Localization of dual norms, local stopping criteria, and fully adaptive solvers

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Outline

1 Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace

2 Localization dual norms
   - Local–global equivalence
   - Numerical results

3 Fully adaptive solvers
   - Setting
   - A posteriori guaranteed upper bound
   - Local stopping criteria, efficiency, and robustness
   - Applications
   - Numerical results

4 Conclusions and ongoing work
1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace

2. Localization dual norms
   - Local–global equivalence
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3. Fully adaptive solvers
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4. Conclusions and ongoing work
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4. Conclusions and ongoing work
The Laplace problem

\[-\Delta u = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

- polytope \( \Omega \subset \mathbb{R}^d, \, d \geq 1, \, f \in L^2(\Omega) \)

Weak formulation
Find \( u \in H^1_0(\Omega) \) such that

\[(\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)\]

Residual \( R(u_h) \in H^{-1}(\Omega) \)

\[\langle R(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H^1_0(\Omega) \quad \text{weak form. misfit} \]
The Laplace problem
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Residual and its dual norm for Laplacian, $u_h \in H_0^1(\Omega)$

The Laplace problem

$$-\Delta u = f \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega$$

- polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H_0^1(\Omega) \quad \text{weak form. misfit}$$
The Laplace problem
\[-\Delta u = f \quad \text{in } \Omega,\]
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\[\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H^1_0(\Omega)\]  weak form. misfit
The Laplace problem

\[-\Delta u = f \quad \text{in } \Omega,\]
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- polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$

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Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$

\[
\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H^1_0(\Omega) \quad \text{weak form. misfit}
\]
The Laplace problem

\[-\Delta u = f \quad \text{in } \Omega,\]
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Weak formulation

Find \( u \in H^1_0(\Omega) \) such that

\[(\nabla u, \nabla v) = (f, v) \quad \forall v \in H^1_0(\Omega)\]

Residual \( \mathcal{R}(u_h) \in H^{-1}(\Omega) \) and its dual norm

\[\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H^1_0(\Omega) \quad \text{weak form. misfit} \]

\[\|\mathcal{R}(u_h)\|_{-1} := \sup_{v \in H^1_0(\Omega), \|\nabla v\|=1} \langle \mathcal{R}(u_h), v \rangle\]
The Laplace problem

\[-\Delta u = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$

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Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ and its dual norm

\[\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \quad v \in H^1_0(\Omega) \quad \text{weak form. misfit}\]

\[\|\mathcal{R}(u_h)\|_{-1} := \sup_{v \in H^1_0(\Omega), \|\nabla v\| = 1} \langle \mathcal{R}(u_h), v \rangle \quad \text{size of the misfit}\]
Equivalence energy error–dual norm of the residual

**Theorem (Equivalence energy error–dual norm of the residual)**

Let \( u_h \in H^1_0(\Omega) \). Then

\[
\| \mathcal{R}(u_h) \|_{-1} = \| \nabla (u - u_h) \|.
\]

**Proof.**

- residual and its dual norm definition
  \[
  \| \mathcal{R}(u_h) \|_{-1} = \sup_{v \in H^1_0(\Omega), \| \nabla v \|=1} \{(f, v) - (\nabla u_h, \nabla v)\}
  \]
- weak solution definition
  \[
  (f, v) = (\nabla u, \nabla v)
  \]
- conformity ((\( u - u_h \) \in \( H^1_0(\Omega) \)) and duality:
  \[
  \sup_{v \in H^1_0(\Omega), \| \nabla v \|=1} (\nabla (u - u_h), \nabla v) = \| \nabla (u - u_h) \|
Theorem (Equivalence energy error–dual norm of the residual)

Let \( u_h \in H^1_0(\Omega) \). Then

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Proof.

- residual and its dual norm definition
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  \| \mathcal{R}(u_h) \|_{-1} = \sup_{\nu \in H^1_0(\Omega), \| \nabla \nu \|=1} \{ (f, \nu) - (\nabla u_h, \nabla \nu) \}
  \]

- weak solution definition
  \[
  (f, \nu) = (\nabla u, \nabla \nu)
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- conformity \(((u - u_h) \in H^1_0(\Omega))\) and duality:
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  \sup_{\nu \in H^1_0(\Omega), \| \nabla \nu \|=1} (\nabla (u - u_h), \nabla \nu) = \| \nabla (u - u_h) \|
Theorem (Equivalence energy error–dual norm of the residual)

Let $u_h \in H^1_0(\Omega)$. Then

$$\| \mathcal{R}(u_h) \|_{-1} = \| \nabla (u - u_h) \|.$$ 

Proof.

