

p -robust global–local equivalence,
 p -stable local (commuting) projectors,
and optimal elementwise hp approximation estimates
in $H^1(\Omega)$ and $\mathbf{H}(\text{div}, \Omega)$

Théophile Chaumont-Frelet, Leszek Demkowicz, and **Martin Vohralík**

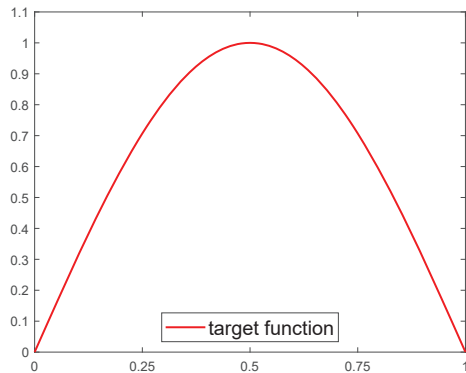
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Bonn, June 11, 2024

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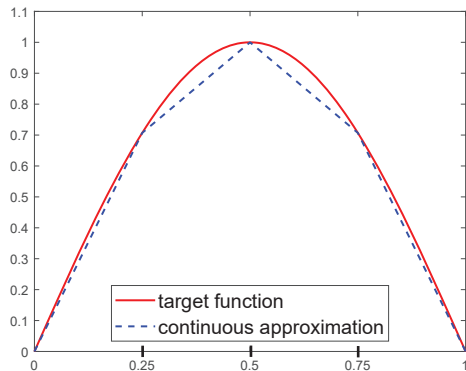


Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D



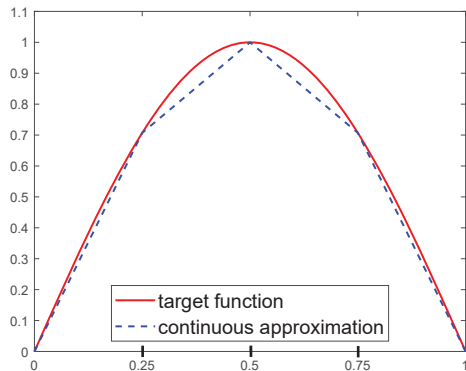
Target function in $H_0^1(\Omega)$

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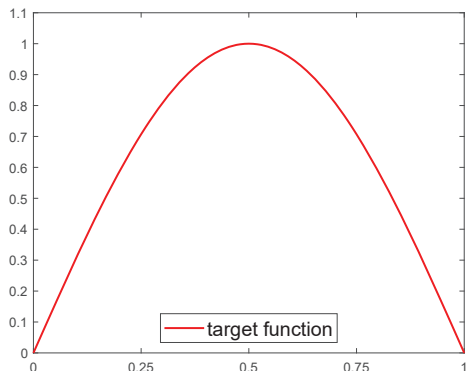


Approximation by **continuous**
 piecewise polynomials in
 $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D

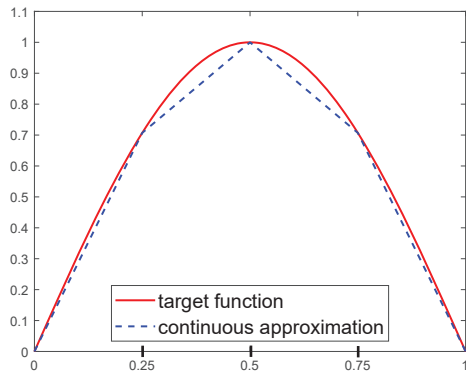


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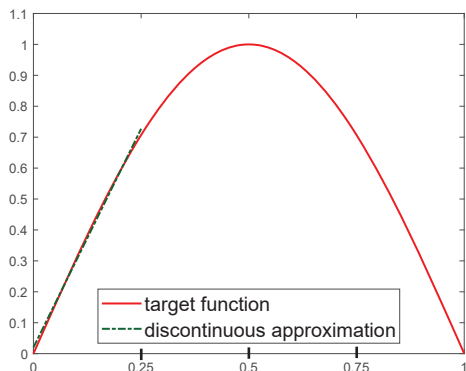


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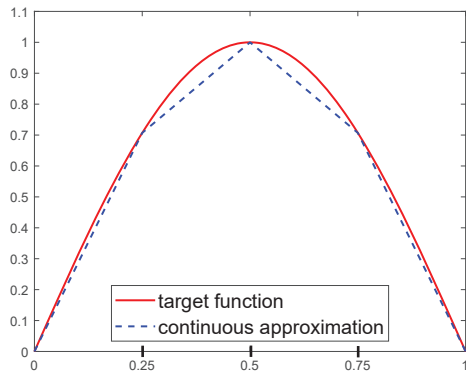


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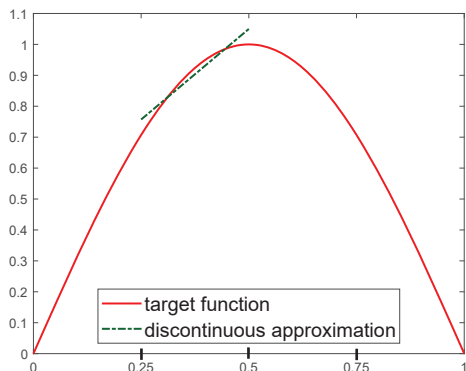


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

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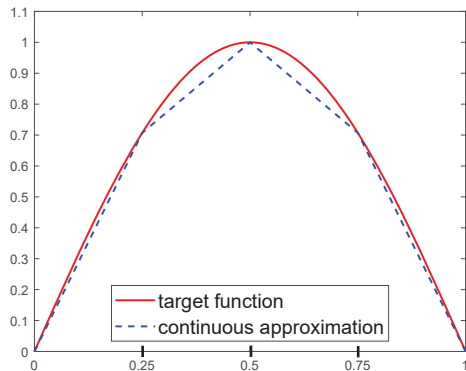


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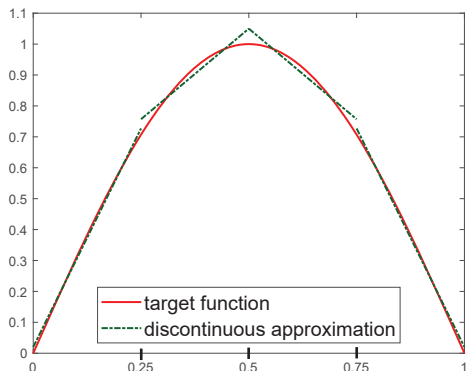


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Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D

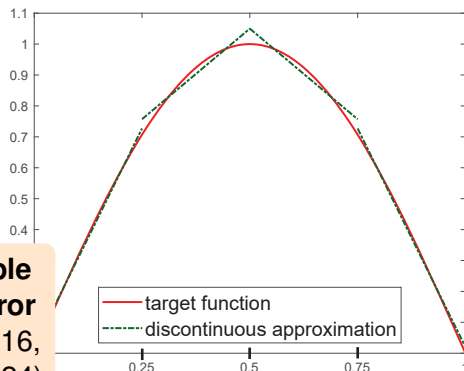
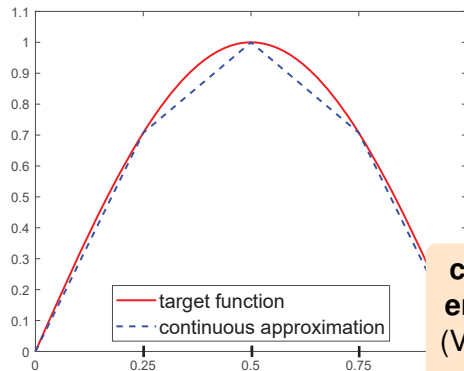


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem



Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of **global-** and **local-best** approximations in $H_0^1(\Omega)$: 1D



comparable energy error
(Veiser 2016, p -rob V. 2024)

Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Outline

- 1 Spaces $H^1(\Omega)$ and $\mathbf{H}(\text{div}, \Omega)$, finite element spaces, and the Laplace equation
 - Sobolev spaces, meshes, and finite element spaces
 - The Laplace equation
- 2 Optimal elementwise *hp* approximation error estimates
 - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
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- 3 *p*-robust global-best-local-best equivalence
 - *p*-robust global-best-local-best equivalence in $H^1(\Omega)$
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- 4 A *p*-stable local commuting projector in $\mathbf{H}(\text{div}, \Omega)$
- 5 Tools
 - Equilibration in $\mathbf{H}(\text{div})$
 - Stable (broken) polynomial extensions
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- 6 Conclusions

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Two key Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $L^2(\Omega)$,
 $H^1(\Omega) := \{v \in L^2(\Omega); \nabla v \in L^2(\Omega)\}$

$H(\text{div}, \Omega)$

vector-valued $L^2(\Omega)$ functions with weak divergences in $L^2(\Omega)$,
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Two key Sobolev spaces with BCs

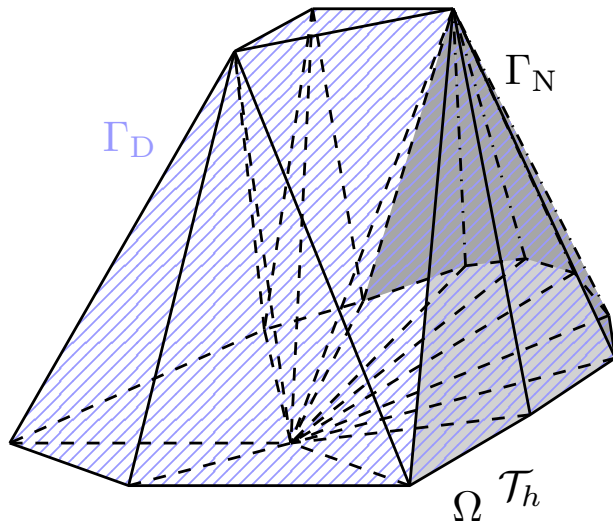
$$H_{0,D}^1(\Omega)$$

$$H_{0,D}^1(\Omega) := \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D\}$$

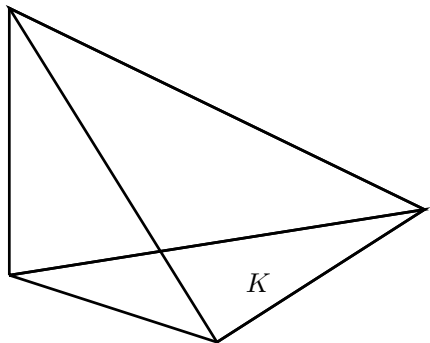
$$H_{0,N}(\text{div}, \Omega)$$

$$H_{0,N}(\text{div}, \Omega) := \{v \in \mathbf{H}(\text{div}, \Omega); v \cdot n_\Omega = 0 \text{ on } \Gamma_N \text{ in appropriate sense}\}$$

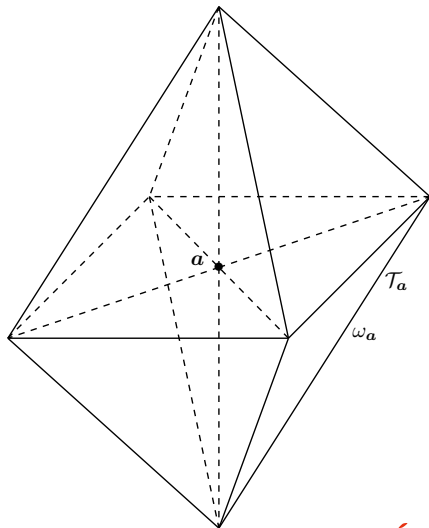
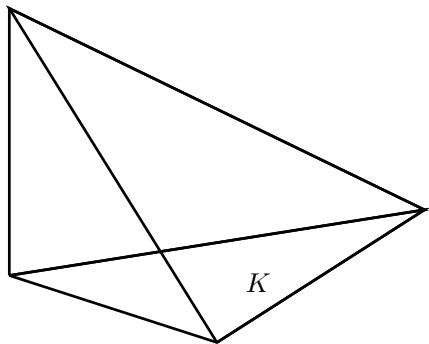
Meshes, elements, and patches



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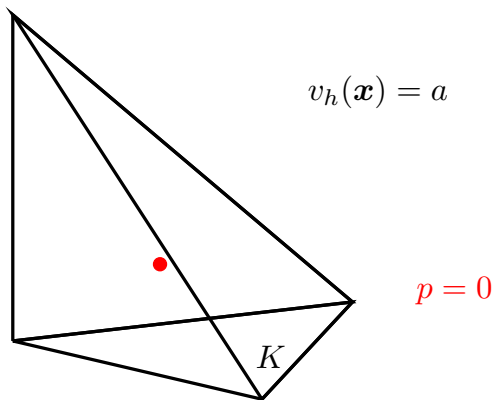


Meshes, elements, and patches

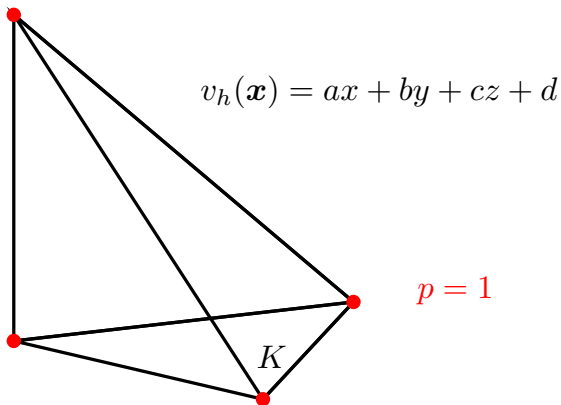


Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

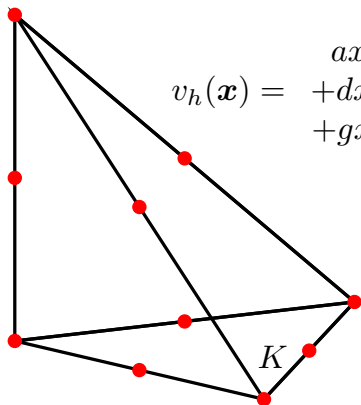
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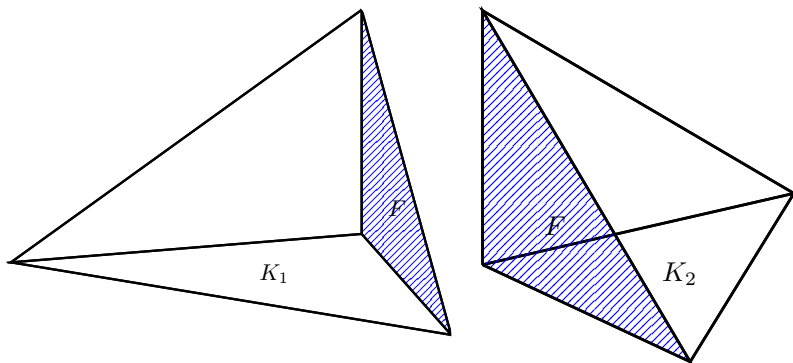


$$v_h(\mathbf{x}) = \begin{aligned} &ax^2 + by^2 + cz^2 \\ &+ dxy + eyz + fxz \\ &+ gx + hy + iz + j \end{aligned}$$

