A posteriori error estimates robust with respect to nonlinearities and orthogonal decomposition based on iterative linearization

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Inria Paris & Ecole des Ponts

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Outline

Setting

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

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\begin{aligned}
-\nabla\cdot(a(|\nabla u|)\nabla u) &= f \quad \text{in} \quad \Omega, \\
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- a strongly monotone and Lipschitz continuous
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- numerical approximation u_ℓ

Goals

Error control

a posteriori error estimates

$$|||u - u_\ell||| \leq \eta(u_\ell)$$

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Guaranteed a posteriori error estimates

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Guaranteed a posteriori error estimates **efficient**

$$|||u - u_\ell||| \leq \eta(u_\ell) \leq C_{\text{eff}}|||u - u_\ell|||,$$

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Guaranteed a posteriori error estimates **efficient** and **robust** with respect to the **strength of nonlinearities**:

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Main question

- **what to choose for $|||\cdot|||$?**

Previous results

Sobolev norm (not robust wrt $\frac{a_c}{a_m}$)

$$a_m \|\nabla(u_\ell - u)\| \leq \eta(u_\ell) \leq C_{\text{eff}} a_c \|\nabla(u_\ell - u)\|$$

- Pousin & Rappaz (1994), Verfürth (1994), Kim (2007), Houston, Süli, & Wihler (2008), Garau, Morin, & Zuppa (2011), Gantner, Haberl, Praetorius, & Stiftner (2018), Heid & Wihler (2020), ...

Energy difference

$$\mathcal{J}(u_\ell) - \mathcal{J}(u) \leq \eta(u_\ell)^2 \leq C_{\text{eff}}^2 \frac{a_c^2}{a_m^2} (\mathcal{J}(u_\ell) - \mathcal{J}(u))$$

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$\mathcal{E}_\ell^k = \mathcal{J}(u_\ell^k) - \mathcal{J}(u)$ at linearization convergence

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Previous use of augmented error norms

- advection-dominated problems: augmenting the energy norm by the dual norm of the skew-symmetric part: robustness wrt advection (Verfürth 2005)
- parabolic pbs: augmenting by temporal jumps: space-time local efficiency

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$$\langle \mathcal{R}(w), v\rangle := (a(|\nabla w|)\nabla w, \nabla v) - (f, v) \quad v \in H_0^1(\Omega)$$

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- **iteration-dependent norm** induced by the **linearization scalar product**

$$|||v|||^2_{1,u_\ell^{k-1}} := ((v,\, v))_{u_\ell^{k-1}} = \|(L_\ell^{k-1})^{1/2} v\|^2 + \|(\boldsymbol{A}_\ell^{k-1})^{1/2} \nabla v\|^2 \quad v \in H_0^1(\Omega)$$

New result #2b

Theorem (Orthogonal decomposition of the total error into linearization and discretization components)

For all linearization steps $k \geq 1$, there holds

$$\underbrace{|||\mathcal{R}(u_\ell^{k-1})|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{total residual/error} \\ |||u_\ell^{k-1}-u^k_{\langle\ell\rangle}|||_{1,u_\ell^{k-1}}}} = \underbrace{|||u_\ell^{k-1} - u_\ell^k|||^2_{1,u_\ell^{k-1}}}_{\substack{\text{linearization} \\ \text{error}}} + \underbrace{|||\mathcal{R}^{u_\ell^{k-1}}_{\mathrm{disc}}(u_\ell^k)|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{discretization residual/error} \\ |||u_\ell^k-u^k_{\langle\ell\rangle}|||_{1,u_\ell^{k-1}}}}.$$

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- **orthogonal decomposition**
- **error components**
- $u^k_{\langle\ell\rangle} \in H^1_0(\Omega)$ such that

$$((u^k_{\langle\ell\rangle} - u_\ell^{k-1}, v))_{u_\ell^{k-1}} = -\langle\underbrace{\mathcal{R}(u_\ell^{k-1})}_{\text{residual}}, v\rangle \qquad \forall v \in H^1_0(\Omega)$$

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1. Introduction
2. Gradient-dependent nonlinearities
   - Setting
   - Iterative linearization
   - A posteriori error estimates for an augmented energy difference
   - Fenchel conjugate, dual energy, flux equilibration, estimator
   - Numerical experiments
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4. Conclusions

A model nonlinear problem

Nonlinear elliptic problem

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Assumption (Nonlinear function a)

Function $a : [0, \infty) \to (0, \infty)$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$|a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}| \leq a_{\mathrm{c}}|\boldsymbol{x} - \boldsymbol{y}| \qquad \text{(Lipschitz continuity)},$$
$$(a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq a_{\mathrm{m}}|\boldsymbol{x} - \boldsymbol{y}|^2 \qquad \text{(strong monotonicity)}.$$

A model nonlinear problem

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot(a(|\nabla u|)\nabla u) = f \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

- $\Omega \subset \mathbb{R}^d$, $1 \leq d \leq 3$, open bounded polytope with Lipschitz boundary $\partial\Omega$
- f piecewise polynomial for simplicity

Assumption (Nonlinear function a)

Function $a : [0,\infty) \to (0,\infty)$, *for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$,

$$|a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}| \leq a_{\rm c}|\boldsymbol{x} - \boldsymbol{y}| \qquad \textit{(Lipschitz continuity)},$$
$$(a(|\boldsymbol{x}|)\boldsymbol{x} - a(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \geq a_{\rm m}|\boldsymbol{x} - \boldsymbol{y}|^2 \qquad \textit{(strong monotonicity)}.$$

- $a_{\rm m} \leq a(r) \leq a_{\rm c}$, $a_{\rm m} \leq (a(r)r)' \leq a_{\rm c}$

Example of the nonlinear function a

Example (Mean curvature nonlinearity)

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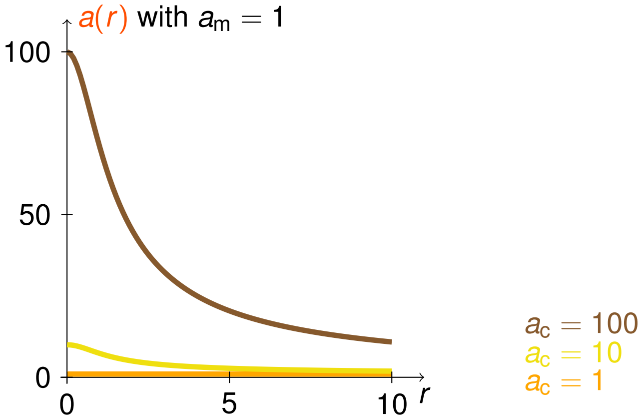