- residual and its dual norm definition
  $$\| \mathcal{R}(u_h) \|_{-1} = \sup_{\nu \in H^1_0(\Omega), \| \nabla \nu \| = 1} \{ (f, \nu) - (\nabla u_h, \nabla \nu) \}$$

- weak solution definition
  $$(f, \nu) = (\nabla u, \nabla \nu)$$

- conformity ($(u - u_h) \in H^1_0(\Omega)$) and duality:
  $$\sup_{\nu \in H^1_0(\Omega), \| \nabla \nu \| = 1} (\nabla (u - u_h), \nabla \nu) = \| \nabla (u - u_h) \|$$
Theorem (Equivalence energy error–dual norm of the residual)

Let $u_h \in H_0^1(\Omega)$. Then

$$\|\mathcal{R}(u_h)\|_{-1} = \|\nabla(u - u_h)\|.$$ 

Proof.

- residual and its dual norm definition
  $$\|\mathcal{R}(u_h)\|_{-1} = \sup_{v \in H_0^1(\Omega), \|\nabla v\| = 1} \{(f, v) - (\nabla u_h, \nabla v)\}$$

- weak solution definition
  $$(f, v) = (\nabla u, \nabla v)$$

- conformity $$(u - u_h) \in H_0^1(\Omega))$$ and duality:
  $$\sup_{v \in H_0^1(\Omega), \|\nabla v\| = 1} (\nabla(u - u_h), \nabla v) = \|\nabla(u - u_h)\|.$$
Theorem (Equivalence energy error–dual norm of the residual)

Let \( u_h \in H^1_0(\Omega) \). Then

\[
\| \mathcal{R}(u_h) \|_{-1} = \| \nabla (u - u_h) \|.
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Proof.

- residual and its dual norm definition

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\| \mathcal{R}(u_h) \|_{-1} = \sup_{v \in H^1_0(\Omega), \| \nabla v \| = 1} \left\{ (f, v) - (\nabla u_h, \nabla v) \right\}
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- weak solution definition

\[
(f, v) = (\nabla u, \nabla v)
\]

- conformity \(((u - u_h) \in H^1_0(\Omega))\) and duality:

\[
\sup_{v \in H^1_0(\Omega), \| \nabla v \| = 1} (\nabla (u - u_h), \nabla v) = \| \nabla (u - u_h) \|
\]
Equivalence energy error–dual norm of the residual

Theorem (Equivalence energy error–dual norm of the residual)

Let $u_h \in H^1_0(\Omega)$. Then

$$\| \mathcal{R}(u_h) \|_{-1} = \| \nabla (u - u_h) \| = \left\{ \sum_{K \in T_h} \| \nabla (u - u_h) \|^2_K \right\}^{\frac{1}{2}}.$$

Proof.

- residual and its dual norm definition
  $$\| \mathcal{R}(u_h) \|_{-1} = \sup_{v \in H^1_0(\Omega), \| \nabla v \| = 1} \{(f, v) - (\nabla u_h, \nabla v)\}$$

- weak solution definition
  $$(f, v) = (\nabla u, \nabla v)$$

- conformity $((u - u_h) \in H^1_0(\Omega))$ and duality:
  $$\sup_{v \in H^1_0(\Omega), \| \nabla v \| = 1} (\nabla (u - u_h), \nabla v) = \| \nabla (u - u_h) \|$$
The nonconforming case, $u_h \not\in H^1_0(\Omega)$

**Theorem (Energy error in the nonconforming case)**

Let $u_h \not\in H^1_0(\Omega)$. Then

$$
\|\nabla (u - u_h)\|^2 = \sup_{v \in H^1_0(\Omega); \|\nabla v\|=1} \left\{ (f, v) - (\nabla u_h, \nabla v) \right\}^2 + \min_{v \in H^1_0(\Omega)} \|\nabla (v - u_h)\|^2.
$$

