Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems

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Outline

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What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE.
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|u - u_h\|_\Omega \leq f(u)h^q$. Dependent on u, not computable. Useful in theory.
- A posteriori error estimate: $\|u - u_h\|_\Omega \lesssim f(u_h)$. Only uses u_h, computable. Great in practice.

Usual form

- $f(u_h)^2 = \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2$, where $\eta_T(u_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|u - u_h\|_\Omega^2 \leq \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|u - u_h\|_\Omega^2 \leq C \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2$

Local efficiency (local error lower bound)

- $\eta_T(u_h)^2 \leq C_{\mathrm{eff},T}^2 \sum_{T' \text{ close to } T} \|u - u_h\|_{T'}^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2 / \|u - u_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},T}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

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Previous results on a posteriori error estimation in DG

DG, pure diffusion case

- Karakashian and Pascal (2003), Becker, Hansbo, and Larson (2003), Houston, Süli, and Wihler (2008) residual-based estimates
- Rivière and Wheeler (2003), L^2-estimates
- Ainsworth (2007), Ainsworth and Rankin (2008, preprint) reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), Lazarov, Repin, and Tomar (2008), Ern, Stephansen, and Vohralík (2007, preprint), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

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- Braess and Schöberl (2008, 2009)

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

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- Verfürth (1998, 2005), conforming finite elements
- Sangalli (2008), conforming finite elements

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Motivations and key points

Motivations

- establish an optimal abstract framework for a posteriori error estimation in potential- and flux-nonconforming methods
- derive estimates satisfying as many as possible of the five optimal properties

Key points

- focus on inhomogeneous and anisotropic diffusion
- case of nonmatching meshes
- **singular regimes** of **dominant convection** or **reaction**

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A model convection–diffusion–reaction problem

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$$\begin{aligned} -\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Bilinear form

$$\mathcal{B}(u,v) := (\mathbf{K}\nabla u, \nabla v) + (\boldsymbol{\beta}\cdot\nabla u, v) + (\mu u, v), \qquad u, v \in H^1(\mathcal{T}_h)$$

Weak solution

Find $u \in H_0^1(\Omega)$ such that $\mathcal{B}(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$.

Energy norm

Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_\mathrm{S} + \mathcal{B}_\mathrm{A}$, where

$$\begin{aligned} \mathcal{B}_\mathrm{S}(u,v) &:= (\mathbf{K}\nabla u, \nabla v) + \left(\left(\mu - \tfrac{1}{2}\nabla\cdot\boldsymbol{\beta}\right)u, v\right), \\ \mathcal{B}_\mathrm{A}(u,v) &:= \left(\boldsymbol{\beta}\cdot\nabla u + \tfrac{1}{2}(\nabla\cdot\boldsymbol{\beta})u, v\right). \end{aligned}$$

- $\mathcal{B}_\mathrm{S}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||v|||^2 := \mathcal{B}_\mathrm{S}(v,v)$
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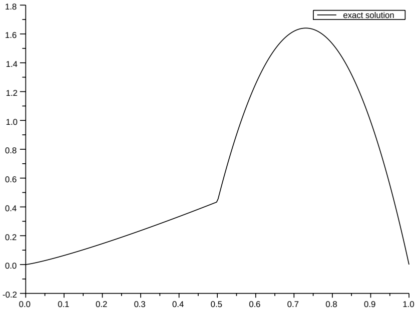
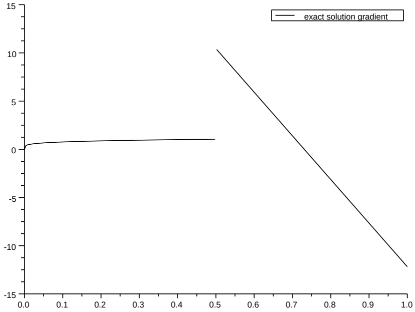
Properties of the weak solution