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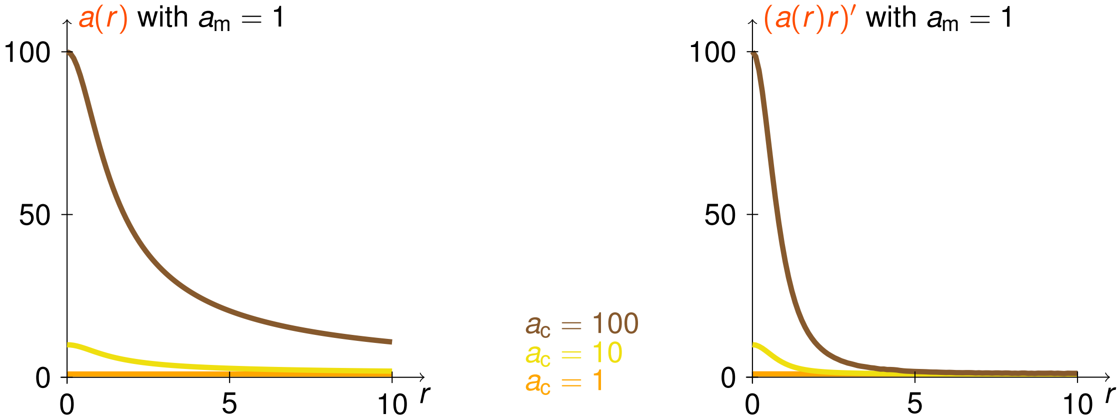

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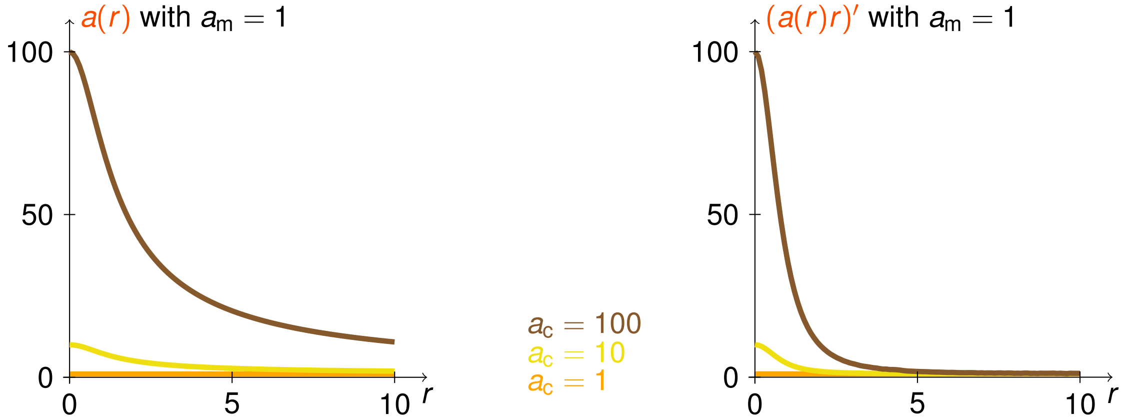
Strength of the nonlinearity

$$\frac{a_{\rm c}}{a_{\rm m}} = \frac{\text{Lipschitz continuity}}{\text{strong monotonicity}}$$



Outline

Weak solution

Definition (Weak solution)

$u \in H_0^1(\Omega)$ such that

$$(a(|\nabla u|)\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega).$$

Energy

Definition (Energy functional)

$\mathcal{J} : H_0^1(\Omega) \to \mathbb{R}$

$$\mathcal{J}(v) := \int_\Omega \phi(|\nabla v|) - (f, v), \quad v \in H_0^1(\Omega),$$

with function $\phi : [0, \infty) \to [0, \infty)$ such that, for all $r \in [0, \infty)$,

$$\phi(r) := \int_0^r a(s) s \, \mathrm{d}s.$$

Equivalently

$$u = \arg \min_{v \in H_0^1(\Omega)} \mathcal{J}(v)$$

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Finite element approximation

Definition (Finite element approximation)

$u_\ell \in V_\ell^p$ such that

$$(a(|\nabla u_\ell|)\nabla u_\ell, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell^p.$$

- $\mathcal{T}_\ell$ simplicial mesh of Ω
- $p \geq 1$ polynomial degree
- $V_\ell^p := \mathcal{P}_p(\mathcal{T}_\ell) \cap H_0^1(\Omega)$
- conforming finite elements

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Energy difference

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$$\boxed{\mathcal{J}(u_\ell) - \mathcal{J}(u)}$$

- $\mathcal{J}(u_\ell) - \mathcal{J}(u) \geq 0$, $\mathcal{J}(u_\ell) - \mathcal{J}(u) = 0$ if and only if $u_\ell = u$
- physically-based error measure

Outline

Iterative linearization

Need to solve a nonlinear system

$$\mathcal{A}_\ell(\mathsf{U}_\ell) = \mathsf{F}_\ell$$

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Definition (Linearized finite element approximation)

$u_\ell^k \in V_\ell^p$ such that

$$(\boldsymbol{A}_\ell^{k-1}\nabla u_\ell^k, \nabla v_\ell) = (f, v_\ell) + (\boldsymbol{b}_\ell^{k-1}, \nabla v_\ell) \qquad \forall v_\ell \in V_\ell^p.$$

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- $u_\ell^0 \in V_\ell^p$ a given initial guess
- iterative linearization index $k \geq 1$
- **linearization**: $\boldsymbol{A}_\ell^{k-1}\colon \Omega \to \mathbb{R}^{d\times d}$ matrix, $\boldsymbol{b}_\ell^{k-1}\colon \Omega \to \mathbb{R}^d$ vector constructed from u_ℓ^{k-1}

Examples

Example (Picard (fixed-point))

$$\mathbf{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\mathbf{I}_d, \quad \mathbf{b}_\ell^{k-1} = \mathbf{0}.$$