$\|R(u_h)\|_{-1}$, dual norm of the residual

distance of $u_h$ to $H^1_0(\Omega)$

**Proof.**

- define $s \in H^1_0(\Omega)$ by (projection)
  $$
  (\nabla s, \nabla v) = (\nabla u_h, \nabla v), \quad \forall v \in H^1_0(\Omega)
  $$
- develop (Pythagoras)
  $$
  \|\nabla (u - u_h)\|^2 = \|\nabla (u - s)\|^2 + \|\nabla (s - u_h)\|^2
  $$
- projection definition of $s$:
  $$
  \|\nabla (s - u_h)\|^2 = \min_{v \in H^1_0(\Omega)} \|\nabla (v - u_h)\|^2
  $$
- norm characterization by duality, definition of $s$:
  $$
  \|\nabla (u - s)\|^2 = \sup_{v \in H^1_0(\Omega); \|\nabla v\|=1} (\nabla (u - ), \nabla v)^2
  $$
The nonconforming case, $u_h \notin H^1_0(\Omega)$

**Theorem (Energy error in the nonconforming case)**

Let $u_h \notin H^1_0(\Omega)$. Then

$$\|\nabla(u - u_h)\|^2 = \sup_{v \in H^1_0(\Omega); \|\nabla v\|=1} \{(f, v) - (\nabla u_h, \nabla v)\}^2 + \min_{v \in H^1_0(\Omega)} \|\nabla(v - u_h)\|^2.$$

- $\|R(u_h)\|_{-1}$, dual norm of the residual
- Distance of $u_h$ to $H^1_0(\Omega)$

**Proof.**

- Define $s \in H^1_0(\Omega)$ by (projection)

  $$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \quad \forall v \in H^1_0(\Omega)$$

- Develop (Pythagoras)

  $$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- Projection definition of $s$:

  $$\|\nabla(s - u_h)\|^2 = \min_{v \in H^1_0(\Omega)} \|\nabla(v - u_h)\|^2$$

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Let $u_h \not\in H^1_0(\Omega)$. Then

$$
\|\nabla(u-u_h)\|^2 = \sup_{\nu \in H^1_0(\Omega); \|\nabla\nu\|=1} \{(f, \nu) - (\nabla u_h, \nabla \nu)\}^2 + \min_{\nu \in H^1_0(\Omega)} \|\nabla(\nu - u_h)\|^2.
$$

- $\|\mathcal{R}(u_h)\|_{-1}$, dual norm of the residual
- distance of $u_h$ to $H^1_0(\Omega)$

**Proof.**

- define $s \in H^1_0(\Omega)$ by (projection)
  $$
  (\nabla s, \nabla \nu) = (\nabla u_h, \nabla \nu) \quad \forall \nu \in H^1_0(\Omega)
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The nonconforming case, $u_h \not\in H_0^1(\Omega)$

**Theorem (Energy error in the nonconforming case)**

Let $u_h \not\in H_0^1(\Omega)$. Then

$$\|\nabla (u - u_h)\|^2 = \sup_{v \in H_0^1(\Omega); \|\nabla v\|=1} \{ (f, v) - (\nabla u_h, \nabla v) \}^2 + \min_{v \in H_0^1(\Omega)} \|\nabla (v - u_h)\|^2.$$

- $\|\mathcal{R}(u_h)\|_{-1}$, *dual norm of the residual*
- *distance of $u_h$ to $H_0^1(\Omega)$*

**Proof.**

- define $s \in H_0^1(\Omega)$ by (projection)
  $$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \quad \forall v \in H_0^1(\Omega)$$
- develop (Pythagoras)
  $$\|\nabla (u - u_h)\|^2 = \|\nabla (u - s)\|^2 + \|\nabla (s - u_h)\|^2$$
- projection definition of $s$:
  $$\|\nabla (s - u_h)\|^2 = \min_{v \in H_0^1(\Omega)} \|\nabla (v - u_h)\|^2$$
- norm characterization by duality, definition of $s$:
  $$\|\nabla (u - s)\|^2 = \sup_{v \in H_0^1(\Omega); \|\nabla v\|=1} (\nabla (u - ), \nabla v)^2$$
The nonconforming case, $u_h \not\in H^1_0(\Omega)$

**Theorem (Energy error in the nonconforming case)**

Let $u_h \not\in H^1_0(\Omega)$. Then

$$
\|\nabla (u - u_h)\|^2 = \sup_{v \in H^1_0(\Omega); \|\nabla v\|=1} \{ (f, v) - (\nabla u_h, \nabla v) \}^2 + \min_{v \in H^1_0(\Omega)} \|\nabla (v - u_h)\|^2.
$$

\[\|R(u_h)\|_{-1}, \text{ dual norm of the residual}\]
\[\text{distance of } u_h \text{ to } H^1_0(\Omega)\]

