Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems

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Outline

1. Introduction and motivation
2. Energy norm setting
   - Optimal energy norm abstract framework
   - A posteriori error estimate
   - Scheme definition
   - Potential and flux reconstructions
   - Local efficiency
3. Augmented norm setting
   - Optimal augmented norm abstract framework
   - A posteriori error estimate and its efficiency
4. Numerical experiments
5. Conclusions and future work
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   - Optimal augmented norm abstract framework
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4 Numerical experiments

5 Conclusions and future work
What is an a posteriori error estimate

A posteriori error estimate

- Let $u$ be a weak solution of a PDE.
- Let $u_h$ be its approximate numerical solution.
- A priori error estimate: $\|u - u_h\|_{\Omega} \leq f(u)h^q$. Dependent on $u$, not computable. Useful in theory.
- A posteriori error estimate: $\|u - u_h\|_{\Omega} \lesssim f(u_h)$. Only uses $u_h$, computable. Great in practice.

Usual form

- $f(u_h)^2 = \sum_{T \in T_h} \eta_T(u_h)^2$, where $\eta_T(u_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.
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What an a posteriori error estimate should fulfill

**Guaranteed upper bound** (global error upper bound)
- \( \| u - u_h \|_\Omega^2 \leq \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2 \)
- no undetermined constant: error control
- remark (reliability): \( \| u - u_h \|_\Omega^2 \leq C \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2 \)

**Local efficiency** (local error lower bound)
- \( \eta_T(u_h)^2 \leq C_{\text{eff}, T} \sum_{T' \text{ close to } T} \| u - u_h \|_{T'}^2 \)
- necessary for optimal mesh refinement

**Asymptotic exactness**
- \( \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2 / \| u - u_h \|_\Omega^2 \to 1 \)
- overestimation factor goes to one with mesh size

**Robustness**
- \( C_{\text{eff}, T} \) does not depend on data, mesh, or solution

**Negligible evaluation cost**
- estimators can be evaluated locally
What an a posteriori error estimate should fulfill

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Previous results on a posteriori error estimation in DG

DG, pure diffusion case

- Karakashian and Pascal (2003), Becker, Hansbo, and Larson (2003), Houston, Süli, and Wihler (2008) residual-based estimates
- Rivière and Wheeler (2003), $L^2$-estimates

DG, convection–diffusion–reaction case

- Sun and Wheeler (2006), $L^2$-estimates
- Schötzau and Zhu (2008), (2009, preprint), res.-based estimates

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Problems with discontinuous coefficients
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- Ainsworth (2005), nonconforming finite elements

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Motivations and key points

Motivations

- establish an optimal abstract framework for a posteriori error estimation in potential- and flux-nonconforming methods
- derive estimates satisfying as many as possible of the five optimal properties

Key points

- focus on inhomogeneous and anisotropic diffusion
- case of nonmatching meshes
- singular regimes of dominant convection or reaction
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- focus on inhomogeneous and anisotropic diffusion
- case of nonmatching meshes
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A model convection–diffusion–reaction problem

\[ -\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{on } \partial \Omega. \]

Bilinear form

\[ B(u, v) := (K \nabla u, \nabla v) + (\beta \cdot \nabla u, v) + (\mu u, v), \quad u, v \in H^1(\mathcal{T}_h) \]

Weak solution

Find \( u \in H^1_0(\Omega) \) such that \( B(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega) \).

Energy norm

Decompose \( B = B_S + B_A \), where

\[ B_S(u, v) := (K \nabla u, \nabla v) + ((\mu - \frac{1}{2} \nabla \cdot \beta) u, v), \]
\[ B_A(u, v) := (\beta \cdot \nabla u + \frac{1}{2} (\nabla \cdot \beta) u, v). \]

- \( B_S \) is symmetric on \( H^1(\mathcal{T}_h) \); put \( \| v \|^2 := B_S(v, v) \)
- \( B_A \) is skew-symmetric on \( H^1_0(\Omega) \)
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Decompose \(\mathcal{B} = \mathcal{B}_S + \mathcal{B}_A\), where

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- \( \mathcal{B}_A \) is skew-symmetric on \( H^1_0(\Omega) \)
Properties of the weak solution

Solution $u$ is in $H^1_0(\Omega)$

Solution gradient $\nabla u$ is not necessarily in $H(\text{div}, \Omega)$
Properties of the weak solution

Diffusive flux $-K \nabla u$ is in $H(\text{div}, \Omega)$

Convective flux $\beta u$ is in $H(\text{div}, \Omega)$
Approximate solution and approximate flux

Approximate solution $u_h$ is not in $H^1_0(\Omega)$

Approximate diffusive and convective fluxes $-K\nabla u_h$ and $\beta u_h$ are not in $H(\text{div}, \Omega)$
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Optimal abstract estimate in the energy norm

Theorem (Optimal abstract framework, energy norm (Vohralík ’07, Ern & Stephansen ’08))

Let \( u \in H^1_0(\Omega) \) and \( u_h \in H^1(\mathcal{T}_h) \) be arbitrary. Then

\[
\| u - u_h \| \leq \inf_{s \in H^1_0(\Omega)} \left\{ \| u_h - s \| + \sup_{\varphi \in H^1_0(\Omega), \| \varphi \| = 1} \left\{ B(u - u_h, \varphi) + B_A(u_h - s, \varphi) \right\} \right\} 
\leq 2\| u - u_h \| 
\]