Example (Zarantonello)

$$\mathbf{A}_\ell^{k-1} = \gamma \mathbf{I}_d, \quad \mathbf{b}_\ell^{k-1} = (\gamma - a(|\nabla u_\ell^{k-1}|))\nabla u_\ell^{k-1},$$

with $\gamma \geq \frac{a_c^2}{a_m}$ *a constant parameter.*

Example (Newton)

$$\mathbf{A}_\ell^{k-1} = a(|\nabla u_\ell^{k-1}|)\mathbf{I}_d + \frac{a'(|\nabla u_\ell^{k-1}|)}{|\nabla u_\ell^{k-1}|}\nabla u_\ell^{k-1} \otimes \nabla u_\ell^{k-1},$$

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A posteriori error estimates for an augmented energy difference

Theorem (A posteriori estimate of augmented energy)

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- ✓ C_ℓ^k **computable**: we can affirm **robustness *a posteriori***, for the given case

A posteriori error estimates for an augmented energy difference

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$$\mathcal{E}_\ell^k = \frac{1}{2}\text{energy difference} + \lambda_\ell^k \times \frac{1}{2}(\text{linearized energy difference})$$

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Practically

$\mathcal{E}_\ell^k = \mathcal{J}(u_\ell^k) - \mathcal{J}(u)$ at convergence

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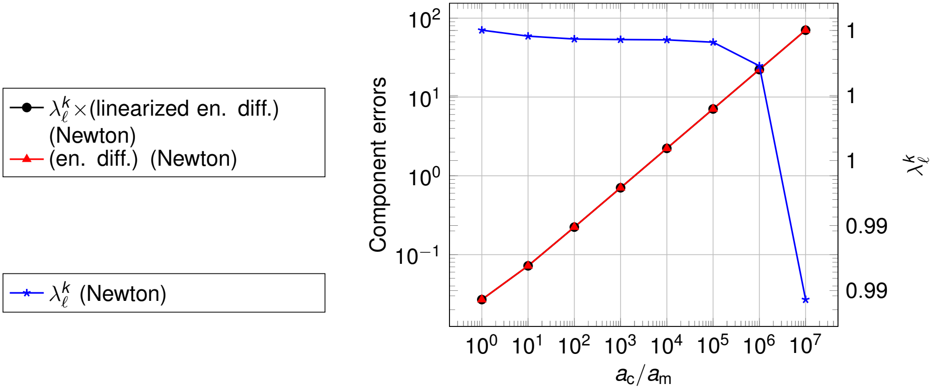

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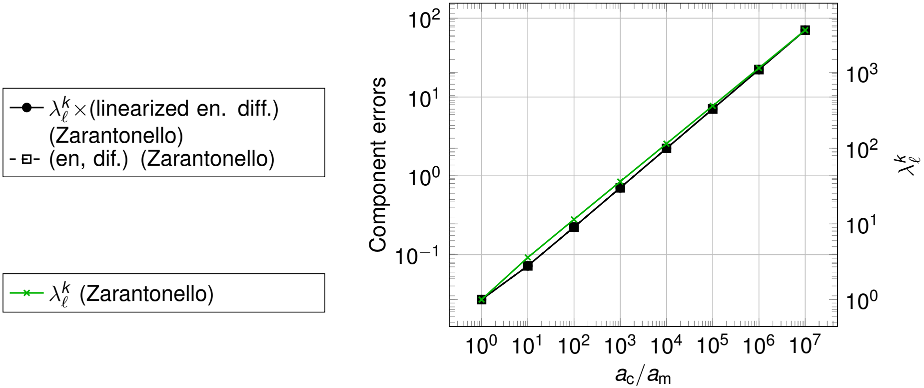

Outline

1. Introduction
2. Gradient-dependent nonlinearities
   - Setting
   - Iterative linearization
   - A posteriori error estimates for an augmented energy difference
   - Fenchel conjugate, dual energy, flux equilibration, estimator
   - Numerical experiments
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   - Setting
   - A posteriori error estimates for an iteration-dependent norm
   - Numerical experiments
4. Conclusions

Fenchel conjugate, dual energy, flux equilibration, estimator

Definition (Fenchel conjugate)

$$\phi^*(\cdot, s) := \sup_{r \in [0,\infty)} (sr - \phi(\cdot, r)).$$

Definition (Dual energy)

$$\mathcal{J}^*(\boldsymbol{v}) := -\int_\Omega \phi^*(\cdot, |\boldsymbol{v}|), \quad \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega).$$

Definition (Flux equilibration: $\boldsymbol{\sigma}_\ell^k = \sum_{\boldsymbol{a} \in \mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\boldsymbol{a},k}$)

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Definition (Estimator)

$$\eta_\ell^k := \frac{1}{2} \underbrace{(\mathcal{J}(u_\ell^k) - \mathcal{J}^*(\boldsymbol{\sigma}_\ell^k))}_{\text{en. diff. estimate}} + \lambda_\ell^k \frac{1}{2} \underbrace{(\mathcal{J}_\ell^{k-1}(u_\ell^k) - \mathcal{J}_\ell^{*,k-1}(\boldsymbol{\sigma}_\ell^k))}_{\text{linearized en. diff. estimate}}$$

Fenchel conjugate, dual energy, flux equilibration, estimator

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Outline

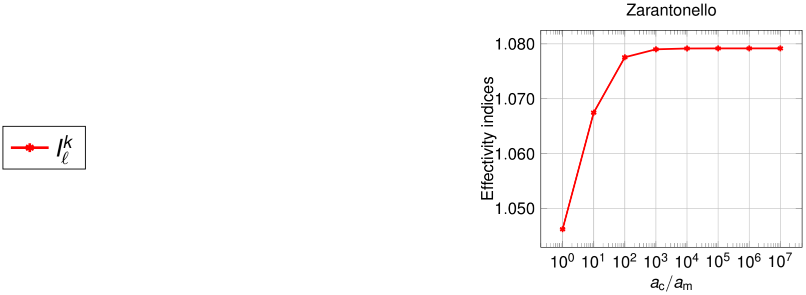
Smooth solution

Setting

- unit square $\Omega = (0,1)^2$
- known smooth solution $u(x,y) := 10\,x(x-1)y(y-1)$
- $p = 1$
- effectivity indices

$$\underbrace{I_\ell^k := \left(\frac{\eta_\ell^k}{\mathcal{E}_\ell^k}\right)^{\frac{1}{2}}}_{\text{total}}, \quad \underbrace{I_{\mathrm{N},\ell}^k := \left(\frac{\mathcal{J}(u_\ell^k) - \mathcal{J}^*(\boldsymbol{\sigma}_\ell^k)}{\mathcal{J}(u_\ell^k) - \mathcal{J}(u)}\right)^{\frac{1}{2}}}_{\text{energy difference}}$$