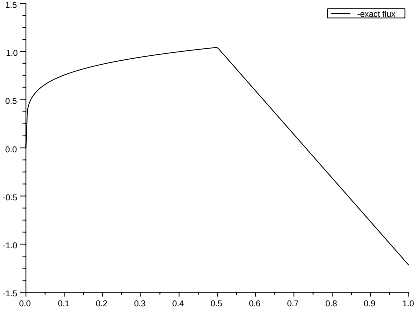


Solution u is in $H_0^1(\Omega)$

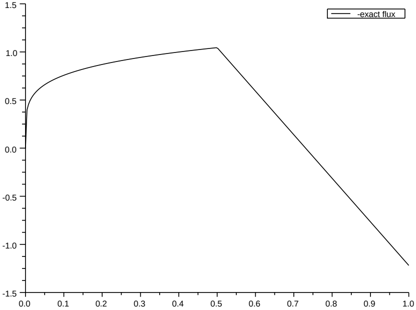


Solution gradient ∇u is not necessarily in $\mathbf{H}(\mathrm{div}, \Omega)$

Properties of the weak solution

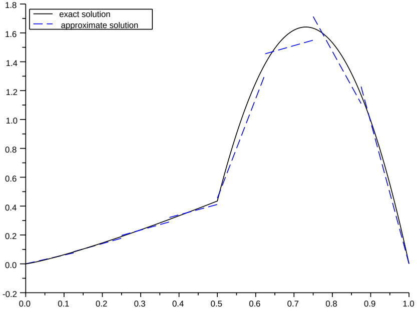


Diffusive flux $-\mathbf{K}\nabla u$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

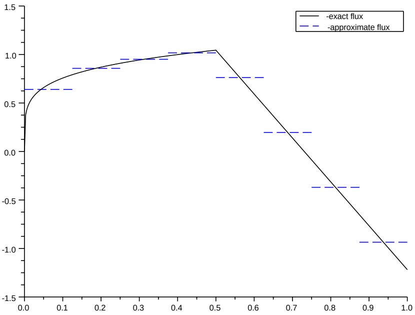


Convective flux βu is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate solution and approximate flux



Approximate solution u_h is not in $H_0^1(\Omega)$



Approximate diffusive and convective fluxes $-\mathbf{K}\nabla u_h$ and βu_h are not in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

1. Introduction and motivation
2. Energy norm setting
   - Optimal energy norm abstract framework
   - A posteriori error estimate
   - Scheme definition
   - Potential and flux reconstructions
   - Local efficiency
3. Augmented norm setting
   - Optimal augmented norm abstract framework
   - A posteriori error estimate and its efficiency
4. Numerical experiments
5. Conclusions and future work

Outline

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract framework, energy norm (Vohralík '07, Ern & Stephansen '08))

Let $u \in H_0^1(\Omega)$ and $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$|||u - u_h||| \leq \inf_{s \in H_0^1(\Omega)} \Big\{ |||u_h - s||| + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ \mathcal{B}(u - u_h, \varphi) + \mathcal{B}_{\mathrm{A}}(u_h - s, \varphi) \} \Big\}$$

$$\leq 2 |||u - u_h|||$$

- specific to the nonconforming and nonsymmetric (CDR) case

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Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of u_h.
- Not computable (infimum over an infinite-dim. space).

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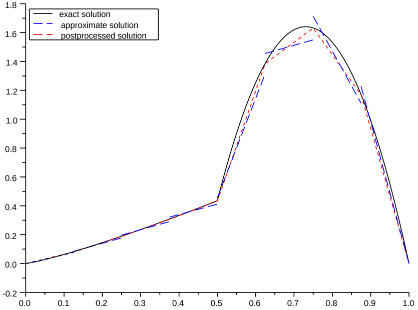
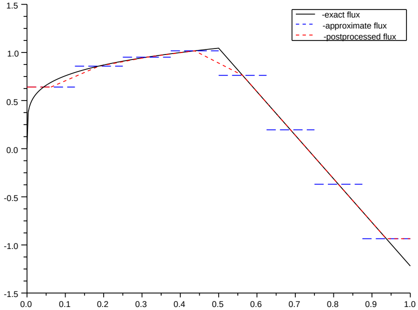


A postprocessed potential s in $H_0^1(\Omega)$



Postprocessed diffusive and convective fluxes $\mathbf{t}$ and $\mathbf{q}$ in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

A post. estimate for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

Theorem (A posteriori error estimate, energy norm)

Let $u_h \in H^1(\mathcal{T}_h)$ (with $-\nabla\cdot(\mathbf{K}\nabla u_h) \in L^2(\Omega)$ and $-(\mathbf{K}\nabla u_h)\cdot\mathbf{n} \in L^2(\mathcal{F}_h)$) be **arbitrary**. *Let $s_h \in H_0^1(\Omega)$ be arbitrary. Let $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{H}(\mathrm{div},\Omega)$ (with $\mathbf{t}_h, \mathbf{q}_h \cdot \mathbf{n} \in L^2(\mathcal{F}_h)$) be such that $(\nabla\cdot\mathbf{t}_h + \nabla\cdot\mathbf{q}_h + (\mu - \nabla\cdot\boldsymbol{\beta})u_h, 1)_{0,T} = (f,1)_{0,T} \forall T \in \mathcal{T}_h$. Then*

$$|||u - u_h||| \le \eta,$$

$$\eta := \left\{\sum_{T\in\mathcal{T}_h} \eta_{\mathrm{NC},T}^2\right\}^{1/2} + \left\{\sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{C},1,T} + \eta_{\mathrm{C},2,T} + \eta_{\mathrm{U},T})^2\right\}^{\frac{1}{2}},$$