• specific to the nonconforming and nonsymmetric (CDR) case

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- specific to the nonconforming and nonsymmetric (CDR) case
Theorem (Optimal abstract estimate, energy norm)

Let $u$ be the weak sol. and let $u_h \in H^1(T_h)$ be arbitrary. Then

$$
|||u - u_h||| \leq \inf_{s \in H^1_0(\Omega)} \left\{ |||u_h - s||| \right. \\
+ \inf_{t,q \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), ||\varphi||=1} \left\{ (f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta) u_h, \varphi) \\
- (K \nabla u_h + t, \nabla \varphi) + (\nabla \cdot q - \nabla \cdot (\beta s), \varphi) - \left( \frac{1}{2} (\nabla \cdot \beta)(u_h - s), \varphi \right) \right\} \\
\leq 2 |||u - u_h|||.
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of $u_h$.
- Not computable (infimum over an infinite-dim. space).
Abstract

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract estimate, energy norm)

Let $u$ be the weak sol. and let $u_h \in H^1(T_h)$ be arbitrary. Then

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\|u - u_h\| \leq \inf_{s \in H_0^1(\Omega)} \left\{ \|u_h - s\| + \inf_{t, q \in H(\text{div}, \Omega)} \sup_{\varphi \in H_0^1(\Omega), \|\varphi\|=1} \left\{ \left( f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta)u_h, \varphi \right) \\
- (K \nabla u_h + t, \nabla \varphi) + (\nabla \cdot q - \nabla \cdot (\beta s), \varphi) - \left( \frac{1}{2} (\nabla \cdot \beta)(u_h - s), \varphi \right) \right\} \right\}
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**Theorem (Optimal abstract estimate, energy norm)**

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$$+ \inf_{t, q \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), \|\| \varphi \|\|=1} \left\{ (f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta) u_h, \varphi) - (K \nabla_h u_h + t, \nabla \varphi) + (\nabla \cdot q - \nabla \cdot (\beta s), \varphi) - \left( \frac{1}{2} (\nabla \cdot \beta)(u_h - s), \varphi \right) \right\}$$

$$\leq 2 \|\| u - u_h \|\|.$$  

**Properties**

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of $u_h$.
- Not computable (infimum over an infinite-dim. space).
Approximate solution and approximate flux

A postprocessed potential $s$ in $H^1_0(\Omega)$

Postprocessed diffusive and convective fluxes $t$ and $q$ in $H(\text{div}, \Omega)$
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Let \( u_h \in H^1(T_h) \) (with \( -\nabla \cdot (K \nabla u_h) \in L^2(\Omega) \) and \( -(K \nabla u_h) \cdot n \in L^2(\mathcal{F}_h) \)) be arbitrary. Let \( s_h \in H^1_0(\Omega) \) be arbitrary. Let \( t_h, q_h \in H(\text{div}, \Omega) \) (with \( t_h, q_h \cdot n \in L^2(\mathcal{F}_h) \)) be such that \( (\nabla \cdot t_h + \nabla \cdot q_h + (\mu - \nabla \cdot \beta)u_h, 1)_{0,T} = (f, 1)_{0,T} \forall T \in T_h \). Then

\[
\| u - u_h \| \leq \eta, \\
\eta := \left\{ \sum_{T \in T_h} \eta_{NC,T}^2 \right\}^{1/2} + \left\{ \sum_{T \in T_h} \left( \eta_{R,T} + \eta_{DF,T} + \eta_{C,1,T} + \eta_{C,2,T} + \eta_{U,T} \right)^2 \right\}^{1/2},
\]

- \( \eta_{NC,T} = \| u_h - s_h \|_T \) (nonconformity),
- \( \eta_{DF,T} = \min \left\{ \eta^{(1)}_{DF,T}, \eta^{(2)}_{DF,T} \right\} \) (diffusive flux),
- \( \eta_{R,T} = m_T \| f - \nabla \cdot t_h - \nabla \cdot q_h - (\mu - \nabla \cdot \beta)u_h \|_{0,T} \) (residual),
- \( \eta_{C,1,T} = m_T \| (Id - \Pi_0)(\nabla \cdot (q_h - \beta s_h)) \|_{0,T} \) (convection),
- \( \eta_{C,2,T} = c_{\beta,\mu,T}^{-1/2} \| \frac{1}{2} (\nabla \cdot \beta)(u_h - s_h) \|_{0,T} \) (convection),
- \( \eta_{U,T} = \sum_{F \in \mathcal{F}_T} m_F \| \Pi_{0,F}((q_h - \beta s_h) \cdot n_F) \|_F \) (upwinding).
A post. estimate for $-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f$

Theorem (A posteriori error estimate, energy norm)

Let $u_h \in H^1(\mathcal{T}_h)$ (with $-\nabla \cdot (K \nabla u_h) \in L^2(\Omega)$ and $-(K \nabla u_h) \cdot n \in L^2(\mathcal{F}_h)$) be arbitrary. Let $s_h \in H^1_0(\Omega)$ be arbitrary. Let $t_h, q_h \in H(\text{div}, \Omega)$ (with $t_h, q_h \cdot n \in L^2(\mathcal{F}_h)$) be such that $(\nabla \cdot t_h + \nabla \cdot q_h + (\mu - \nabla \cdot \beta) u_h, 1)_{0,T} = (f, 1)_{0,T} \forall T \in \mathcal{T}_h$. Then