How **large** is the error? **Robustness** wrt the nonlinearities $\left(a(r) = a_{\mathrm{m}} + \frac{a_{\mathrm{c}} - a_{\mathrm{m}}}{\sqrt{1+r^2}}\right)$



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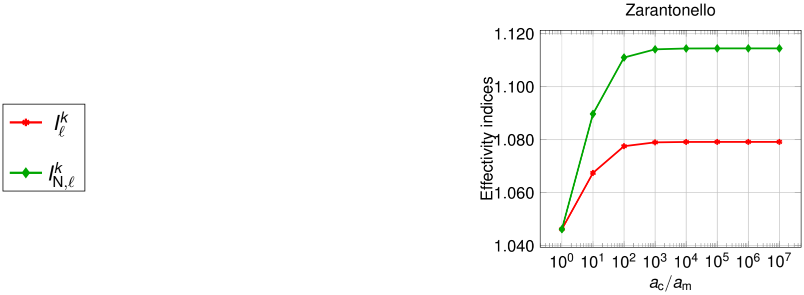

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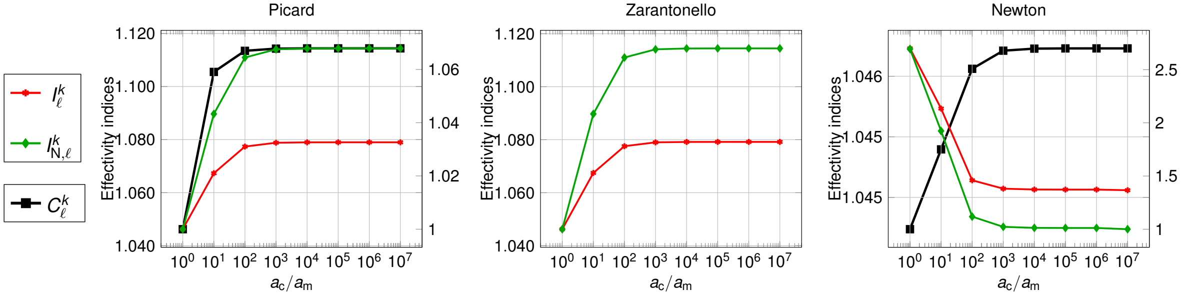
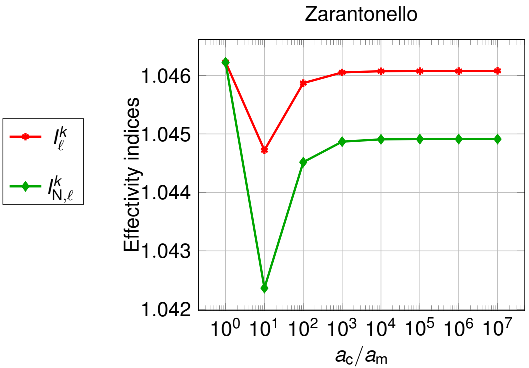


A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, preprint (2023)

How **large** is the error? **Robustness** wrt the nonlinearities

$$\left(a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}})\frac{1-e^{-\frac{3}{2}r^2}}{1+2e^{-\frac{3}{2}}}\right)$$



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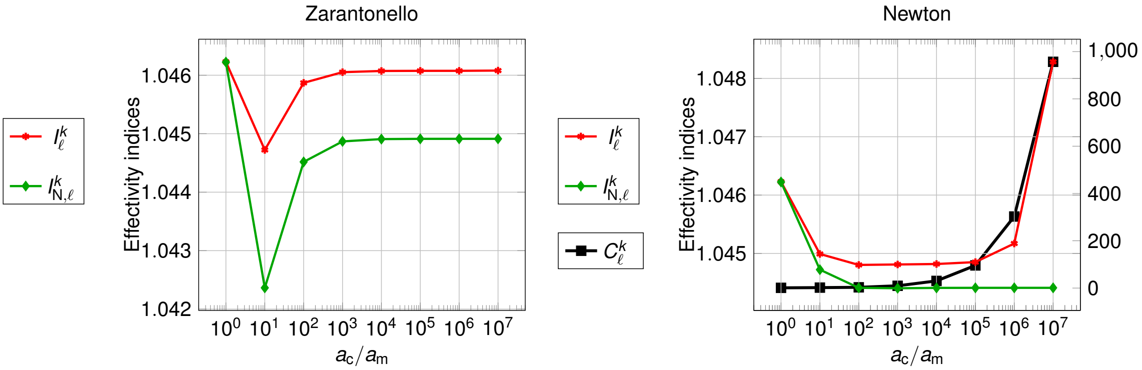
$(a(r) = a_{\rm m} + (a_{\rm c} - a_{\rm m})\frac{1-e^{-\frac{3}{2}r^2}}{1+2e^{-\frac{3}{2}}}$, robustness only for Zarantonello)



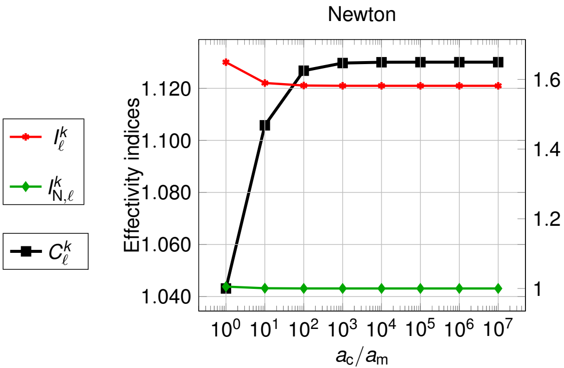
A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, preprint (2023)