- $\eta_{\mathrm{NC},T} = |||u_h - s_h|||_T$ (*nonconformity*),
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A post. estimate for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

Theorem (A posteriori error estimate, energy norm)

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A post. estimate for $-\nabla\cdot(\mathbf{K}\nabla u) = f$

Theorem (A posteriori error estimate, pure diffusion case)

Let $u_h \in H^1(\mathcal{T}_h)$ be **arbitrary**. *Let $s_h \in H^1_0(\Omega)$ be arbitrary. Let $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ be such that $(\nabla\cdot\mathbf{t}_h, 1)_{0,T} = (f, 1)_{0,T} \forall T \in \mathcal{T}_h$. Then*

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Properties of the estimate

Principal properties

- **no scheme definition needed!**
- **only local conservativity necessary!**
- guaranteed upper bound
- no constants in principal estimators, known constants in the other ones
- cutoff functions of local Péclet ($h_T \|\beta\|_{\infty,T} c_{\mathbf{K},T}^{-1}$) and Damköhler ($h_T^2 c_{\beta,\mu,T} c_{\mathbf{K},T}^{-1}$) numbers (here $c_{\beta,\mu,T}$ is the (essential) minimum of $(\mu - \frac{1}{2}\nabla\cdot\beta)$) Verfürth (1998, 2005)
- valid for arbitrary polynomial degree and data (even nonpolynomial)
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$$m_T := \min\{C_{\mathrm{P}}^{1/2} h_T c^{-1/2}_{\mathbf{K},T}, c^{-1/2}_{\beta,\mu,T}\},$$

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Individual estimators

Upwinding estimator $\eta_{\mathrm{U},T}$

- $\eta_{\mathrm{U},T} = \sum_{F\in\mathcal{F}_T} m_F \|\Pi_{0,F}((\mathbf{q}_h - \beta s_h)\cdot\mathbf{n}_F)\|_F$
- cutoff function of local Péclet and Damköhler numbers:

$$m_F^2 = \min\left\{\max_{T\in\mathcal{T}_F}\left\{C_{\mathrm{F},T,F}\frac{|F|h_T^2}{|T|c_{\mathbf{K},T}}\right\}, \max_{T\in\mathcal{T}_F}\left\{\frac{|F|}{|T|c_{\beta,\mu,T}}\right\}\right\}$$

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Outline

Discontinuous Galerkin method

Discontinuous Galerkin method for the CDR case

Find $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $v_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$
\begin{aligned}
(f, v_h) &= (\mathbf{K}\nabla u_h, \nabla v_h) + ((\mu - \nabla\cdot\boldsymbol{\beta})u_h, v_h) - (u_h, \boldsymbol{\beta}\cdot\nabla v_h) \\
&- \sum_{F\in\mathcal{F}_h} \{(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega, [\![v_h]\!])_F + \theta(\mathbf{n}_F\cdot\{\!\{\mathbf{K}\nabla v_h\}\!\}_\omega, [\![u_h]\!])_F\} \\
&+ \sum_{F\in\mathcal{F}_h} \left\{((\alpha_F\gamma_{\mathbf{K},F}h_F^{-1} + \gamma_{\boldsymbol{\beta},F})[\![u_h]\!], [\![v_h]\!])_F + (\boldsymbol{\beta}\cdot\mathbf{n}_F\{\!\{u_h\}\!\}, [\![v_h]\!])_F\right\}.
\end{aligned}
$$

- jump operator $[\![\varphi]\!]_F = \varphi^- - \varphi^+$
- average operator $\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi^- + \varphi^+)$
- harmonic-weighted average op. $\{\!\{\varphi\}\!\}_\omega = (\omega^-\varphi^- + \omega^+\varphi^+)$
- diff.-dep. penalties $\gamma_{\mathbf{K},F}$ (Ern, Stephansen, and Zunino 08)
- θ: different scheme types (SIPG/NIPG/IIPG)
- $\gamma_{\boldsymbol{\beta},F}$: upwind-weighting stabilization
- $u_h \notin H_0^1(\Omega)$, $-\mathbf{K}\nabla u_h \notin \mathbf{H}(\mathrm{div},\Omega)$, $\boldsymbol{\beta}u_h \notin \mathbf{H}(\mathrm{div},\Omega)$