$\|u - u_h\| \leq \eta,$

$\eta := \left\{ \sum_{T \in \mathcal{T}_h} \eta_{NC,T}^2 \right\}^{1/2} + \left\{ \sum_{T \in \mathcal{T}_h} \left( \eta_R,T + \eta_{DF,T} + \eta_{C,1,T} + \eta_{C,2,T} + \eta_{U,T} \right)^2 \right\}^{1/2},$

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- $\eta_{R,T} = m_T \|f - \nabla \cdot t_h - \nabla \cdot q_h - (\mu - \nabla \cdot \beta) u_h\|_{0,T}$ (residual),
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A post. estimate for $-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f$

Theorem (A posteriori error estimate, energy norm)

Let $u_h \in H^1(T_h)$ (with $-\nabla \cdot (K \nabla u_h) \in L^2(\Omega)$ and $-(K \nabla u_h) \cdot n \in L^2(\mathcal{F}_h)$) be arbitrary. Let $s_h \in H^1_0(\Omega)$ be arbitrary. Let $t_h, q_h \in H(\text{div}, \Omega)$ (with $t_h, q_h \cdot n \in L^2(\mathcal{F}_h)$) be such that $(\nabla \cdot t_h + \nabla \cdot q_h + (\mu - \nabla \cdot \beta) u_h, 1)_{0,T} = (f, 1)_{0,T} \forall T \in T_h$. Then

$$||| u - u_h ||| \leq \eta,$$

where

$$\eta := \left\{ \sum_{T \in T_h} \eta_{NC,T}^2 \right\}^{1/2} + \left\{ \sum_{T \in T_h} (\eta_{R,T} + \eta_{DF,T} + \eta_{C,1,T} + \eta_{C,2,T} + \eta_{U,T})^2 \right\}^{1/2},$$

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Abstract Estimate Scheme Reconstructions Efficiency

Energy norm Augmented norm Numerical experiments C

A post. estimate for $-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f$

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Let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $s_h \in H^1_0(\Omega)$ be arbitrary. Let $t_h \in H(\text{div}, \Omega)$ be such that $(\nabla \cdot t_h, 1)_{0,T} = (f, 1)_{0,T} \forall T \in \mathcal{T}_h$. Then

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A. Ern, A. F. Stephansen & M. Vohralík
Guaranteed and robust estimates for DG
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Properties of the estimate

Principal properties

- **no scheme definition needed!**
- **only local conservativity necessary!**
- guaranteed upper bound
- **no constants in principal estimators, known constants in the other ones**
- cutoff functions of local Péclet \( h_T \| \beta \|_{\infty, T} c_{K, T}^{-1} \) and Damköhler \( h_T^2 c_{\beta, \mu, T} c_{K, T}^{-1} \) numbers (here \( c_{\beta, \mu, T} \) is the (essential) minimum of \( (\mu - \frac{1}{2} \nabla \cdot \beta) \)) Verfürth (1998, 2005)
- valid for arbitrary polynomial degree and data (even nonpolynomial)
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- residual estimator \( \eta_{R, T} \) is evaluated for the diffusive and convective fluxes and will be a higher-order term (data oscillation)
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Individual estimators

Diffusive flux estimator $\eta_{DF,T}$

- $\eta_{DF,T} = \min \left\{ \eta_{DF,T}^{(1)}, \eta_{DF,T}^{(2)} \right\}$
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- cutoff functs of local Péclet and Damköhler numbers in $\eta_{DF,T}^{(2)}$:
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- $\eta_{DF,T}^{(1)}$ alone cannot be shown semi-robust (Verfürth ’08)
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- cutoff fcts of local Péclet and Damköhler numbers in $\eta_{DF,T}^{(2)}$:

  $$m_T := \min \{ C_P^{1/2} h_T c_{K,T}^{-1/2}, c_{\beta,\mu,T}^{-1/2} \},$$
  $$\tilde{m}_T := \min \{ (C_P + C_P^{1/2}) h_T c_{K,T}^{-1}, h_T^{-1} c_{\beta,\mu,T}^{-1} + c_{\beta,\mu,T}^{-1/2} c_{K,T}^{-1/2} / 2 \}$$

- $\eta_{DF,T}^{(1)}$ alone cannot be shown semi-robust (Verfürth ’08)
- the idea of defining of $\eta_{DF,T}$ using a min has recently been proposed in Cheddadi, Fučík, Prieto, Vohralík ’08 in context of conforming FEM and reaction–diffusion problems


**Upwinding estimator** $\eta_{U,T}$

- $\eta_{U,T} = \sum_{F \in \mathcal{T}} m_F \|\Pi_{0,F}(q_h - \beta s_h) \cdot n_F\|_F$
- cutoff function of local Péclet and Damköhler numbers:

$$m_F^2 = \min \left\{ \max_{T \in \mathcal{T}_F} \left\{ C_{F,T,F} \frac{|F|}{|T| c_{K,T}} \right\}, \max_{T \in \mathcal{T}_F} \left\{ \frac{|F|}{|T| c_{\beta,\mu,T}} \right\} \right\}$$
Individual estimators

Upwinding estimator $\eta_{U,T}$

- $\eta_{U,T} = \sum_{F \in \mathcal{F}_T} m_F \| \Pi_0,F ((q_h - \beta s_h) \cdot n_F) \|_F$
- cutoff function of local Péclet and Damköhler numbers:

$$m_F^2 = \min \left\{ \max_{T \in \mathcal{T}_F} \left\{ C_{F,T,F} \frac{|F|}{|T| c_{\mathcal{K},T}} \right\}, \max_{T \in \mathcal{T}_F} \left\{ \frac{|F|}{|T| c_{\beta,\mu,T}} \right\} \right\}$$
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   - A posteriori error estimate
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   - Optimal augmented norm abstract framework
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Discontinuous Galerkin method