Singular solution

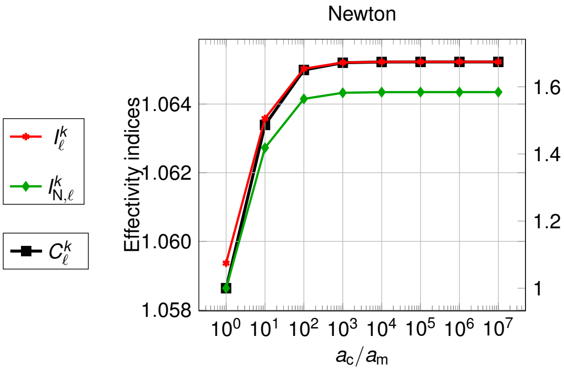
Setting

- L-shaped domain $\Omega = (-1,1)^2 \setminus ([0,1) \times (-1,0])$
- known singular solution $u(\rho,\theta) = \rho^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$
- $a(r) = a_{\mathrm{m}} + (a_{\mathrm{c}} - a_{\mathrm{m}}) \frac{1 - e^{-\frac{3}{2} r^2}}{1 + 2e^{-\frac{3}{2}}}$
- $p = 1$
- uniform or adaptive mesh refinement

How **large** is the error? **Robustness** wrt the nonlinearities



Uniform mesh refinement



Adaptive mesh refinement

A. Harnist, K. Mitra, A. Rappaport, M. Vohralík, preprint (2023)

Outline

Observation

Observation

Not all nonlinear problems admit an energy minimization structure.

A model nonlinear problem

Nonlinear elliptic problem

Find $u : \Omega \to \mathbb{R}$ such that

$$-\nabla\cdot\Big(\tau \boldsymbol{K}(\boldsymbol{x})\big(\underbrace{\mathcal{D}(\boldsymbol{x},u)}_{\text{diffusion}}\nabla u + \underbrace{\boldsymbol{q}(\boldsymbol{x},u)}_{\text{advection}}\big)\Big) + \underbrace{f(\boldsymbol{x},u)}_{\text{reaction}} = 0 \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial\Omega.$$

- $\tau > 0$ a parameter (time step size in transient problems: applies to Richards on each time step)

Assumption (Nonlinear functions $\mathcal{D}$, $\boldsymbol{q}$, and f)

$|\mathcal{D}(\boldsymbol{x}_1, u_1) - \mathcal{D}(\boldsymbol{x}_2, u_2)| \le \mathcal{D}_{\mathrm{M}}(|\boldsymbol{x}_1 - \boldsymbol{x}_2| + |u_1 - u_2|) \quad \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}$,

$0 \le f(\boldsymbol{x}, u_2) - f(\boldsymbol{x}, u_1) \le f_{\mathrm{M}}(u_2 - u_1) \quad \forall \boldsymbol{x} \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}, u_2 \ge u_1$,

$\boldsymbol{q}$ *is "small" wrt* $\boldsymbol{K}\mathcal{D}$.

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Outline

Finite element discretization and iterative linearization

Definition (Linearized finite element approximation)

$u_\ell^k \in V_\ell^p$ such that

$$((u_\ell^k - u_\ell^{k-1}, v_\ell))_{u_\ell^{k-1}} = -\langle \underbrace{\mathcal{R}(u_\ell^{k-1})}_{\text{residual}}, v_\ell \rangle \qquad \forall v_\ell \in V_\ell^p.$$

- covers most linearization schemes: Picard (fixed-point), L & M-schemes, ...
- linearization: **reaction–diffusion scalar product**

$$((w, v))_{u_\ell^{k-1}} := (\underbrace{L_\ell^{k-1}}_{\text{reaction coef.}} w, v) + (\underbrace{A_\ell^{k-1}}_{\text{diffusion coef.}} \nabla w, \nabla v), \qquad w, v \in H_0^1(\Omega)$$

Iteration-dependent norm

- $\|v\|^2_{1, u_\ell^{k-1}} := ((v, v))_{u_\ell^{k-1}} = \|(L_\ell^{k-1})^{1/2} v\|^2 + \|(A_\ell^{k-1})^{1/2} \nabla v\|^2, \qquad v \in H_0^1(\Omega)$
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An orthogonal decomposition of the total residual/error

Theorem (Orthogonal decomposition of the total error into linearization and discretization components)

For all linearization steps $k \geq 1$, there holds

$$\underbrace{|||\mathcal{R}(u_\ell^{k-1})|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{total residual/error} \\ |||u_\ell^{k-1} - u^k_{\langle\ell\rangle}|||_{1,u_\ell^{k-1}}}} = \underbrace{|||u_\ell^{k-1} - u_\ell^k|||^2_{1,u_\ell^{k-1}}}_{\substack{\text{linearization} \\ \text{error}}} + \underbrace{|||\mathcal{R}^{u_\ell^{k-1}}_{\text{disc}}(u_\ell^k)|||^2_{-1,u_\ell^{k-1}}}_{\substack{\text{discretization residual/error} \\ |||u_\ell^k - u^k_{\langle\ell\rangle}|||_{1,u_\ell^{k-1}}}}.$$

- orthogonal decomposition
- error components
- $u^k_{\langle\ell\rangle} \in H^1_0(\Omega)$ such that

$$((u^k_{\langle\ell\rangle} - u_\ell^{k-1}, v))_{u_\ell^{k-1}} = -\underbrace{\langle \mathcal{R}(u_\ell^{k-1}), v\rangle}_{\text{residual}} \qquad \forall v \in H^1_0(\Omega)$$

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Outline

A posteriori error estimates for an iteration-dependent norm

Theorem (A posteriori estimate of iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta(u_\ell^k).$$

A posteriori error estimates for an iteration-dependent norm

Theorem (A posteriori estimate of iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta(u_\ell^k).$$

Moreover, for all linearization steps $k \geq 1$, there holds

$$\eta(u_\ell^k) \leq C_{\text{eff}}(d, \kappa_{\mathcal{T}}, p) C_\ell^k |||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \quad + \textit{quadrature error terms},$$

A posteriori error estimates for an iteration-dependent norm

Theorem (A posteriori estimate of iteration-dependent norm)

For all linearization steps $k \geq 1$,

$$|||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1}} \leq \eta(u_\ell^k).$$

Moreover, for all linearization steps $k \geq 1$ and **for each element** $K \in \mathcal{T}_\ell$*, there holds*

$$\eta_K(u_\ell^k) \leq C_{\mathrm{eff}}(d, \kappa_{\mathcal{T}}, p) C_K^k |||\mathcal{R}(u_\ell^{k-1})|||_{-1,u_\ell^{k-1},\omega_K} + \text{quadrature error terms},$$