**Proof.**

- define $s \in H^1_0(\Omega)$ by (projection)
  $$
  (\nabla s, \nabla v) = (\nabla u_h, \nabla v) \quad \forall v \in H^1_0(\Omega)
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The nonconforming case, \( u_h \not\in H^1_0(\Omega) \)

**Theorem (Energy error in the nonconforming case)**

Let \( u_h \not\in H^1_0(\Omega) \). Then

\[
\|\nabla (u-u_h)\|^2 = \sup_{\nu \in H^1_0(\Omega); \|\nabla \nu\|_1 = 1} \{ (f, \nu) - (\nabla u_h, \nabla \nu) \}^2 + \min_{\nu \in H^1_0(\Omega)} \|\nabla (\nu - u_h)\|^2.
\]

\( \|\mathcal{R}(u_h)\|_1 \), dual norm of the residual

\( \text{distance of } u_h \text{ to } H^1_0(\Omega) \)

**Proof.**

- Define \( s \in H^1_0(\Omega) \) by (projection)
  \[
  (\nabla s, \nabla \nu) = (\nabla u_h, \nabla \nu) \quad \forall \nu \in H^1_0(\Omega)
  \]

- Develop (Pythagoras)
  \[
  \|\nabla (u-u_h)\|^2 = \|\nabla (u-s)\|^2 + \|\nabla (s-u_h)\|^2
  \]

- Projection definition of \( s \):
  \[
  \|\nabla (s-u_h)\|^2 = \min_{\nu \in H^1_0(\Omega)} \|\nabla (\nu - u_h)\|^2
  \]

- Norm characterization by duality, definition of \( s \):
  \[
  \|\nabla (u-s)\|^2 = \sup_{\nu \in H^1_0(\Omega); \|\nabla \nu\|_1 = 1} (\nabla (u-u_h), \nabla \nu)^2
  \]
Outline

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4. Conclusions and ongoing work
Quasi-linear elliptic problem

\[-\nabla \cdot \sigma(u, \nabla u) = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

- \(p > 1, \quad q := \frac{p}{p-1}, \quad f \in L^q(\Omega)\)
- example: \(p\)-Laplacian with \(\sigma(u, \nabla u) = |\nabla u|^{p-2} \nabla u\)

Weak formulation

Find \(u \in W^{1,p}_0(\Omega)\) such that

\[(\sigma(u, \nabla u), \nabla v) = (f, v) \quad \forall v \in W^{1,p}_0(\Omega)\]

Residual \(R(u^{k,i}_h) \in W^{1,p}_0(\Omega)'\) and its dual norm

\[\langle R(u^{k,i}_h), v \rangle := (f, v) - (\sigma(u^{k,i}_h, \nabla u^{k,i}_h), \nabla v), \quad v \in W^{1,p}_0(\Omega)\]

\[\|R(u^{k,i}_h)\|_{W^{1,p}_0(\Omega)'} := \sup_{v \in W^{1,p}_0(\Omega), \|\nabla v\|_p = 1} \langle R(u^{k,i}_h), v \rangle\]
Nonlinear Laplacian

Quasi-linear elliptic problem

\[-\nabla \cdot \sigma(u, \nabla u) = f \quad \text{in } \Omega,\]
\[u = 0 \quad \text{on } \partial \Omega\]

- \( p > 1, \ q := \frac{p}{p-1}, \ f \in L^q(\Omega) \)
- example: \( p \)-Laplacian with \( \sigma(u, \nabla u) = |\nabla u|^{p-2} \nabla u \)

Weak formulation

Find \( u \in \mathcal{W}^{1,p}_0(\Omega) \) such that

\[(\sigma(u, \nabla u), \nabla v) = (f, v) \quad \forall v \in \mathcal{W}^{1,p}_0(\Omega)\]
Nonlinear Laplacian, \( u_h^{k,i} \in W_0^{1,p}(\Omega) \) (Newton linearization step \( k \), algebraic solver step \( i \))

**Quasi-linear elliptic problem**

\[
-\nabla \cdot \sigma(u, \nabla u) = f \quad \text{in } \Omega, \\
\quad u = 0 \quad \text{on } \partial \Omega
\]

- \( p > 1, \ q := \frac{p}{p-1}, \ f \in L^q(\Omega) \)
- **example:** \( p \)-Laplacian with \( \sigma(u, \nabla u) = |\nabla u|^{p-2} \nabla u \)