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Outline

Potential- and flux-conforming reconstructions

Choice of s_h: the Oswald interpolate of u_h

- Karakashian and Pascal (2003)
- $\mathcal{I}_{\mathrm{Os}} : \mathbb{P}_k(\mathcal{T}_h) \to \mathbb{P}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$
- prescribed at Lagrange nodes by arithmetic averages

$$\mathcal{I}_{\mathrm{Os}}(v_h)(V) = \frac{1}{\#(\mathcal{T}_V)} \sum_{T \in \mathcal{T}_V} v_h|_T(V)$$

- one can also use diffusivity-weighted averages (Ainsworth '05)

Choice of $\mathbf{t}_h$: a $\mathbf{H}(\mathrm{div}, \Omega)$ flux reconstruction

- Kim (2007) (matching meshes)
- Ern, Nicaise & Vohralík (2007) (matching meshes)
- Ern & Vohralík (2009) (nonmatching meshes)
- related to Ainsworth (2007) and Ainsworth and Rankin (2008)

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Diffusive flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $\qquad\qquad$ $l = 1$

Construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,

$$(\mathbf{t}_h \cdot \mathbf{n}_F, q_h)_F = \left(-\mathbf{n}_F \cdot \{\!\{\mathbf{K}\nabla u_h\}\!\}_\omega + \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} [\![u_h]\!], q_h\right)_F$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(T)$,

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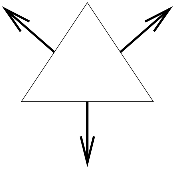
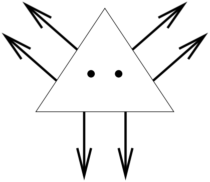


$l = 0$



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Diffusive flux reconstruction property ($\beta = \mu = 0$)

Crucial diffusive flux reconstruction property

- note that all the terms of the DG scheme are used in the construction of $\mathbf{t}_h$
- denote by Π_l the L^2-orthogonal projection onto $\mathbb{P}_k(\mathcal{T}_h)$
- the above construction yields $\nabla\cdot\mathbf{t}_h = \Pi_l(f)$

  Proof: $(\nabla\cdot\mathbf{t}_h, \xi_h)_T = -(\mathbf{t}_h, \nabla\xi_h)_T + \langle\mathbf{t}_h\cdot\mathbf{n}, \xi_h\rangle_{\partial T} = \mathcal{B}_h(u_h, \xi_h) = (f, \xi_h)_T$
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Nonmatching grids ($\beta = \mu = 0$)

Oswald interpolate on nonmatching grids

- consider a matching simplicial submesh $\widehat{\mathcal{T}}_h$ of $\mathcal{T}_h$
- consider $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ as function in $\mathbb{P}_k(\widehat{\mathcal{T}}_h)$
- take $\mathcal{I}_{\mathrm{Os}}(u_h)$ on $\widehat{\mathcal{T}}_h$

Reconstruction of $\mathbf{t}_h$ by direct prescription

- directly prescribe $\mathbf{t}_h \in \mathbf{RTN}^l(\widehat{\mathcal{T}}_h)$ by the values of u_h
- this gives $(\nabla\cdot\mathbf{t}_h, \xi_h)_T = (f, \xi_h)_T$ for all $T \in \mathcal{T}_h$ and all $\xi_h \in \mathbb{P}_l(T)$

Reconstruction of $\mathbf{t}_h$ by solving local linear systems

- consider the simplicial submesh $\mathfrak{R}_T$ of each T
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Convective flux reconstruction

Convective flux reconstruction $\mathbf{q}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k-1$

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Crucial property

$$(\nabla{\cdot}\mathbf{t}_h + \nabla{\cdot}\mathbf{q}_h + (\mu - \nabla{\cdot}\boldsymbol{\beta})u_h, \xi_h)_T = (f, \xi_h)_T \qquad \forall T \in \mathcal{T}_h,\ \forall \xi_h \in \mathbb{P}_l(T)$$

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Outline

Loc. efficiency for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

Theorem (Local efficiency, energy norm)

There holds

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Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers
- result comparable to that of Verfürth (1998)

Loc. efficiency for $-\nabla\cdot(\mathbf{K}\nabla u) + \boldsymbol{\beta}\cdot\nabla u + \mu u = f$