Discontinuous Galerkin method for the CDR case
Find $u_h \in P_k(T_h)$ such that for all $v_h \in P_k(T_h)$

$$(f, v_h) = (K \nabla u_h, \nabla v_h) + ((\mu - \nabla \cdot \beta) u_h, v_h) - (u_h, \beta \cdot \nabla v_h)$$

$$- \sum_{F \in \mathcal{F}_h} \left\{ (n_F \cdot \{K \nabla u_h\}_\omega, [v_h])_F + \theta (n_F \cdot \{K \nabla v_h\}_\omega, [u_h])_F \right\}$$

$$+ \sum_{F \in \mathcal{F}_h} \left\{ (\alpha_F \gamma_{K,F} h_F^{-1} + \gamma_{\beta,F}) [u_h], [v_h])_F + (\beta \cdot n_F \{u_h\}, [v_h])_F \right\}.$$ 

- jump operator $[\varphi]_F = \varphi^- - \varphi^+$
- average operator $\{\varphi\} = \frac{1}{2}(\varphi^- + \varphi^+)$
- harmonic-weighted average op. $\{\varphi\}_\omega = (\omega^- \varphi^- + \omega^+ \varphi^+)$
- diff.-dep. penalties $\gamma_{K,F}$ (Ern, Stephansen, and Zunino 08)
- $\theta$: different scheme types (SIPG/NIPG/IIPG)
- $\gamma_{\beta,F}$: upwind-weighting stabilization
- $u_h \notin H^1_0(\Omega)$, $-K \nabla u_h \notin H(\text{div}, \Omega)$, $\beta u_h \notin H(\text{div}, \Omega)$
Discontinuous Galerkin method

Discontinuous Galerkin method for the CDR case

Find $u_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $v_h \in \mathbb{P}_k(\mathcal{T}_h)$

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$$+ \sum_{F \in \mathcal{F}_h} \left\{(\alpha_F \gamma_{K,F} h_F^{-1} + \gamma_{\beta,F}) [u_h], [v_h]_F + (\beta \cdot n_F \{u_h\}, [v_h])_F\right\}.$$ 

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+ \sum_{F \in \mathcal{F}_h} \left\{ ((\alpha_F \gamma_K, F h_F^{-1} + \gamma_{\beta, F})[u_h], [v_h])_F + (\beta \cdot n_F \{ u_h \}, [v_h])_F \right\}.
\]

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- average operator \{\varphi\} = \frac{1}{2}(\varphi^- + \varphi^+)
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- \(u_h \not\in H^1_0(\Omega), -K \nabla u_h \not\in H(\text{div}, \Omega), \beta u_h \not\in H(\text{div}, \Omega)\)

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Guaranteed and robust estimates for DG
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Guaranteed and robust estimates for DG
Potential- and flux-conforming reconstructions

Choice of $s_h$: the **Oswald interpolate** of $u_h$

- Karakashian and Pascal (2003)
- $\mathcal{I}_{Os}: \mathbb{P}_k(\mathcal{T}_h) \to \mathbb{P}_k(\mathcal{T}_h) \cap H^1_0(\Omega)$
- prescribed at Lagrange nodes by arithmetic averages

\[ \mathcal{I}_{Os}(v_h)(V) = \frac{1}{\#(\mathcal{T}_V)} \sum_{T \in \mathcal{T}_V} v_h|_T(V) \]

- one can also use diffusivity-weighted averages (Ainsworth '05)

Choice of $t_h$: a $H(\text{div}, \Omega)$ flux reconstruction

- Kim (2007) (matching meshes)
- Ern, Nicaise & Vohralík (2007) (matching meshes)
- Ern & Vohralík (2009) (nonmatching meshes)
- related to Ainsworth (2007) and Ainsworth and Rankin (2008)
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Diffusive flux reconstruction

$\text{RTN}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree $l$

Construction of $t_h \in \text{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$
- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,
  $$(t_h \cdot n_F, q_h)_F = (-n_F \cdot \{K \nabla u_h\})_\omega + \alpha_F \gamma_{K,F} h_F^{-1} [u_h], q_h)_F$$
- on each element (only for $l \geq 1$): $\forall r_h \in \mathbb{P}^d_{l-1}(T)$,
  $$(t_h, r_h)_T = -(K \nabla u_h, r_h)_T + \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (n_F \cdot Kr_h, [u_h])_F$$
Diffusive flux reconstruction

**RTN**\(^l(\mathcal{I}_h)\): Raviart–Thomas–Nédélec spaces of degree \(l\)

\[ l = 0 \quad \text{and} \quad l = 1 \]

**Construction of** \(t_h \in \text{RTN}^l(\mathcal{I}_h), \ l = k \ \text{or} \ l = k - 1\)

- normal components on each side: \(\forall q_h \in \mathbb{P}_l(F),\)
  \[
  (t_h \cdot n_F, q_h)_F = (-n_F \cdot \{K \nabla u_h\})_F + \alpha_F \gamma_K,F h_F^{-1} \{u_h\}, q_h)_F
  \]
- on each element (only for \(l \geq 1\)): \(\forall r_h \in \mathbb{P}^d_{l-1}(T),\)
  \[
  (t_h, r_h)_T = -(K \nabla u_h, r_h)_T + \theta \sum_{F \in \mathcal{F}_T} \omega_{T,F} (n_F \cdot Kr_h, \{u_h\})_F
  \]
Crucial diffusive flux reconstruction property