A posteriori error estimates for an iteration-dependent norm

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- ✓ **local efficiency**

Outline

One time step of the Richards equation

Setting

- unit square $\Omega = (0,1)^2$
- realistic data

$$f(\boldsymbol{x},u) = S(u) - S(u_\ell^{n-1}(\boldsymbol{x})), \quad \mathcal{D}(\boldsymbol{x},u) = \kappa(S(u)), \quad \boldsymbol{q}(\boldsymbol{x},u) = -\kappa(S(u))\,\boldsymbol{g},$$

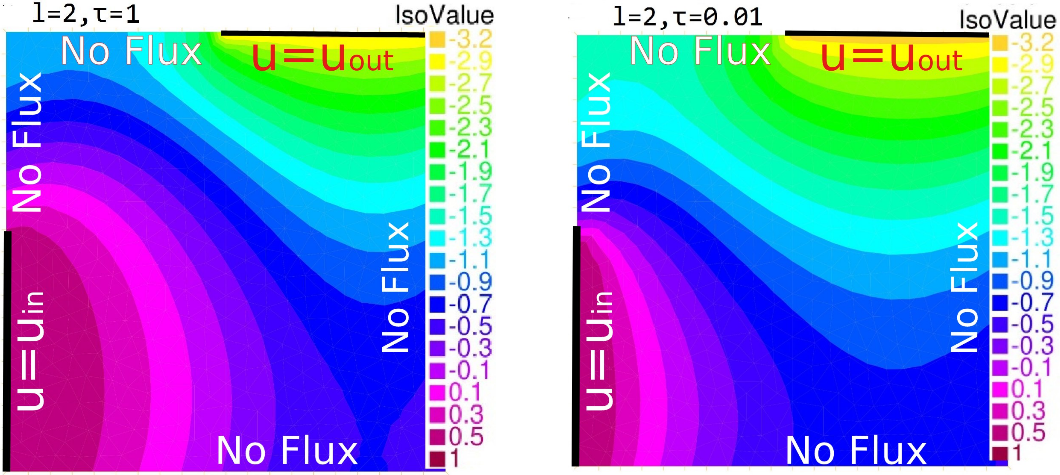
$$\boldsymbol{K} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **van Genuchten saturation** and **permeability** laws

$$S(u) := \left(1 + (2-u)^{\frac{1}{1-\lambda}}\right)^{-\lambda}, \quad \kappa(s) := \sqrt{s}\left(1 - (1 - s^{\frac{1}{\lambda}})^{\lambda}\right)^2, \quad \lambda = 0.5$$

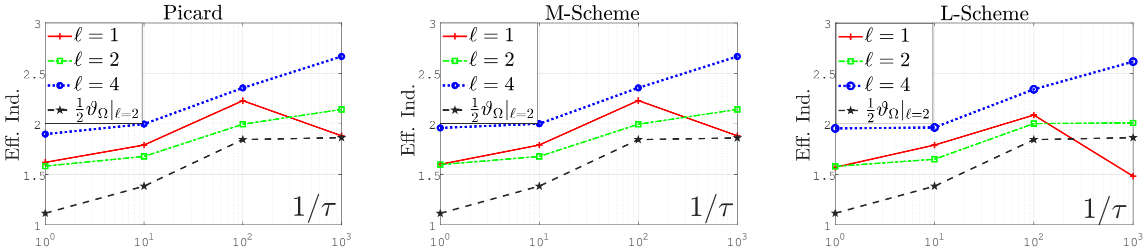
- time step length $\tau \in [10^{-3}, 1]$

One time step of the Richards equation: saturation u



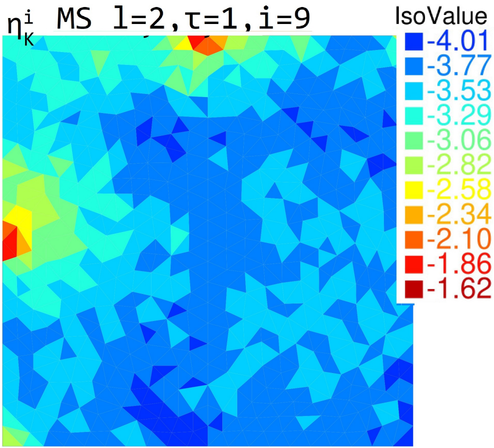
Time step length $\tau = 1$ Time step length $\tau = 0.01$

How **large** is the error? **Robustness** wrt the nonlinearities

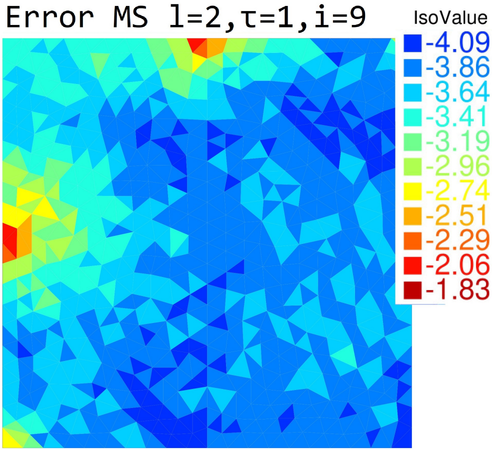


K. Mitra, M. Vohralík, preprint (2023)

Where is the error **localized**?



Estimated local error, $\tau = 1$



Exact local error, $\tau = 1$

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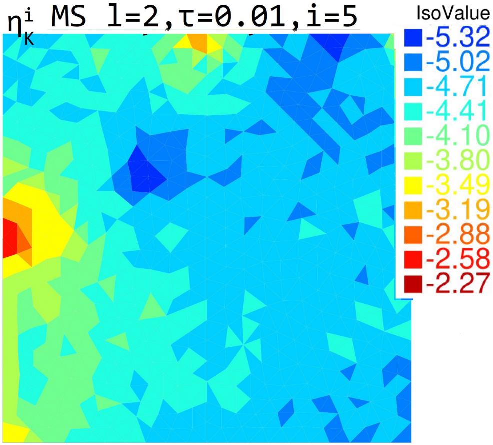
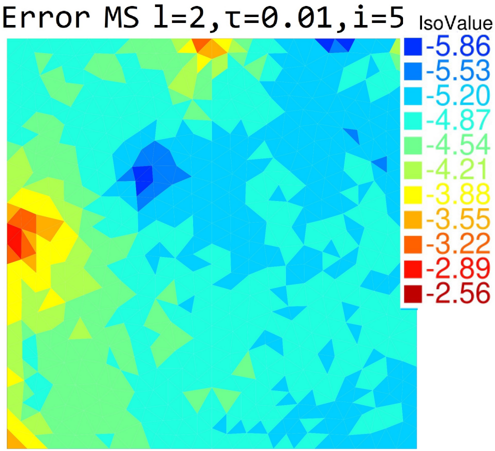