$p = 2$

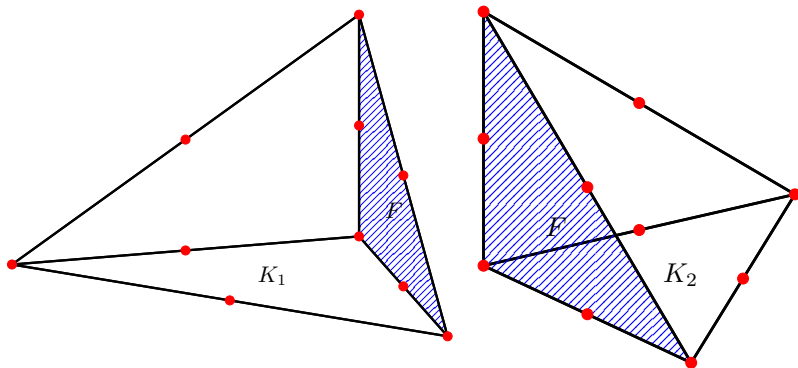
Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

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- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- \Rightarrow ensure this by putting sufficient DoFs at the face F

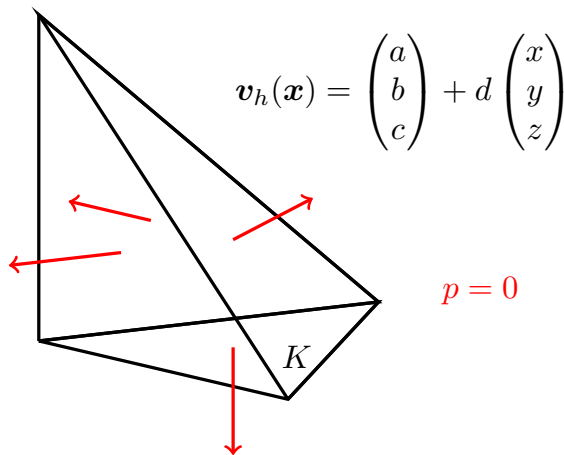
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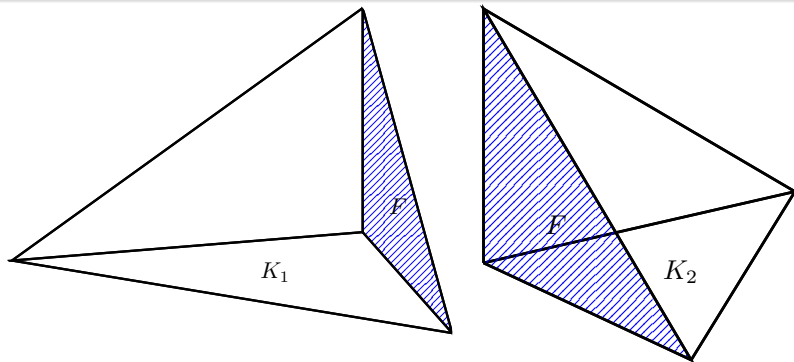
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Raviart–Thomas space $\mathcal{RT}_p(K) := [\mathcal{P}_p(K)]^3 + \mathcal{P}_p(K)\mathbf{x}$, $p \geq 0$

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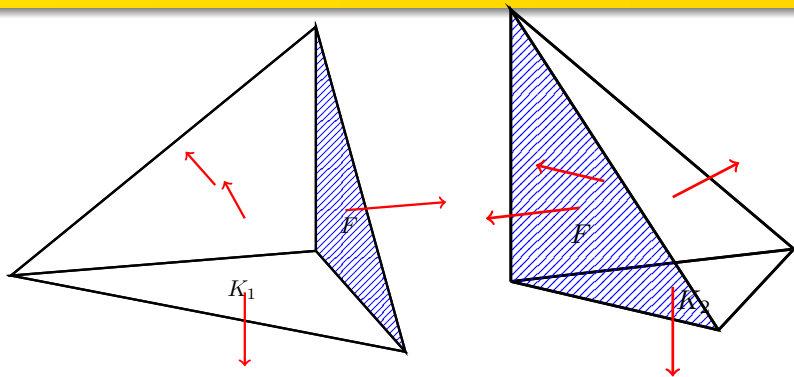


Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega)$, $p \geq 0$



- $\mathbf{v} \in \mathbf{H}(\text{div}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{div}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{div}, K_2)$, and $(\mathbf{v}|_{K_1} \cdot \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \cdot \mathbf{n}_F)|_F$ in appropriate sense
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The Laplace equation (source term $f \in L^2(\Omega)$)

The Laplace equation

Find $u : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ -\nabla u \cdot \mathbf{n}_\Omega &= 0 && \text{on } \Gamma_N. \end{aligned}$$

The Laplace equation (source term $f \in L^2(\Omega)$)

Primal weak formulation

$u \in H_{0,D}^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_{0,D}^1(\Omega).$$

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Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)$, $p \geq 1$, s.t.

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega).$$

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Error characterisation

$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)} \|\nabla(u - v_h)\|$$

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$u \in H_{0,D}^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_{0,D}^1(\Omega).$$

Dual weak formulation

$\sigma \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ with $\nabla \cdot \sigma = f$ such that

$$(\sigma, \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_{0,N}(\text{div}, \Omega) \text{ with } \nabla \cdot \mathbf{v} = 0.$$

Primal finite element approximation

$u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)$, $p \geq 1$, s.t.

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Dual finite element approximation

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Error characterisation

$$\|\sigma - \sigma_h\| = \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_h^p f}} \|\sigma - \mathbf{v}_h\|$$

The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations**

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$$\|\nabla(u - u_h)\| = \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)} \|\nabla(u - v_h)\|$$

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The Laplace equation (source term $f \in L^2(\Omega)$): **global-best approximations (constrained)**

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Approximation error estimates in the $H^1(\Omega)$ context

h approximation estimate

Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005) ($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

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Let $v \in H^s(\Omega)$, $s > d/2$. Then

$$\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)} \|\nabla(u - v_h)\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p, s-1\}} |v|_{H^s(\Omega)}.$$

- Ciarlet (1978), Ern and Guermond (2021)
- Babuška and Suri (1987, $d = 2$)
- Demkowicz and Buffa (2005) ($d = 3$, commutes, under a conjecture on polynomial extension operators proved in 2009–2012)
- Melenk (2005), Karkulik and Melenk (2015), varying polynomial degree, local patchwise regularity

Approximation error estimates in the $H^1(\Omega)$ context

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Approximation error estimates in the $H^1(\Omega)$ context

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Main result

Theorem (Local hp -optimal approximation under minimal Sobolev regularity)

Let $v \in H_{0,D}^1(\Omega)$ with

$$v|_K \in H^{s_K}(K) \quad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- $P_h^p : H_{0,D}^1(\Omega) \rightarrow \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)$: a locally defined projector
- $\underline{p}_K := \min_{L \in \tilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\tilde{\mathcal{T}}_K$

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$$\|\nabla(v - P_h^p v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s) \sum_{L \in \tilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(p_K, s_L - 1)}}{p_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \quad \forall K \in \mathcal{T}_h.$$

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Outline

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Approximation error estimates in the $\mathbf{H}(\text{div}, \Omega)$ context

h approximation estimate

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega) \cap \mathbf{H}^s(\Omega)$, $s > 1/2$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_h^p(\nabla \cdot \mathbf{v})}} \|\mathbf{v} - \mathbf{v}_h\| \leq C(\kappa_{\mathcal{T}_h}, d, s, p) h^{\min\{p+1, s\}} \|\mathbf{v}\|_{\mathbf{H}^s(\Omega)}.$$

- Raviart and Thomas (1977), Nédélec (1980), Boffi, Brezzi, and Fortin (2013)
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Main result

Theorem (**Local** *hp*-optimal approximation under **minimal Sobolev regularity**)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ with

$$\mathbf{v}|_K \in \mathbf{H}^{s_K}(K), \quad (\nabla \cdot \mathbf{v})|_K \in [\mathcal{P}_p(K)]^3 \quad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$.