- note that all the terms of the DG scheme are used in the construction of $t_h$
- denote by $\Pi_l$ the $L^2$-orthogonal projection onto $\mathbb{P}_k(T_h)$
- the above construction yields $\nabla \cdot t_h = \Pi_l(f)$

Proof: $(\nabla \cdot t_h, \xi_h)_T = -(t_h, \nabla \xi_h)_T + \langle t_h \cdot n, \xi_h \rangle_{\partial T} = B_h(u_h, \xi_h) = (f, \xi_h)_T$

diffusive flux as in the Raviart–Thomas–Nédélec mixed finite element method of order $l$, by local postprocessing
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Proof: $\langle \nabla \cdot t_h, \xi_h \rangle_T = -\langle t_h, \nabla \xi_h \rangle_T + \langle t_h \cdot n, \xi_h \rangle_{\partial T} = B_h(u_h, \xi_h) = (f, \xi_h)_T$

- diffusive flux as in the Raviart–Thomas–Nédélec mixed finite element method of order $l$, by local postprocessing
Diffusive flux reconstruction property \((\beta = \mu = 0)\)

Crucial diffusive flux reconstruction property

- Note that all the terms of the DG scheme are used in the construction of \(t_h\)
- Denote by \(\Pi_l\) the \(L^2\)-orthogonal projection onto \(\mathbb{P}_k(T_h)\)
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**Proof:**

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(\nabla \cdot t_h, \xi_h)_T = -(t_h, \nabla \xi_h)_T + \langle t_h \cdot n, \xi_h \rangle_{\partial T} = B_h(u_h, \xi_h) = (f, \xi_h)_T
\]

- Diffusive flux as in the Raviart–Thomas–Nédélec mixed finite element method of order \(l\), by local postprocessing.
**Crucial diffusive flux reconstruction property**

- note that all the terms of the DG scheme are used in the construction of $t_h$
- denote by $\Pi_l$ the $L^2$-orthogonal projection onto $P_k(T_h)$
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- diffusive flux as in the Raviart–Thomas–Nédélec mixed finite element method of order $l$, by local postprocessing
Nonmatching grids ($\beta = \mu = 0$)

Oswald interpolate on nonmatching grids

- consider a matching simplicial submesh $\hat{T}_h$ of $T_h$
- consider $u_h \in P_k(T_h)$ as function in $P_k(\hat{T}_h)$
- take $I_{Os}(u_h)$ on $\hat{T}_h$

Reconstruction of $t_h$ by direct prescription

- directly prescribe $t_h \in RTN^l(\hat{T}_h)$ by the values of $u_h$
- this gives $(\nabla \cdot t_h, \xi_h)_T = (f, \xi_h)_T$ for all $T \in T_h$ and all $\xi_h \in P_l(T)$

Reconstruction of $t_h$ by solving local linear systems

- consider the simplicial submesh $R_T$ of each $T$
- solve a local minimization problem (local linear system) on each $T$
- get in particular $\nabla \cdot t_h = \hat{\Pi}_l f$
Nonmatching grids ($\beta = \mu = 0$)

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Reconstruction of \(t_h\) by solving local linear systems
- consider the simplicial submesh \(R_T\) of each \(T\)
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Convective flux reconstruction $q_h \in RTN^l(I_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in P_l(F),
  \langle q_h \cdot n_F, q_h \rangle_F = (\beta \cdot n_F \{u_h\} + \gamma_{\beta,F}[u_h], q_h)_F$
  - on each element (only for $l \geq 1$): $\forall r_h \in P^d_{l-1}(T),
  \langle q_h, r_h \rangle_T = (u_h, \beta \cdot r_h)_T$

Crucial property

\[
(\nabla \cdot t_h + \nabla \cdot q_h + (\mu - \nabla \cdot \beta) u_h, \xi_h)_T = (f, \xi_h)_T \quad \forall T \in I_h, \forall \xi_h \in P_l(T)
\]
Convective flux reconstruction $q_h \in RTN^l(T_h), \ l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in P_l(F),$

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- on each element (only for $l \geq 1$): $\forall r_h \in P_{d-1}^l(T),$

$$\langle q_h, r_h \rangle_T = (u_h, \beta \cdot r_h)_T$$

Crucial property

$$\langle \nabla \cdot t_h + \nabla \cdot q_h + (\mu - \nabla \cdot \beta)u_h, \xi_h \rangle_T = (f, \xi_h)_T \quad \forall T \in T_h, \ \forall \xi_h \in P_l(T)$$
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Theorem (Local efficiency, energy norm)

There holds

$$\eta_{\text{NC}, T} + \eta_{\text{DF}, T} + \eta_{\text{R}, T} + \eta_{\text{C}, 1, T} + \eta_{\text{C}, 2, T} + \eta_{\text{U}, T} \leq C_{\text{eff}, T} \| u - u_h \|_{*, \tilde{E}_T}.$$ 

Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers
- result comparable to that of Verfürth (1998)
Abstract

Loc. efficiency for $-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f$

Theorem (Local efficiency, energy norm)

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$\eta_{NC,T} + \eta_{DF,T} + \eta_R,T + \eta_{C,1,T} + \eta_{C,2,T} + \eta_U,T \leq C_{\text{eff},T} \| u - u_h \|_{*,\tilde{E}_T}$.