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Outline

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A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(u,v) := -\sum_{F\in\mathcal{F}_h} (\beta\cdot\mathbf{n}_F[\![u]\!], \{\!\{\Pi_0 v\}\!\})_F.$$

- introduce the **augmented norm**

$$|\!|\!|v|\!|\!|_{\oplus} := |\!|\!|v|\!|\!| + \sup_{\varphi\in H_0^1(\Omega),\, |\!|\!|\varphi|\!|\!|=1} \{\mathcal{B}_{\mathrm{A}}(v,\varphi) + \mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\beta\|_{\infty,T}$ is controlled by $(\mu - \frac{1}{2}\nabla\cdot\beta)$ on T for all T and when $v \in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

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Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let u be the weak sol. and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||u - u_h|||_{\oplus} \le 2 \inf_{s \in H_0^1(\Omega)} \Big\{ & |||u_h - s||| \\
& + \inf_{\mathbf{t},\mathbf{q} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ (f - \nabla\cdot\mathbf{t} - \nabla\cdot\mathbf{q} - (\mu - \nabla\cdot\boldsymbol{\beta})u_h, \varphi) \\
& - (\mathbf{K}\nabla_h u_h + \mathbf{t}, \nabla\varphi) + (\nabla\cdot\mathbf{q} - \nabla\cdot(\boldsymbol{\beta}s), \varphi) - (\tfrac{1}{2}(\nabla\cdot\boldsymbol{\beta})(u_h - s), \varphi) \} \Big\} \\
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\end{aligned}
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Comments

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case

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Outline

Augmented norm a posteriori error estimate

Estimator

$$\widetilde{\eta} := 2\eta + \left\{ \sum_{T\in\mathcal{T}_h} (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \widetilde{\eta}_{\mathrm{C},1,T} + \widetilde{\eta}_{\mathrm{U},T})^2 \right\}^{1/2}$$

- η, $\eta_{\mathrm{R},T}$, and $\eta_{\mathrm{DF},T}$ defined previously for the energy norm
- $\widetilde{\eta}_{\mathrm{C},1,T}$ and $\widetilde{\eta}_{\mathrm{U},T}$ – slight modifications of $\eta_{\mathrm{C},1,T}$ and $\eta_{\mathrm{U},T}$

Global jump seminorm

- define

$$|||v|||^2_{\#,\mathcal{F}_h} = \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathfrak{F}_T} \frac{1}{\#(\mathfrak{T}_F)} \left\{ \frac{c_{\mathbf{K},T}}{c_{\mathbf{K},\mathfrak{T}_T}} \alpha_F \gamma_{\mathbf{K},F} h_F^{-1} \|[\![v]\!]\|_F^2 + c_{\beta,\mu,T} h_F \|[\![v]\!]\|_F^2 + m^2_{\mathcal{T}_T} \|\beta\|^2_{\infty,\mathcal{T}_T} h_F^{-1} \|[\![v]\!]\|^2_{0,\mathcal{F}_F\cap\mathfrak{F}_T} \right\},$$

- the first two terms are natural for DG methods
- the third term at least contains the cutoff factor $m_{\mathcal{T}_T}$

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Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h} \leq \widetilde{\eta} + |||u_h|||_{\#,\mathcal{F}_h} \leq \tilde{C}(|||u - u_h|||_{\oplus} + |||u - u_h|||_{\#,\mathcal{F}_h}).$$

- **guaranteed** and **fully robust** with respect to **convection dominance**, **reaction dominance**, and **diffusion inhomogeneities** (no “monotonicity” assumption!)
- sharper than Schötzau & Zhu ’08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and is difficult to evaluate
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

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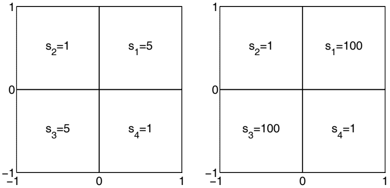
Outline

Discontinuous diffusion tensor and finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot (\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

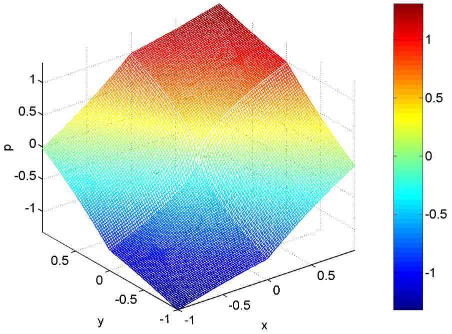


- analytical solution: singularity at the origin

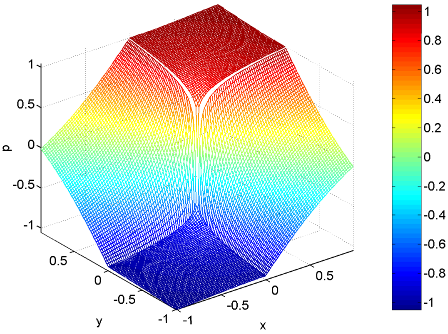
$$p(r,\theta)|_{\Omega_i} = r^\alpha (a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