- varying polynomial degree can be added

Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

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for $s_K \geq 0$. Then

$$\begin{aligned} & \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \nabla \cdot \mathbf{v}}} \|\mathbf{v} - \mathbf{v}_h\|^2 \\ & \leq C(\kappa_{\mathcal{T}_h}, d, s)^2 \sum_{K \in \mathcal{T}_h} \left(\frac{h_K^{\min\{p+1, s_K\}}}{(p+1)^{s_K}} \|\mathbf{v}\|_{\mathbf{H}^{s_K}(K)} \right)^2. \end{aligned}$$

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Main result

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ with

$$\mathbf{v}|_K \in \mathbf{H}^{s_K}(K), \quad (\nabla \cdot \mathbf{v})|_K \in H^{t_K}(K) \quad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ and $t_K \geq 0$.

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Main result

Theorem (Local *hp-optimal* approximation under minimal Sobolev regularity)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ with

$$\mathbf{v}|_K \in \mathbf{H}^{s_K}(K), \quad (\nabla \cdot \mathbf{v})|_K \in \mathbf{H}^{t_K}(K) \quad \forall K \in \mathcal{T}_h$$

for $s_K \geq 0$ and $t_K \geq 0$. Then

$$\min_{\substack{\mathbf{v}_h \in \mathcal{RT}_\rho(\mathcal{T}_h) \cap \mathbf{H}(\text{div}, \Omega) \\ \nabla \cdot \mathbf{v}_h = \Pi_h^\rho(\nabla \cdot \mathbf{v})}} \left[\|\mathbf{v} - \mathbf{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{\rho+1} \|\nabla \cdot \mathbf{v} - \Pi_h^\rho(\nabla \cdot \mathbf{v})\|_K \right)^2 \right]$$

$$\leq C(\kappa_{\mathcal{T}_h}, d, s, t)^2 \sum_{K \in \mathcal{T}_h} \left[\left(\frac{h_K^{\min\{\rho+1, s_K\}}}{(\rho+1)^{s_K}} \|\mathbf{v}\|_{\mathbf{H}^{s_K}(K)} \right)^2 + \left(\frac{h_K}{\rho+1} \frac{h_K^{\min\{\rho+1, t_K\}}}{(\rho+1)^{t_K}} \|\nabla \cdot \mathbf{v}\|_{\mathbf{H}^{t_K}(K)} \right)^2 \right].$$

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for $s_K \geq 0$ and $t_K \geq 0$. Then

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- varying polynomial degree can be added

Outline

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Global-best approximation \approx local-best approximation

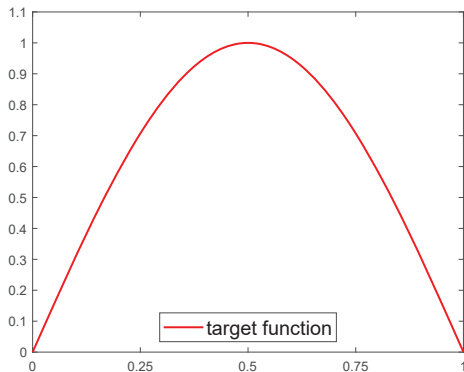
Previous contributions

- Carstensen, Peterseim, and Schedensack (2012): H^1 (lowest-order case $p = 1$)
- Aurada, Feischl, Kemetmüller, Page, and Praetorius (2013): H^1 (boundary approximation context)
- Veerer (2016): H^1 (any p , p -dependent constant)
- Canuto, Nochetto, Stevenson, and Verani (2017): H^1 (improvement of the p dependence of the equivalence constant in 2D)
- Ern, Gudi, Smears, and Vohralík (2022): $H(\text{div})$ (any p , p -dependent constant)
- Chaumont-Frelet and Vohralík (2021, 2022): $H(\text{curl})$ (any p , p -dependent constant)
- Gawlik, Holst, and Licht (2021): finite element exterior calculus context (any p , p -dependent constant)

Outline

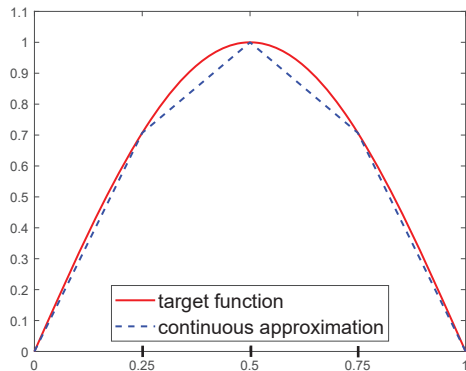
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Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D



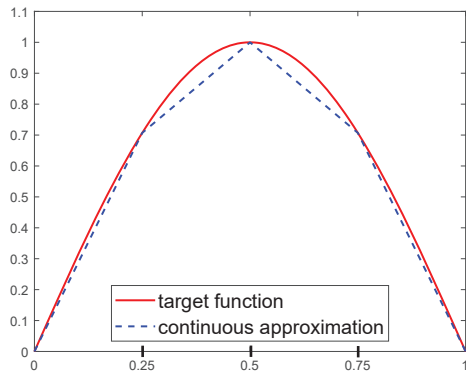
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

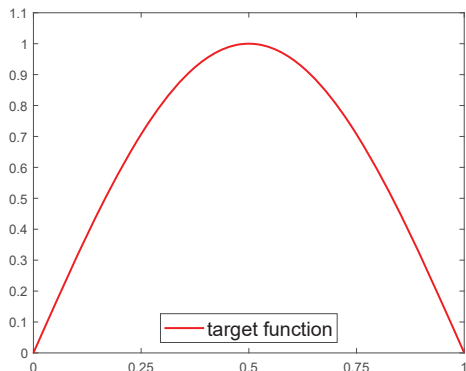


Approximation by **continuous**
 piecewise polynomials in
 $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