Properties

- the estimates are locally efficient
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Loc. efficiency for \(-\nabla \cdot (K \nabla u) + \beta \cdot \nabla u + \mu u = f\)

**Theorem (Local efficiency, energy norm)**

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**Properties**

- the estimates are **locally** efficient
- only **semi-robustness**: overestimation is a function of local Péclet and Damköhler numbers
- result comparable to that of Verfürth (1998)
Theorem (Local efficiency, energy norm)

There holds

\[ \eta_{\text{NC}}, T + \eta_{\text{DF}}, T + \eta_{\text{R}}, T + \eta_{\text{C},1}, T + \eta_{\text{C},2}, T + \eta_{\text{U}}, T \leq C_{\text{eff}}, T \| u - u_h \|_{*, \bar{\tilde{E}}_T}. \]

Properties

- the estimates are **locally** efficient
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2. Energy norm setting

3. Augmented norm setting

4. Numerical experiments

5. Conclusions and future work

A. Ern, A. F. Stephansen & M. Vohralík
Guaranteed and robust estimates for DG
A dual norm augmented by the convective derivative

- define
  \[ B_D(u, v) := - \sum_{F \in \mathcal{F}_h} (\beta \cdot n_F[u], \{\Pi_0 v\})_F. \]

- introduce the augmented norm
  \[ \|v\|_\oplus := \|v\| + \sup_{\varphi \in H_0^1(\Omega), \|\varphi\| = 1} \{B_A(v, \varphi) + B_D(v, \varphi)\} \]

- when \(\|\nabla \cdot \beta\|_\infty, T\) is controlled by \((\mu - \frac{1}{2} \nabla \cdot \beta)\) on \(T\) for all \(T\) and when \(v \in H_0^1(\Omega)\), recover the augmented norm introduced by Verfürth ’05

- \(B_D\) contribution is new and specific to the nonconforming case
A dual norm augmented by the convective derivative

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- when \( \|\nabla \cdot \beta\|_\infty, T \) is controlled by \( (\mu - \frac{1}{2} \nabla \cdot \beta) \) on \( T \) for all \( T \) and when \( v \in H_0^1(\Omega) \), recover the augmented norm introduced by Verfürth ’05

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- when \( \|\nabla \cdot \beta\|_{\infty, T} \) is controlled by \( (\mu - \frac{1}{2} \nabla \cdot \beta) \) on \( T \) for all \( T \) and when \( v \in H_0^1(\Omega) \), recover the augmented norm introduced by Verfürth ’05

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- \( B_D \) contribution is new and specific to the nonconforming case
Optimal abstract estimate in the augmented norm

**Theorem (Optimal abstract estimate, augmented norm)**

Let $u$ be the weak sol. and let $u_h \in H^1(T_h)$ be arbitrary. Then

$$
\|\| u - u_h \|\|_\oplus \leq 2 \inf_{s \in H^1_0(\Omega)} \left\{ \| u_h - s \| \right. \\
+ \inf_{t, q \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), \| \varphi \| = 1} \left\{ (f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta) u_h, \varphi) \\
- (K \nabla u_h + t, \nabla \varphi) + (\nabla \cdot q - \nabla \cdot (\beta s), \varphi) - \left( \frac{1}{2} (\nabla \cdot \beta) (u_h - s), \varphi \right) \right\} \\
+ \inf_{t \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), \| \varphi \| = 1} \left\{ (f - \nabla \cdot t - \beta \cdot \nabla u_h - \mu u_h, \varphi) \\
- (K \nabla u_h + t, \nabla \varphi) - B_D(u_h, \varphi) \right\} \leq 5 \| u - u_h \|_\oplus.
$$

**Comments**

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case
Optimal abstract estimate in the augmented norm

**Theorem (Optimal abstract estimate, augmented norm)**

Let $u$ be the weak sol. and let $u_h \in H^1(I_h)$ be arbitrary. Then

$$\|\|u - u_h\|\|_\oplus \leq 2 \inf_{s \in H^1_0(\Omega)} \{ \|u_h - s\|$$

$$+ \inf_{t,q \in H(\text{div},\Omega)} \sup_{\varphi \in H^1_0(\Omega), \|\varphi\| = 1} \left\{ (f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta)u_h, \varphi)\right. $$

$$- (K \nabla_h u_h + t, \nabla \varphi) + (\nabla \cdot q - \nabla \cdot (\beta s), \varphi) - \left(\frac{1}{2}(\nabla \cdot \beta)(u_h - s), \varphi\right) \right\}$$

$$+ \inf_{t \in H(\text{div},\Omega)} \sup_{\varphi \in H^1_0(\Omega), \|\varphi\| = 1} \left\{ (f - \nabla \cdot t - \beta \cdot \nabla_h u_h - \mu u_h, \varphi) $$

$$- (K \nabla_h u_h + t, \nabla \varphi) - B_D(u_h, \varphi) \right\} \leq 5\|\|u - u_h\|\|_\oplus.$$

**Comments**

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case
Theorem (Optimal abstract estimate, augmented norm)