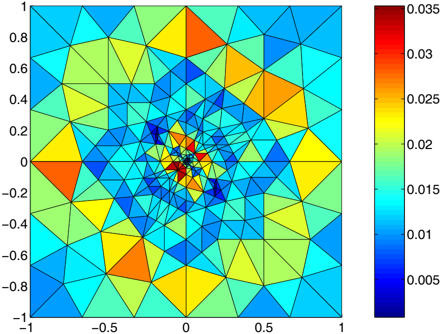


Case 1

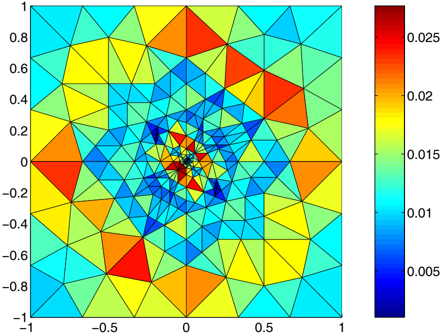


Case 2

Error distribution on an adaptively refined mesh, case 1

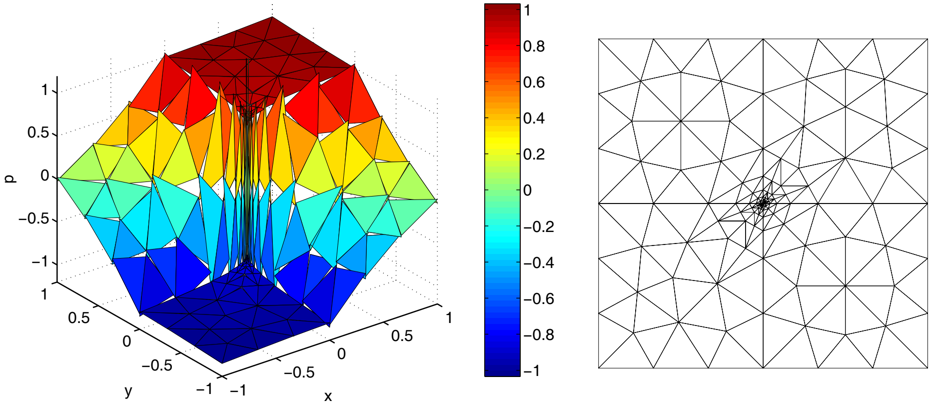


Estimated error distribution

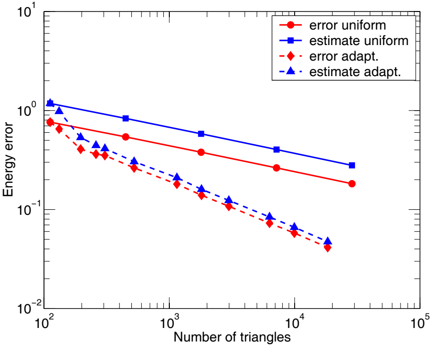


Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, case 2



Estimated and actual errors in uniformly/adaptively refined meshes



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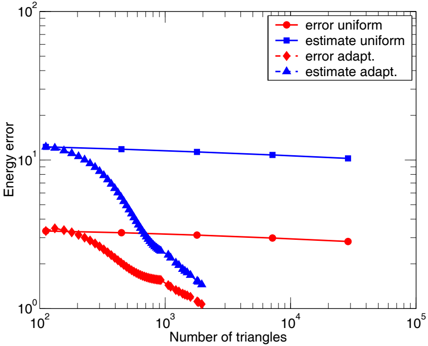
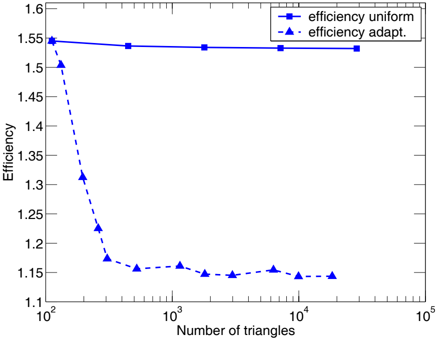
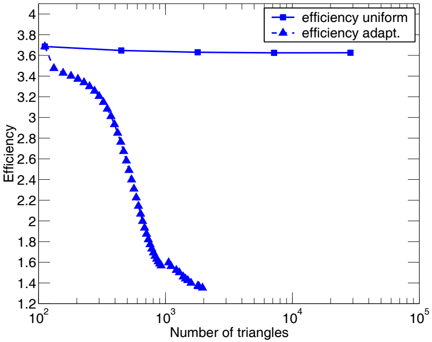