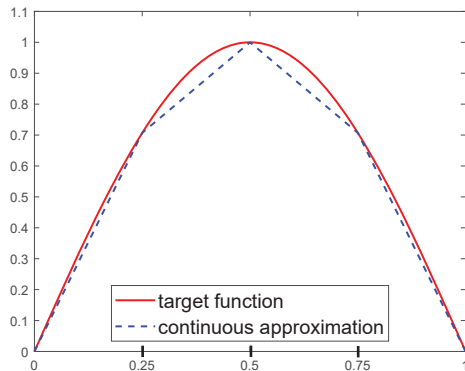


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

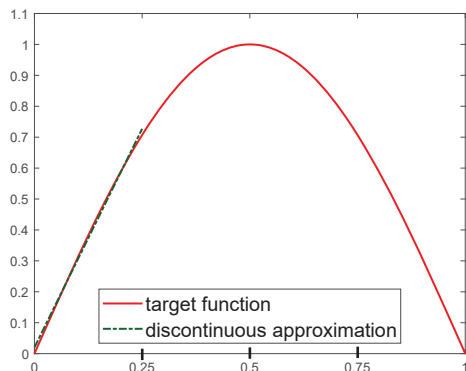


Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

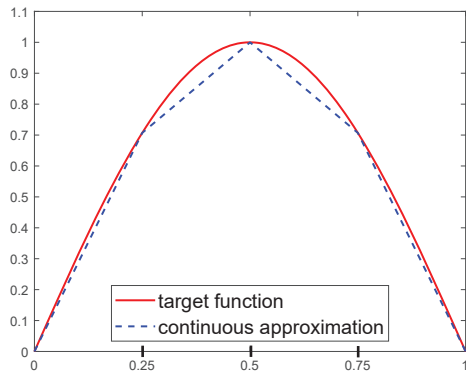


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

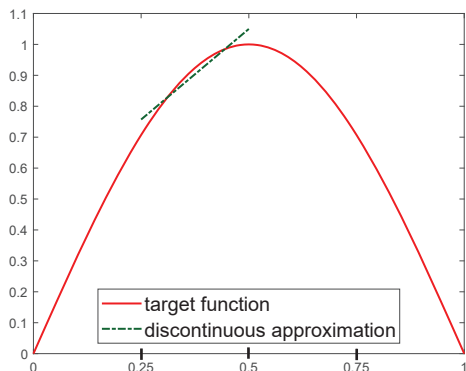


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

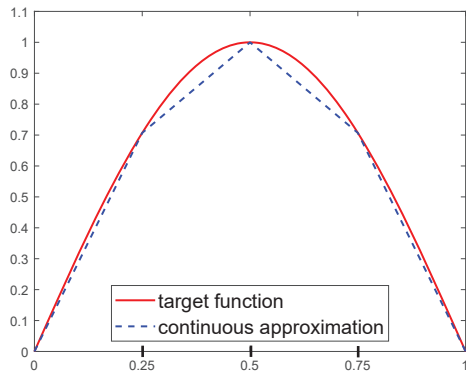


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

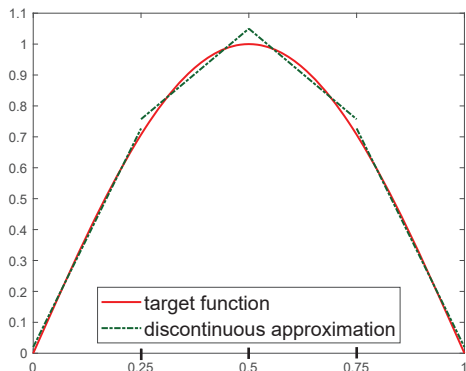


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D

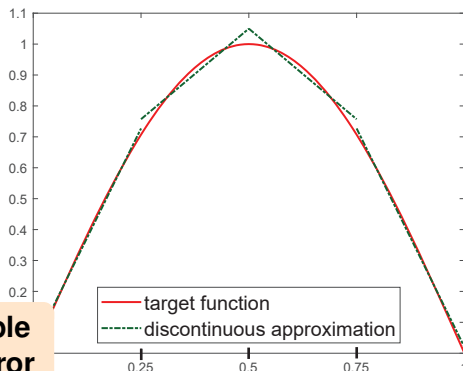
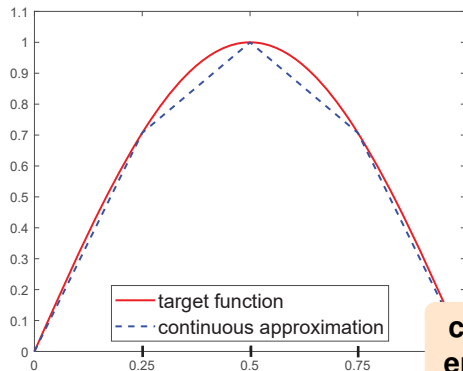


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem



Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D



comparable energy error

Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veeseer (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

bigger \approx_p smaller

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$$\min_{\text{smaller space}} \approx_p \min_{\text{bigger space}}$$

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$$\min_{CG \text{ space}} \approx_p \min_{DG \text{ space}}$$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1 , Carstensen, Peterseim, & Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, & Praetorius (2013), Veerer (2016), Canuto, Nochetto, Stevenson, & Verani (2017))

Let $v \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}$$

- \approx_p : up to a generic constant that only depends on space dimension d , shape-regularity of the mesh \mathcal{T}_h , and polynomial degree p

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Main result

Theorem (Equivalence in $H_{0,D}^1(\Omega)$)

Let $v \in H_{0,D}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}$$

- \approx : up to a generic constant that only depends on space dimension d and shape-regularity of the mesh \mathcal{T}_h
- also for varying polynomial degree

Main result

Theorem (Equivalence in $H_{0,D}^1(\Omega)$)

Let $v \in H_{0,D}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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- \approx : up to a generic constant that only depends on space dimension d and shape-regularity of the mesh \mathcal{T}_h
- also for varying polynomial degree

Main result

Theorem (Equivalence in $H_{0,D}^1(\Omega)$)

Let $v \in H_{0,D}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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- also for varying polynomial degree

Main result

Theorem (Equivalence in $H_{0,D}^1(\Omega)$)

Let $v \in H_{0,D}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_{0,D}^1(\Omega)} \|\nabla(v - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

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Main result

Theorem (Equivalence in $H_{0,D}^1(\Omega)$)

Let $v \in H_{0,D}^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

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- also for **varying polynomial degree**

Outline

- 1 Spaces $H^1(\Omega)$ and $\mathbf{H}(\text{div}, \Omega)$, finite element spaces, and the Laplace equation
 - Sobolev spaces, meshes, and finite element spaces
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Equivalence of global- and local-best approximations in $\mathbf{H}(\text{div}, \Omega)$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2022))

bigger \approx_p smaller

Equivalence of global- and local-best approximations in $\mathbf{H}(\text{div}, \Omega)$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2022))

$$\min_{\text{smaller space with constraints}} \approx_p \min_{\text{bigger space without constraints}}$$

Equivalence of global- and local-best approximations in $\mathbf{H}(\text{div}, \Omega)$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2022))

$$\min_{\text{MFE space with constraints}} \approx_p \min_{\text{broken MFE space without constraints}}$$

Equivalence of global- and local-best approximations in $\mathbf{H}(\text{div}, \Omega)$

Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$, Ern, Gudi, Smears, & V. (2022))

Let $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

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global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K \in \mathcal{T}_h} \left[\min_{\mathbf{v}_h \in \mathcal{RTN}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla \cdot \mathbf{v} - \Pi_h^p \nabla \cdot \mathbf{v}\|_K^2 \right].$$

local-best on each K
no normal trace-continuity constraint
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- \approx_p : only depends on d , shape-regularity of \mathcal{T}_h , and p

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Theorem (Constrained equivalence in $\mathbf{H}(\text{div})$)

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Commuting de Rham diagram with operator $P_h^{p,\text{div}}$

Commuting de Rham diagram

$$\begin{array}{ccccccc}
 H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & L_*^2(\Omega) \\
 \downarrow P_h^{p+1,\text{grad}} & & \downarrow P_h^{p,\text{curl}} & & \downarrow P_h^{p,\text{div}} & & \downarrow \Pi_h^p \\
 \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
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 \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_{0,N}^1(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h) \cap H_{0,N}(\text{curl}, \Omega) & \xrightarrow{\nabla \times} & \mathcal{RT}_p(\mathcal{T}_h) \cap H_{0,N}(\text{div}, \Omega) & \xrightarrow{\nabla \cdot} & \mathcal{P}_p(\mathcal{T}_h) \cap L_*^2(\Omega)
 \end{array}$$

Properties of $P_h^{p,\text{div}}$

- 1 is defined over the **entire** $H_{0,N}(\text{div}, \Omega)$ (**minimal regularity**)
- 2 is defined **locally** (in neighborhood of mesh elements)
- 3 is defined **simply** (starting from the **elementwise L^2 orthogonal projection**)
- 4 has **optimal hp** approximation properties, that of **elementwise div-unconstrained L^2 -orthogonal projector** (global-local equivalence)
- 5 is **stable in $L^2(\Omega)$** (up to data oscillation)
- 6 satisfies the **commuting properties** expressed by the arrows
- 7 is **projector**, i.e., leaves intact piecewise polynomials

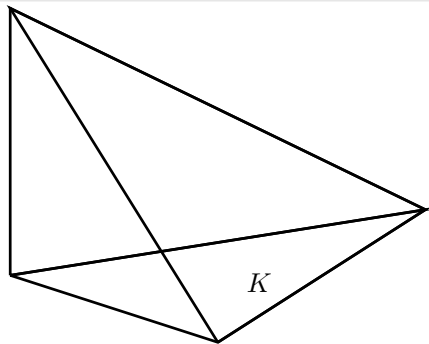
Stable local commuting projectors defined on $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$
- Falk and Winther (2014): local and $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$ -stable but not L^2 -stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on part of $\partial\Omega$
- Arnold and Guzmán (2021): L^2 -stable
- Ern, Gudi, Smears, and Vohralík (2022): all the above properties in $\mathbf{H}(\text{div})$
- none is p -robust

Stable local commuting projectors defined on $\mathbf{H}(\text{div})/\mathbf{H}(\text{curl})$