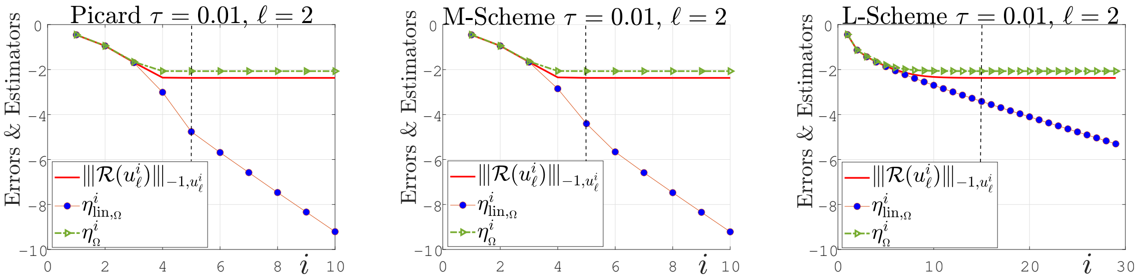
Estimated local error, $\tau = 0.01$



Exact local error, $\tau = 0.01$

K. Mitra, M. Vohralík, preprint (2023)

Error components and adaptivity via stopping criteria



Time step length $\tau = 0.01$

K. Mitra, M. Vohralík, preprint (2023)

Outline

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- a posteriori **certification** of the **error** for nonlinear problems
- **robustness** with respect to the **strength of nonlinearities**
- augmenting the energy difference by the (discretization) error on the given linearization step
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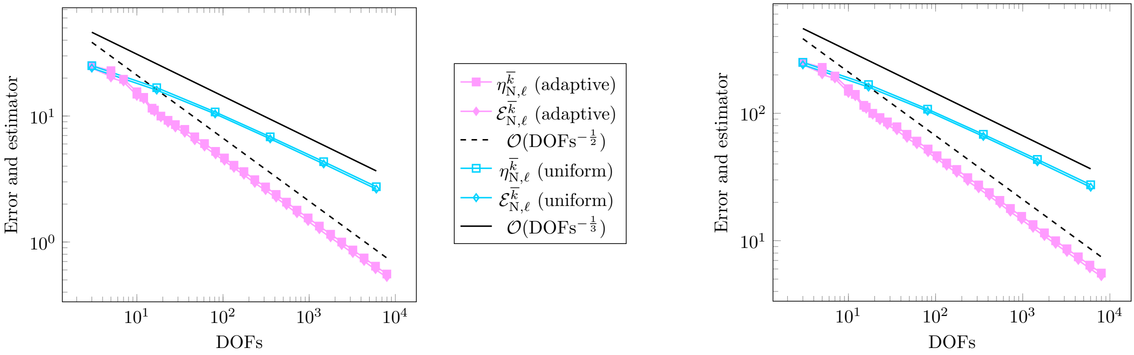
Thank you for your attention!

Outline

Decreasing the **error efficiently**: **optimal decay rate** wrt DoFs

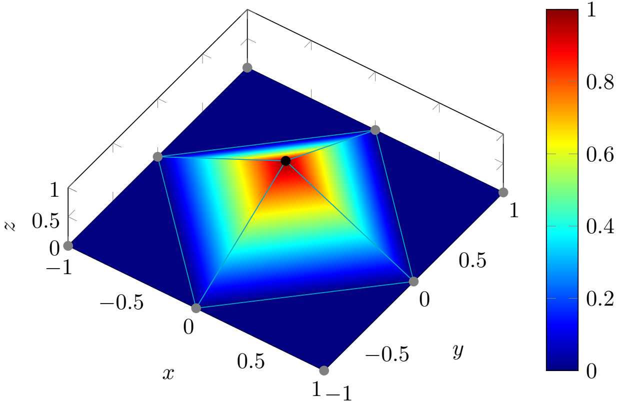


$\frac{a_{\mathrm{c}}}{a_{\mathrm{m}}} = 10^3$ $\qquad$ $\frac{a_{\mathrm{c}}}{a_{\mathrm{m}}} = 10^6$

Outline

Partition of unity

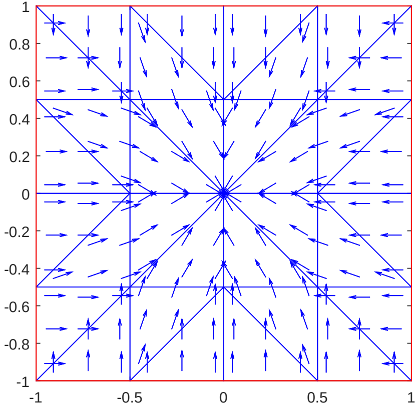
$$\sum_{\boldsymbol{a} \in \mathcal{V}_\ell} \psi^{\boldsymbol{a}} = 1$$



Hat basis function $\psi^{\boldsymbol{a}}$

Equilibrated flux reconstruction

Destuynder and Métivet (1998), Braess & Schöberl (2008), Ern & Vohralík (2013)



Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$ (e.g. FE flux $-\nabla u_\ell$)

Equilibrated flux reconstruction

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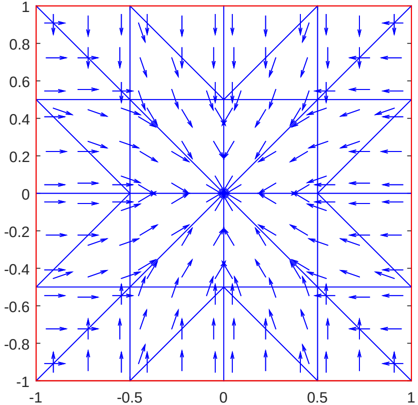

Flux $\boldsymbol{\iota}_\ell \notin \boldsymbol{H}(\mathrm{div})$, $\nabla \cdot \boldsymbol{\iota}_\ell \neq f$

