83	26.73	1.48E-01	5.19
10	1,396	9.63972	9.67581	8.77643	9.67225	10.12	24.92	9.71E-02	4.98
14	2,792	9.63972	9.65137	9.37756	9.65016	9.63	23.51	2.87E-02	4.80
18	7,538	9.63972	9.64438	9.53634	9.64389	9.44	23.19	1.12E-02	4.60
22	20,071	9.63972	9.64137	9.60336	9.64122	10.30	23.01	3.93E-03	4.16

Adaptively refined meshes

Eigenvalue and eigenvector errors and estimators

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
30	0.1038	826	9.63972	9.72744	6.88126	9.72064	12.90	32.45	3.42E-01	5.72
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Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
1	176	9.63972	10.0518	5.43638	9.99630	7.43	11.20	5.91E-01	3.33
6	190	9.63972	9.94166	7.12891	9.89532	6.52	9.32	3.25E-01	3.04
11	426	9.63972	9.72012	9.04493	9.70628	5.81	8.40	7.05E-02	2.90
16	1,533	9.63972	9.66102	9.48546	9.65725	5.65	8.24	1.79E-02	2.87
21	5,671	9.63972	9.64535	9.59920	9.64435	5.61	8.20	4.69E-03	2.75
26	20,587	9.63972	9.64125	9.62872	9.64101	6.14	8.19	1.28E-03	2.45

Adaptively refined meshes, λ_h in place of λ_1, $\lambda_{h,2}$ in place of λ_2

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Nonconforming discretizations

Nonconforming setting

- $u_h \notin V$, $\|u_h\| \neq 1$
- $\|\nabla u_h\|^2 \neq \lambda_h$

Main tools

- conforming projection, scaling

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in V; \quad \tilde{s} := \frac{s}{\|s\|}$$

- conforming eigenvector reconstruction

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in W_h^{\mathbf{a}} \subset H_0^1(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}, \qquad s_h := \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}$$

Unified framework

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin elements
- mixed finite elements

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Conclusions and future directions

Conclusions

- guaranteed upper and lower bounds for the first eigenvalue
- guaranteed and polynomial-degree robust bounds for the associated eigenvector
- general framework

Ongoing work

- extension to nonlinear eigenvalue problems

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Thank you for your attention!