Case 2

Effectivity indices in uniformly/adaptively refined meshes



Case 1



Case 2

Convection-dominated problem, FVs, energy estimates

- consider the convection–diffusion–reaction equation

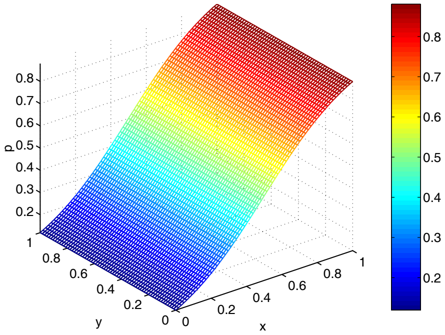
$$-\varepsilon\triangle p + \nabla\cdot(p(0,1)) + p = f \quad \text{in} \quad \Omega = (0,1)\times(0,1)$$

- analytical solution: layer of width a

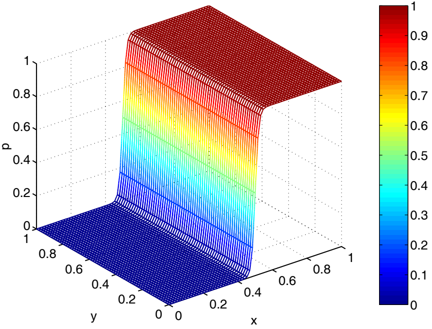
$$p(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5-x}{a}\right)\right)$$

- consider
  - $\varepsilon = 1$, $a = 0.5$
  - $\varepsilon = 10^{-2}$, $a = 0.05$
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- unstructured grid of 46 elements given, uniformly/adaptively refined

Analytical solutions

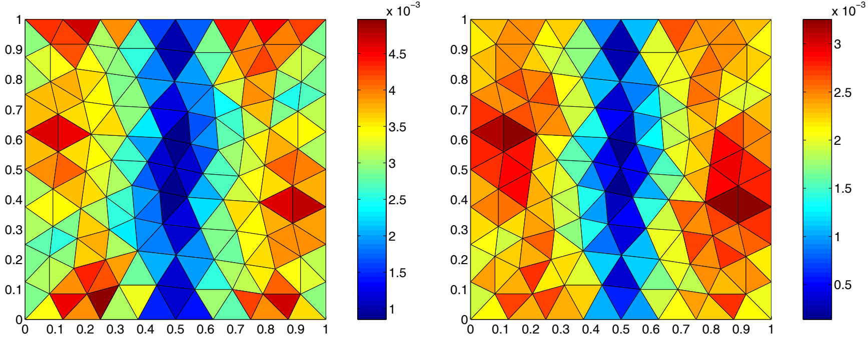


Case $\varepsilon = 1$, $a = 0.5$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

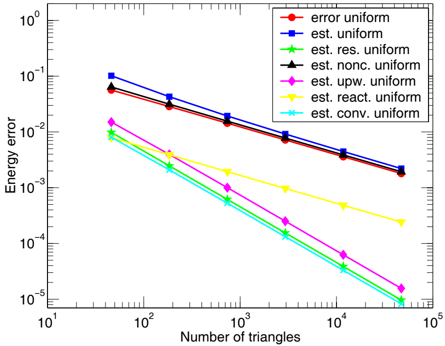
Error distribution on a uniformly refined mesh, $\varepsilon = 1$, $a = 0.5$



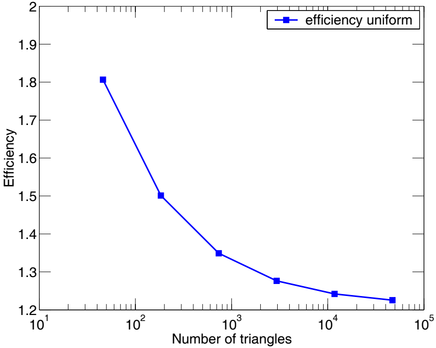
Estimated error distribution

Exact error distribution

Estimated and actual errors and the effectivity index, $\varepsilon = 1$, $a = 0.5$