Let $u$ be the weak sol. and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$||| u - u_h |||_{\oplus} \leq 2 \inf_{s \in H^1_0(\Omega)} \left\{ ||| u_h - s ||| \right\}$$

$$+ \inf_{t, q \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), |||\varphi||| = 1} \left\{ (f - \nabla \cdot t - \nabla \cdot q - (\mu - \nabla \cdot \beta) u_h, \varphi) \right. - (K \nabla_h u_h + t, \nabla \varphi) + \left( \nabla \cdot q - \nabla \cdot (\beta s), \varphi \right) - \left( \frac{1}{2} (\nabla \cdot \beta)(u_h - s), \varphi \right) \right\}$$

$$+ \inf_{t \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), |||\varphi||| = 1} \left\{ (f - \nabla \cdot t - \beta \cdot \nabla_h u_h - \mu u_h, \varphi) \right. - (K \nabla_h u_h + t, \nabla \varphi) - B_D(u_h, \varphi) \right\} \leq 5 ||| u - u_h |||_{\oplus}.$$

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Augmented norm a posteriori error estimate

Estimator

\[ \tilde{\eta} := 2\eta + \left\{ \sum_{T \in \mathcal{T}_h} \left( \eta_{R,T} + \eta_{DF,T} + \tilde{\eta}_{C,1,T} + \tilde{\eta}_{U,T} \right)^2 \right\}^{1/2} \]

- \( \eta, \eta_{R,T}, \) and \( \eta_{DF,T} \) defined previously for the energy norm
- \( \tilde{\eta}_{C,1,T} \) and \( \tilde{\eta}_{U,T} \) – slight modifications of \( \eta_{C,1,T} \) and \( \eta_{U,T} \)

Global jump seminorm

define

\[ \|v\|_{\#,F_h}^2 = \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{S}_T} \frac{1}{\#(\mathcal{S}_F)} \left\{ \begin{array}{c} c_{K,T} \frac{\alpha_F \gamma_{K,F} h_F^{-1}}{c_{K,\mathcal{S}_T}} \|v\|_F^2 \\ + c_{\beta,\mu,T} h_F \|v\|_F^2 + m_{\mathcal{J}_T}^2 \|\beta\|_{\mathcal{J}_T}^2 h_F^{-1} \|v\|_0,\mathcal{F}_F \cap \mathcal{S}_T \end{array} \right\} \]

- the first two terms are natural for DG methods
- the third term at least contains the cutoff factor \( m_{\mathcal{J}_T} \)

A. Ern, A. F. Stephansen & M. Vohralík
Guaranteed and robust estimates for DG
Augmented norm a posteriori error estimate

Estimator

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- $\tilde{\eta}_{C,1,T}$ and $\tilde{\eta}_{U,T}$ – slight modifications of $\eta_{C,1,T}$ and $\eta_{U,T}$

Global jump seminorm

- define

$$\|\|v\|\|_{\#,F_h}^2 = \sum_{T \in T_h} \sum_{F \in \mathcal{F}_T} \frac{1}{\#(\mathcal{S}_F)} \left\{ \frac{c_{K,T}}{c_{K,T}} \alpha_F \gamma_{K,F} h_F^{-1} \|\|v\|\|_F^2 + c_{\beta,\mu,T} h_F \|\|v\|\|_F^2 + m_{\mathcal{I}_T}^2 \|\|v\|\|_F^2 \right\},$$

- the first two terms are natural for DG methods
- the third term at least contains the cutoff factor $m_{\mathcal{I}_T}$
Theorem (Fully robust a posteriori estimate)

There holds

$$\|\| u - u_h \|\|_{\oplus} + \|\| u - u_h \|\|_{\#,F_h} \leq \tilde{\eta} + \|\| u_h \|\|_{\#,F_h}$$

$$\leq \tilde{C}(\|\| u - u_h \|\|_{\oplus} + \|\| u - u_h \|\|_{\#,F_h}).$$

- guaranteed and fully robust with respect to convection dominance, reaction dominance, and diffusion inhomogeneities (no “monotonicity” assumption!)
- sharper than Schötzau & Zhu ’08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $\|\| \cdot \|\|_{\oplus}$ is a dual norm and is difficult to evaluate
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)
Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

\[ \| u - u_h \|_\oplus + \| u - u_h \|_{\#, \mathcal{F}_h} \leq \tilde{\eta} + \| u_h \|_{\#, \mathcal{F}_h} \leq \tilde{\mathcal{C}}(\| u - u_h \|_\oplus + \| u - u_h \|_{\#, \mathcal{F}_h}). \]

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Theorem (Fully robust a posteriori estimate)

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\[
\|u - u_h\|_\oplus + \|u - u_h\|_{\#,\mathcal{F}_h} \leq \tilde{\eta} + \|u_h\|_{\#,\mathcal{F}_h}
\]

\[
\leq \tilde{C}(\|u - u_h\|_\oplus + \|u - u_h\|_{\#,\mathcal{F}_h}).
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$$\left\| \left\| u - u_h \right\| \right\|_{\oplus} + \left\| \left\| u - u_h \right\| \right\|_{\#} \leq \tilde{\eta} + \left\| u_h \right\|_{\#, \mathcal{F}_h}$$

$$\leq \tilde{C} \left( \left\| u - u_h \right\|_{\oplus} + \left\| u - u_h \right\|_{\#} \right).$$

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\[ \leq \tilde{C}(\|\| u - u_h \|\|_\oplus + \|\| u - u_h \|\|_{\#, \mathcal{F}_h}). \]

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Theorem (Fully robust a posteriori estimate)