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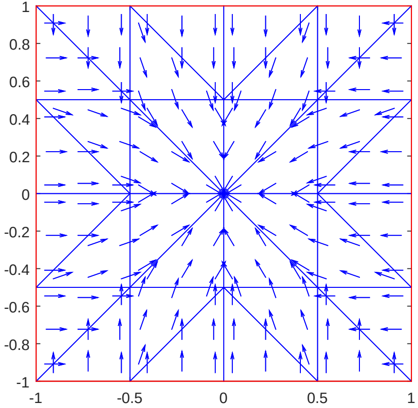

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$$\underbrace{\boldsymbol{\iota}_\ell \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\ell), f \in \mathcal{P}_p(\mathcal{T}_\ell)}$$

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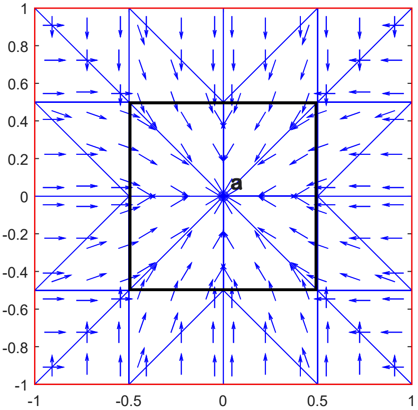

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$$\underbrace{\boldsymbol{\iota}_\ell \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_\ell,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_\ell^{\mathrm{int}}}$$

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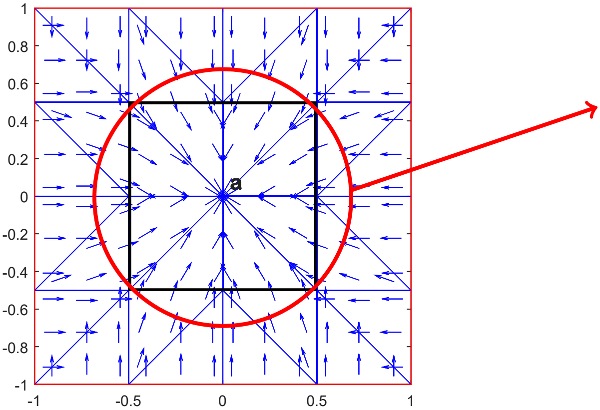

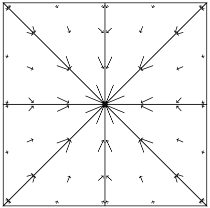

$\psi^{\boldsymbol{a}} \boldsymbol{\iota}_\ell$

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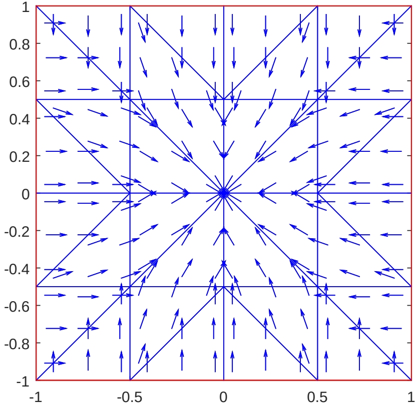

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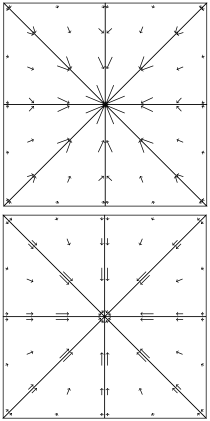

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$\boldsymbol{\sigma}_\ell^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\iota}_\ell \in \mathcal{RT}_p(\mathcal{T}_\ell),\, f \in \mathcal{P}_p(\mathcal{T}_\ell)} \qquad \boldsymbol{\sigma}_\ell^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_\ell \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_\ell = f\psi^{\boldsymbol{a}} + \boldsymbol{\iota}_\ell\cdot\nabla\psi^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_\ell - \boldsymbol{v}_\ell\|^2_{\omega_{\boldsymbol{a}}}$$

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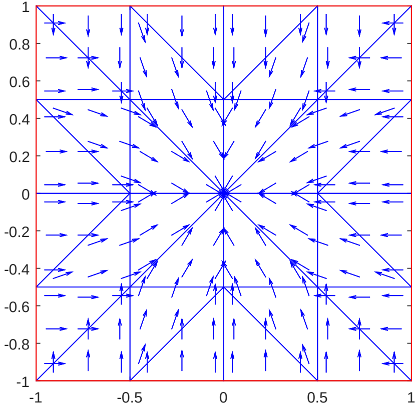

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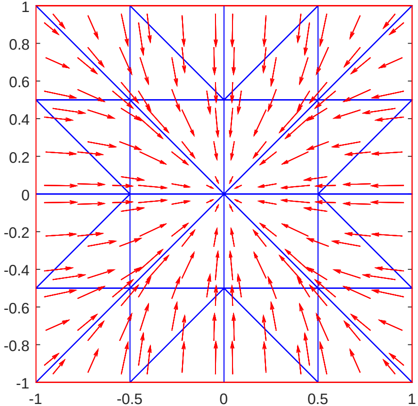

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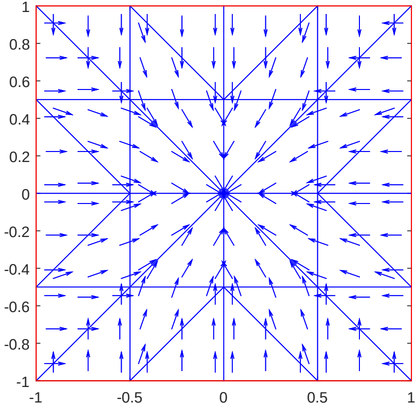

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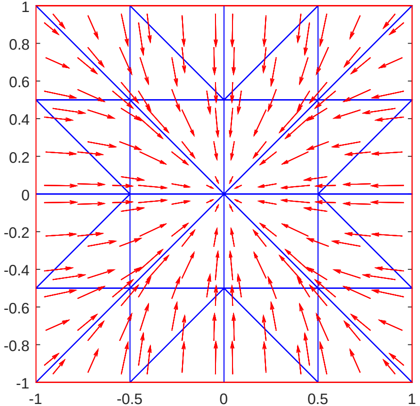

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