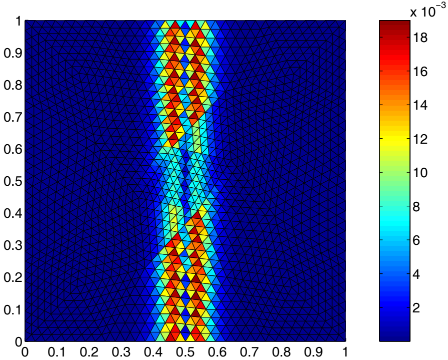


The different estimators

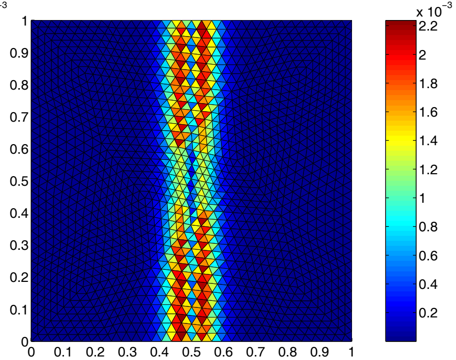


Effectivity index

Error distribution on a uniformly refined mesh, $\varepsilon = 10^{-2}$, $a = 0.05$

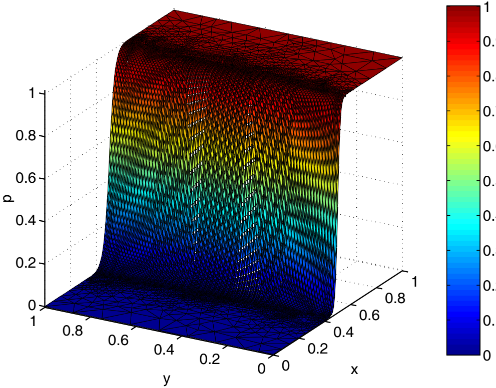


Estimated error distribution

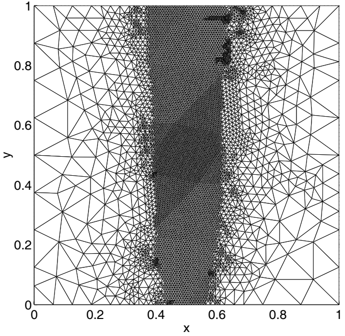


Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$



Approximate solution



Adaptively refined mesh

Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-2}$

	energy norm			augmented norm			
N	err.	est.	eff.	err.	est.	eff.	$\|\|\|p_h\|\|\|_{\#,\mathcal{E}_h}$
128	7.74e-3	1.10e-1	14	1.40e-1	3.28e-1	2.3	3.40e-2
512	4.03e-3	4.35e-2	11	3.97e-2	1.29e-1	3.3	1.16e-2
2048	1.88e-3	1.43e-2	7.6	9.77e-3	4.14e-2	4.2	2.72e-3
8192	9.30e-4	3.58e-3	3.8	2.98e-3	1.02e-2	3.4	8.25e-4
order	1.0	2.0	-	1.7	2.0	-	1.7

Errors ($|||p - p_h|||$ and $|||p - p_h|||_{\oplus'} + |||p - p_h|||_{\#,\mathcal{E}_h}$), estimates ($\eta$ and $\tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-2}$

Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-4}$

	energy norm			augmented norm			
N	err.	est.	eff.	err.	est.	eff.	$\|\|\|p_h\|\|\|_{\#,\mathcal{E}_h}$
128	1.70e-3	1.34e-1	79	3.67e-1	4.05e-1	1.10	4.02e-2
512	5.65e-4	7.01e-2	124	1.44e-1	2.11e-1	1.47	2.11e-2
2048	2.14e-4	3.09e-2	144	5.35e-2	9.36e-2	1.75	9.99e-3
8192	1.00e-4	1.25e-2	125	2.14e-2	3.89e-2	1.82	4.96e-3
order	1.1	1.3	-	1.3	1.3	-	1.0

Errors ($|||p - p_h|||$ and $|||p - p_h|||_{\oplus'} + |||p - p_h|||_{\#,\mathcal{E}_h}$), estimates ($\eta$ and $\tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-4}$

Outline

Conclusions

Conclusions

- guaranteed, locally efficient, and robust a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes (FDs, FVs, FEs, NCFEs, MFEs)
- based on local conservativity

Open questions and future work

Open questions

- are the energy/augmented norms optimal?
- can a robust estimate without the jump seminorm be obtained?
- can a robust estimate in the energy norm be obtained?

Future work

- nonlinear (degenerate) cases
- parabolic convection–reaction–diffusion case

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Thank you for your attention!