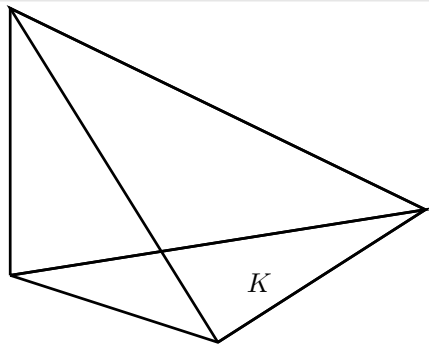
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- Licht (2019): **essential boundary conditions** on part of $\partial\Omega$
- Arnold and Guzmán (2021): **L^2 -stable**
- Ern, Gudi, Smears, and Vohralík (2022): all the above properties in **$\mathbf{H}(\text{div})$**
- **none is p -robust**

Classical elementwise interpolation



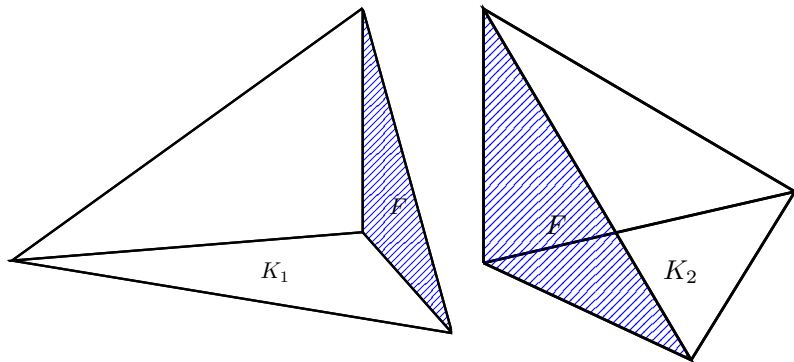
- $\|\mathbf{v} - \mathbf{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{v} - \mathbf{v}_h\|_K^2$
- $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega) \Rightarrow \mathbf{v}|_K \in \mathbf{H}(\text{div}, K) \Rightarrow$ so interpolate $\mathbf{v}|_K$

Classical elementwise interpolation



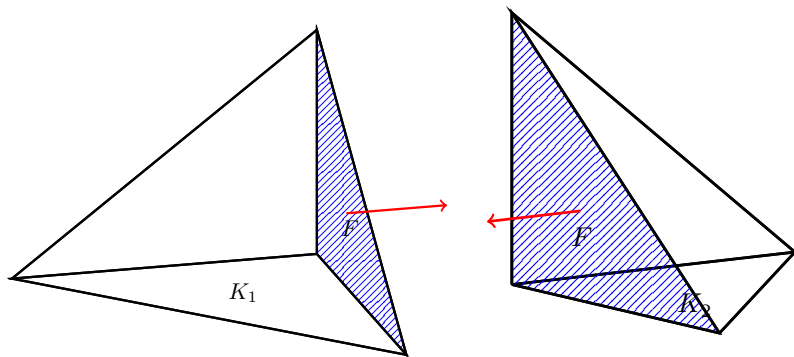
- $\|\mathbf{v} - \mathbf{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{v} - \mathbf{v}_h\|_K^2$
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Classical elementwise interpolation: conformity enforcement



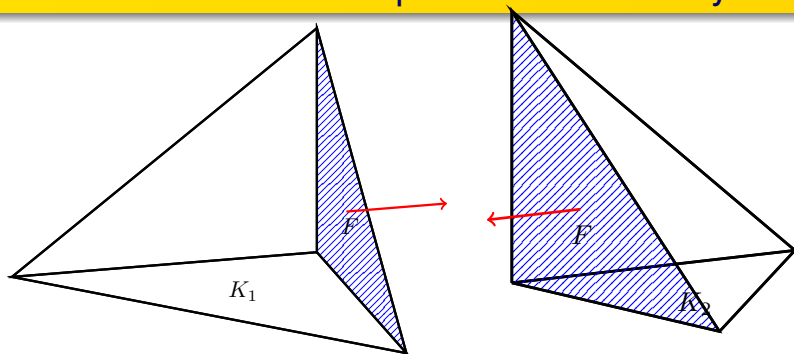
- $\mathbf{v} \in \mathbf{H}(\text{div}, K_1 \cup K_2)$ iff $\mathbf{v} \in \mathbf{H}(\text{div}, K_1)$, $\mathbf{v} \in \mathbf{H}(\text{div}, K_2)$, and $(\mathbf{v}|_{K_1} \cdot \mathbf{n}_F)|_F = (\mathbf{v}|_{K_2} \cdot \mathbf{n}_F)|_F$ in appropriate sense ($p = 0$)

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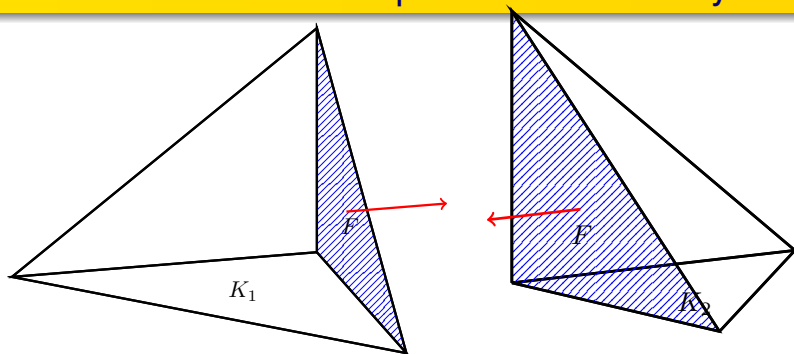


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Clash

Face normal trace integrals not available in $\mathbf{H}(\text{div})$.

Classical elementwise interpolation: conformity enforcement

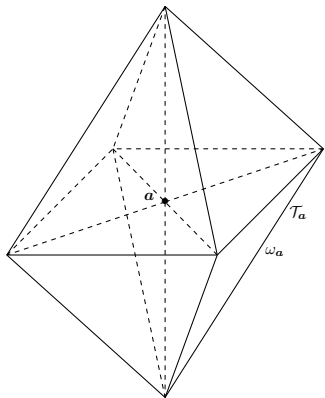


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Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\mathbf{v} \in \mathbf{H}(\text{div}, \Omega)$ requested.

Classical patchwise interpolation (Clément)

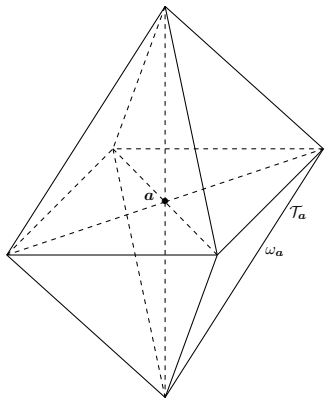


- some local-best polynomial approximation on ω_a
- values on ω_a as coefficients for basis functions supported on ω_a

Conclusion

Allows the **minimal regularity** but breaks the **projection property**, the **elementwise structure**, and the **commuting diagram**.

Classical patchwise interpolation (Clément)



- some local-best polynomial approximation on ω_a
- values on ω_a as coefficients for basis functions supported on ω_a

Conclusion

Allows the **minimal regularity** but breaks the **projection property**, the **elementwise structure**, and the **commuting diagram**.

A p -stable local commuting projector $\mathbf{P}_h^{p,\text{div}}$

Definition (A p -stable local commuting projector $\mathbf{P}_h^{p,\text{div}}$)

Let $\mathbf{v} \in \mathbf{H}_{0,N}(\text{div}, \Omega)$ be given (minimal regularity).



- For each $K \in \mathcal{T}_h$, define $\iota_h|_K$ by the **elementwise unconstrained projection**

$$\iota_h|_K := \arg \min_{\mathbf{v}_h \in \mathcal{RT}_p(K)} \|\mathbf{v} - \mathbf{v}_h\|_K$$

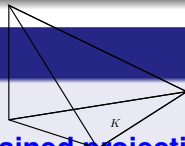
(discrete but nonconforming (normal-trace discontinuous)).