There holds
\[
\|\| u - u_h \|\|_\oplus + \|\| u - u_h \|\|_{\#, \mathcal{F}_h} \leq \tilde{\eta} + \|\| u_h \|\|_{\#, \mathcal{F}_h} \\
\leq \tilde{C}(\|\| u - u_h \|\|_\oplus + \|\| u - u_h \|\|_{\#, \mathcal{F}_h}).
\]

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consider the pure diffusion equation
\[-\nabla \cdot (\mathbf{S} \nabla p) = 0 \quad \text{in} \quad \Omega = (-1, 1) \times (-1, 1)\]
discontinuous and inhomogeneous $\mathbf{S}$, two cases:

analytical solution: singularity at the origin
\[p(r, \theta) \big|_{\Omega_i} = r^\alpha (a_i \sin(\alpha \theta) + b_i \cos(\alpha \theta))\]

$(r, \theta)$ polar coordinates in $\Omega$
$a_i, b_i$ constants depending on $\Omega_i$
$\alpha$ regularity of the solution
Analytical solutions

Case 1

Case 2

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Guaranteed and robust estimates for DG
Error distribution on an adaptively refined mesh, case 1

Estimated error distribution

Exact error distribution
Approximate solution and the corresponding adaptively refined mesh, case 2
Estimated and actual errors in uniformly/adaptively refined meshes

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of triangles</td>
<td>Energy error</td>
</tr>
<tr>
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<td>$10^0$</td>
</tr>
<tr>
<td>$10^1$</td>
<td>$10^1$</td>
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<tr>
<td>$10^2$</td>
<td>$10^2$</td>
</tr>
<tr>
<td>$10^3$</td>
<td>$10^3$</td>
</tr>
<tr>
<td>$10^4$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>$10^5$</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>

Estimated and actual errors in uniformly/adaptively refined meshes

A. Ern, A. F. Stephansen & M. Vohralík
Guaranteed and robust estimates for DG
Effectivity indices in uniformly/adaptively refined meshes

Case 1

Case 2
Convection-dominated problem, FVs, energy estimates

- consider the convection–diffusion–reaction equation
  \[
  -\varepsilon \Delta p + \nabla \cdot (p(0, 1)) + p = f \quad \text{in} \quad \Omega = (0, 1) \times (0, 1)
  \]

- analytical solution: layer of width \( a \)
  \[
  p(x, y) = 0.5 \left( 1 - \tanh\left( \frac{0.5 - x}{a} \right) \right)
  \]

- consider
  - \( \varepsilon = 1, \ a = 0.5 \)
  - \( \varepsilon = 10^{-2}, \ a = 0.05 \)
  - \( \varepsilon = 10^{-4}, \ a = 0.02 \)

- unstructured grid of 46 elements given, uniformly/adaptively refined
Analytical solutions

Case $\varepsilon = 1, \ a = 0.5$

Case $\varepsilon = 10^{-4}, \ a = 0.02$
Error distribution on a uniformly refined mesh, $\varepsilon = 1$, $a = 0.5$

Estimated error distribution

Exact error distribution

A. Ern, A. F. Stephansen & M. Vohralík
Guaranteed and robust estimates for DG
Estimated and actual errors and the effectivity index, $\varepsilon = 1, a = 0.5$

The different estimators

Effectivity index
Error distribution on a uniformly refined mesh, 
\( \varepsilon = 10^{-2}, \ a = 0.05 \)
Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$
### Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-2}$

<table>
<thead>
<tr>
<th>$N$</th>
<th>Energy norm</th>
<th>Augmented norm</th>
<th>$||p_h||_{#,\varepsilon_h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
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<td>1.40e-1</td>
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<td>512</td>
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<td>4.35e-2</td>
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<td>8192</td>
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<tr>
<td>order</td>
<td>1.0</td>
<td>2.0</td>
<td>1.7</td>
</tr>
</tbody>
</table>

Errors ($\|\|p - p_h\|\|$ and $\|\|p - p_h\|\|_{\#,\varepsilon}^\prime + \|\|p - p_h\|\|_{\#,\varepsilon_h}$), estimates ($\eta$ and $\tilde{\eta} + \|\|p_h\|\|_{\#,\varepsilon_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-2}$
Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-4}$

<table>
<thead>
<tr>
<th>$N$</th>
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<tbody>
<tr>
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<td>3.09e-2</td>
</tr>
<tr>
<td>8192</td>
<td>1.00e-4</td>
<td>1.25e-2</td>
</tr>
</tbody>
</table>

order 1.1 1.3 - 1.3 1.3 - 1.0 1.0

Errors ($\|\|p - p_h\|\|$ and $\|\|p - p_h\|\|_{\oplus} + \|\|p - p_h\|\|_{\#,\epsilon_h}$), estimates ($\eta$ and $\tilde{\eta} + \|p_h\|_{\#,\epsilon_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-4}$
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Conclusions

- guaranteed, locally efficient, and robust a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes (FDs, FVs, FEs, NCFEs, MFEs)
- based on local conservativity
Open questions and future work

Open questions

- Are the energy/augmented norms optimal?
- Can a robust estimate without the jump seminorm be obtained?
- Can a robust estimate in the energy norm be obtained?

Future work

- Nonlinear (degenerate) cases
- Parabolic convection–reaction–diffusion case
Open questions and future work

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Bibliography


ERN A., VOHRLÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, submitted, http://hal.archives-ouvertes.fr/hal-00383692/.

Thank you for your attention!