- Obtain $\sigma_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega)$ by applying the **flux equilibration procedure** to ι_h ; in particular, $\sigma_h := \sum_{a \in \mathcal{V}_h} \sigma_h^a$, where σ_h^a are obtained by local energy minimizations on the patch subdomains ω_a .
- Apply a p -stable decomposition to conforming local projections of the reminder $\iota_h - \sigma_h$, yielding a correction ζ_h . Obtain $\mathbf{P}_h^{p,\text{div}}(\mathbf{v}) := \sigma_h + \zeta_h$.

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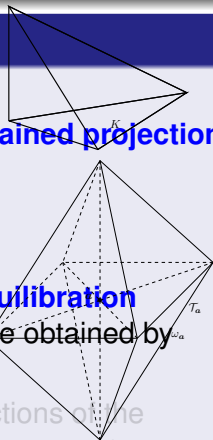
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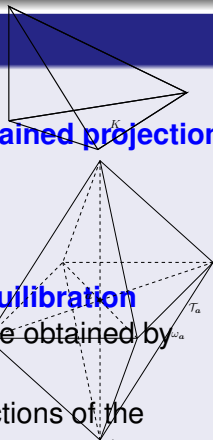
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A p -stable local commuting projector $\mathbf{P}_h^{\rho, \text{div}}$

Theorem ($\mathbf{P}_h^{\rho, \text{div}} : \mathbf{H}_{0, \text{N}}(\text{div}, \Omega) \rightarrow \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0, \text{N}}(\text{div}, \Omega)$)

$\mathbf{P}_h^{\rho, \text{div}}$ is a **commuting projector** since

$$\begin{aligned} \nabla \cdot \mathbf{P}_h^{\rho, \text{div}}(\mathbf{v}) &= \Pi_h^\rho(\nabla \cdot \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{0, \text{N}}(\text{div}, \Omega), \\ \mathbf{P}_h^{\rho, \text{div}}(\mathbf{v}) &= \mathbf{v} & \forall \mathbf{v} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0, \text{N}}(\text{div}, \Omega). \end{aligned}$$

Moreover, it has **local-best approximation properties** and is L^2 stable up to data oscillation, since, for all $\mathbf{v} \in \mathbf{H}_{0, \text{N}}(\text{div}, \Omega)$ and $K \in \mathcal{T}_h$,

$$\begin{aligned} & \|\mathbf{v} - \mathbf{P}_h^{\rho, \text{div}}(\mathbf{v})\|_K^2 + \left(\frac{h_K}{\rho + 1} \|\nabla \cdot (\mathbf{v} - \mathbf{P}_h^{\rho, \text{div}}(\mathbf{v}))\|_K \right)^2 \\ & \lesssim \sum_{L \in \tilde{\mathcal{T}}_K} \left\{ \min_{\mathbf{v}_h \in \mathcal{RT}_p(L)} \|\mathbf{v} - \mathbf{v}_h\|_L^2 + \left(\frac{h_L}{\rho + 1} \|\nabla \cdot \mathbf{v} - \Pi_h^\rho(\nabla \cdot \mathbf{v})\|_L \right)^2 \right\}, \\ & \|\mathbf{P}_h^{\rho, \text{div}}(\mathbf{v})\|_K^2 \lesssim \sum_{L \in \tilde{\mathcal{T}}_K} \left\{ \|\mathbf{v}\|_L^2 + \left(\frac{h_L}{\rho + 1} \|\nabla \cdot \mathbf{v} - \Pi_h^\rho(\nabla \cdot \mathbf{v})\|_L \right)^2 \right\}. \end{aligned}$$

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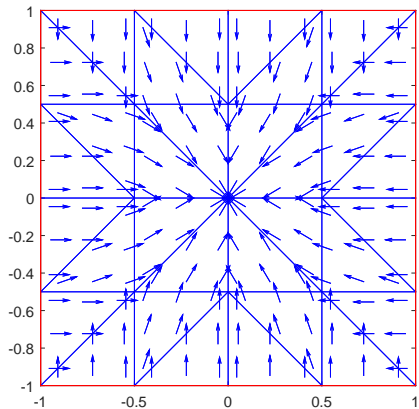
Outline

- 1 Spaces $H^1(\Omega)$ and $\mathbf{H}(\text{div}, \Omega)$, finite element spaces, and the Laplace equation
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 - Stable (broken) polynomial extensions
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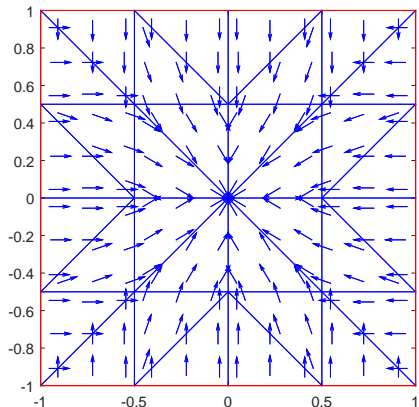
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Equilibration in $H(\text{div})$ Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



Flux $\iota_h \notin H(\text{div}), \nabla \cdot \iota_h \neq f$

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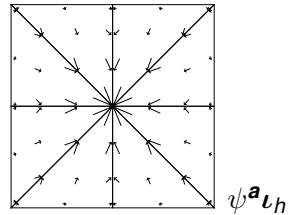
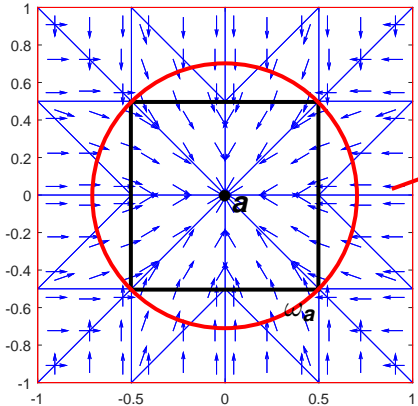


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$$(f, \psi^a)_{\omega_a} + (\iota_h, \nabla \psi^a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}$$

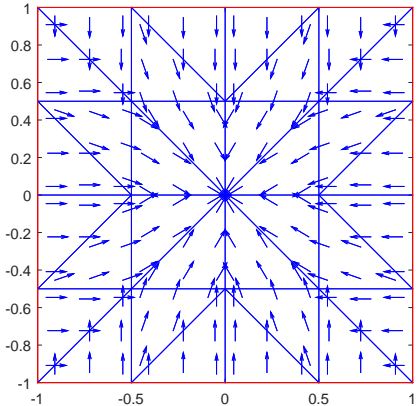
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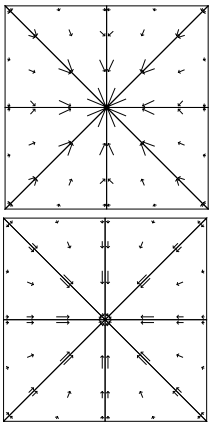
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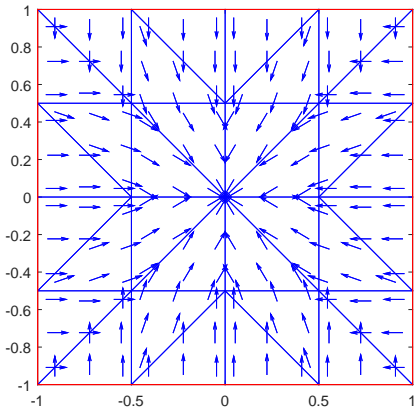


ψ^a

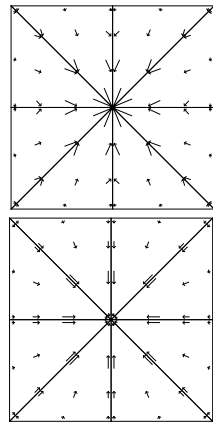
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$\psi^a \boldsymbol{\iota}_h$

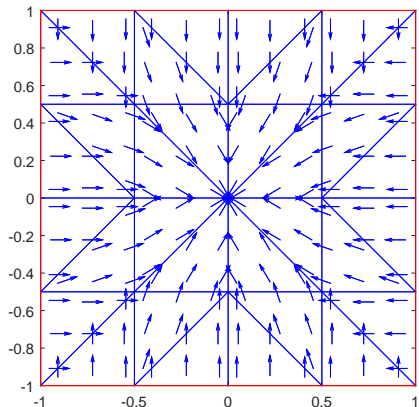
σ_h^a

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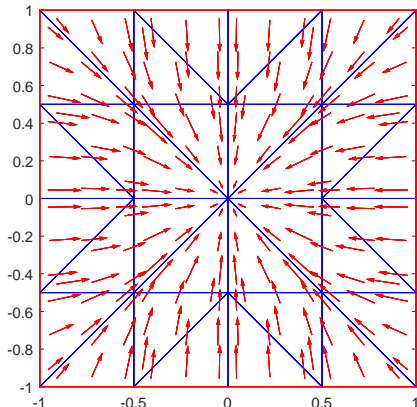
$$\sigma_h^a := \arg \min_{\mathbf{v}_h \in \mathcal{RT}_{p+1}(\mathcal{T}_a) \cap \mathbf{H}_0(\text{div}, \omega_a)} \|\psi^a \boldsymbol{\iota}_h - \mathbf{v}_h\|_{\omega_a}^2$$

$$\nabla \cdot \mathbf{v}_h = f \psi^a + \boldsymbol{\iota}_h \cdot \nabla \psi^a$$

Equilibration in $H(\text{div})$ Destuynder & Métivet (1998), Braess & Schöberl (2008), Ern & V. (2013)



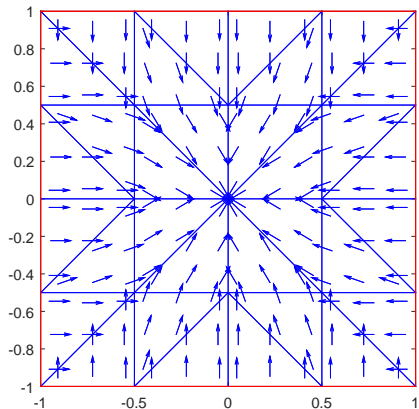
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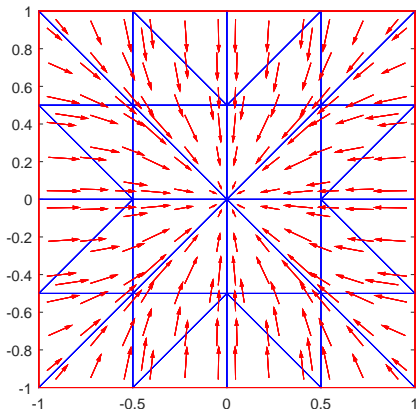
Equilibrated flux rec. σ_h

$$\underbrace{\iota_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_p(\mathcal{T}_h)}_{(f, \psi^a)_{\omega_a} + (\iota_h, \nabla \psi^a)_{\omega_a} = 0 \quad \forall a \in \mathcal{V}_h^{\text{int}}}} \rightarrow \sigma_h := \sum_{a \in \mathcal{V}_h} \sigma_h^a \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap H(\text{div}), \nabla \cdot \sigma_h = f$$

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Equilibrated flux rec. $\sigma_h \in H(\text{div})$, $\nabla \cdot \sigma_h = f$

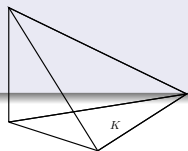
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H(div) polynomial extensions on a tetrahedron

Theorem (**H**(div) polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2012); Braess, Pillwein, & Schöberl (2009); Ern & V. (2020))

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K . Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_{\mathcal{F}} \in \mathcal{P}_p(\Gamma_{\mathcal{F}})$, there holds



$$\min_{\substack{\mathbf{v}_p \in \mathcal{RT}_p(K) \\ \nabla \cdot \mathbf{v}_p = r_K \\ \mathbf{v}_p \cdot \mathbf{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\mathbf{v}_p\|_K \leq C_{\text{st}} \min_{\substack{\mathbf{v} \in \mathbf{H}(\text{div}, K) \\ \nabla \cdot \mathbf{v} = r_K \\ \mathbf{v} \cdot \mathbf{n}_{\mathcal{F}} = r_{\mathcal{F}}}} \|\mathbf{v}\|_K.$$

Comments

- C_{st} only depends on the **shape-regularity** of K
- **p -robustness**: for (pw) p -polynomial data $r_K, r_{\mathcal{F}}$, minimization over $\mathcal{RT}_p(K)$ is up to C_{st} as good as minimization over the entire $\mathbf{H}(\text{div}, K)$
- extension to a **vertex patch**: Braess, Pillwein, & Schöberl (2009); Ern & Vohralík (2020)

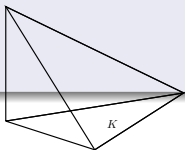


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Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K . Then, for every polynomial degree $p \geq 0$, for all $r_K \in \mathcal{P}_p(K)$, and for all $r_{\mathcal{F}} \in \mathcal{P}_p(\Gamma_{\mathcal{F}})$, there holds



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Comments

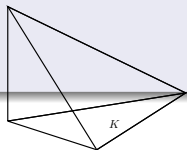
- C_{st} only depends on the **shape-regularity** of K
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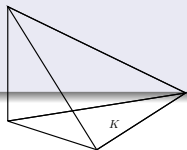
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$H(\text{div})$ polynomial extensions on a tetrahedron and on patches

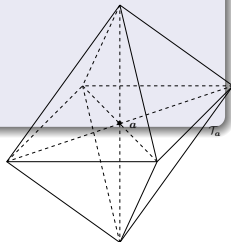
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 - Sobolev spaces, meshes, and finite element spaces
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- 2 Optimal elementwise *hp* approximation error estimates
 - Optimal elementwise *hp* approximation error estimates in $H^1(\Omega)$
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 - Stable (broken) polynomial extensions
 - **Stable decompositions**
- 6 Conclusions

$H(\text{div})$ stable decomposition

Theorem ($H(\text{div})$ stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008))

Let $d = 2$ and let $\bar{\Omega}$ be contractible. Let

$$\begin{aligned} \delta_p \in \mathcal{RT}_p(\mathcal{T}_h) \cap \mathbf{H}_{0,N}(\text{div}, \Omega) \quad \text{with} \quad \nabla \cdot \delta_p = 0, \quad \text{div-free} \\ (\delta_p, \mathbf{r}_h)_K = 0 \quad \forall \mathbf{r}_h \in [\mathcal{P}_0(K)]^d, \quad \forall K \in \mathcal{T}_h. \quad \text{vanishing means} \end{aligned}$$

Then there exists a decomposition of δ_p as

$$\delta_p = \sum_{\mathbf{a} \in \mathcal{V}_h} \delta_p^{\mathbf{a}}, \quad \text{decomposition}$$

where

$\delta_p^{\mathbf{a}}$ are supported on the vertex patch subdomains $\omega_{\mathbf{a}}$, linearly
depend on δ_p on the extended vertex patch subdomains $\tilde{\omega}_{\mathbf{a}}$,

and satisfy

$$\begin{aligned} \delta_p^{\mathbf{a}} \in \mathcal{RT}_p(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\text{div}, \omega_{\mathbf{a}}) \quad \text{with} \quad \nabla \cdot \delta_p^{\mathbf{a}} = 0, \quad \text{local div-free} \\ \|\delta_p^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \lesssim \|\delta_p\|_{\tilde{\omega}_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_h. \quad p\text{-stable} \end{aligned}$$

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Thank you for your attention!