**Weak formulation**

Find \( u \in W_0^{1,p}(\Omega) \) such that

\[
(\sigma(u, \nabla u), \nabla v) = (f, v) \quad \forall v \in W_0^{1,p}(\Omega)
\]

**Residual** \( R(u_h^{k,i}) \in W_0^{1,p}(\Omega)' \) and its dual norm

\[
\langle R(u_h^{k,i}), v \rangle := (f, v) - (\sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla v), \quad v \in W_0^{1,p}(\Omega)
\]

\[
\| R(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)'} := \sup_{v \in W_0^{1,p}(\Omega); \| \nabla v \|_p = 1} \langle R(u_h^{k,i}), v \rangle
\]
Nonlinear Laplacian, $u_h^{k,i} \in W_0^{1,p}(\Omega)$ (Newton linearization step $k$, algebraic solver step $i$)

Quasi-linear elliptic problem

$$-\nabla \cdot \sigma(u, \nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial \Omega$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$
- example: $p$-Laplacian with $\sigma(u, \nabla u) = |\nabla u|^{p-2}\nabla u$

Weak formulation
Find $u \in W_0^{1,p}(\Omega)$ such that

$$\langle \sigma(u, \nabla u), \nabla v \rangle = (f, v) \quad \forall v \in W_0^{1,p}(\Omega)$$

Residual $\mathcal{R}(u_h^{k,i}) \in W_0^{1,p}(\Omega)'$ and its dual norm

$$\langle \mathcal{R}(u_h^{k,i}), v \rangle := (f, v) - \langle \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla v \rangle, \quad v \in W_0^{1,p}(\Omega)$$

$$\| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)'} := \sup_{v \in W_0^{1,p}(\Omega); \| \nabla v \|_p = 1} \| \nabla v \|_{p=1}$$
Is it possible to localize the dual norm of the residual

\[ \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)}' \approx \left\{ \sum_{a \in \mathcal{V}_h} \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\omega_a)}' \right\}^{\frac{1}{q}} \]

- \( \mathcal{V}_h \) vertices, \( \omega_a \) patches of elements of a partition \( T_h \) of \( \Omega \);
- the constant hidden in \( \approx \) must not depend on \( p, \Omega, \) and the regularity of \( u \).

How to give tight and robust computable bounds on \( \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)}' \) on each Newton step \( k \) and algebraic step \( i \)?

How to steer adaptively (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to take into account nonconforming discretizations?
The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

\[ \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)'} \approx \left\{ \sum_{a \in \mathcal{V}_h} \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\omega_a)'} \right\}^{\frac{1}{q}} \]

- \( \mathcal{V}_h \) vertices, \( \omega_a \) patches of elements of a partition \( \mathcal{T}_h \) of \( \Omega \);
- the constant hidden in \( \approx \) must not depend on \( p, \Omega \), and the regularity of \( u \).

How to give tight and robust **computable bounds** on \( \| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)'} \) on each Newton step \( k \) and algebraic step \( i \)?

How to **steer adaptively** (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to take into account **nonconforming discretizations**?
The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

\[ \| \mathcal{R}(u_h^{k,i}) \|_{W_0^1,p(\Omega)'} \approx \left\{ \sum_{a \in \mathcal{V}_h} \| \mathcal{R}(u_h^{k,i}) \|_{W_0^1,p(\omega_a)'}^q \right\}^{\frac{1}{q}} \]

- \( \mathcal{V}_h \) vertices, \( \omega_a \) patches of elements of a partition \( \mathcal{T}_h \) of \( \Omega \);
- the constant hidden in \( \approx \) must not depend on \( p, \Omega \), and the regularity of \( u \).

How to give tight and robust **computable bounds** on \( \| \mathcal{R}(u_h^{k,i}) \|_{W_0^1,p(\Omega)'} \) on each Newton step \( k \) and algebraic step \( i \)?

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The game

Is it possible to **localize** the dual norm of the residual

$$\| \mathcal{R}(u_h^k,i) \|_{W_0^1,^p(\Omega)'} \approx \left\{ \sum_{a \in \mathcal{N}_h} \| \mathcal{R}(u_h^k,i) \|_{W_0^1,^p(\omega_a)'}^q \right\}^{1/q}?$$

- $\mathcal{N}_h$ vertices, $\omega_a$ patches of elements of a partition $\mathcal{T}_h$ of $\Omega$;
- the constant hidden in $\approx$ must not depend on $p$, $\Omega$, and the regularity of $u$.

How to give tight and robust **computable bounds** on $\| \mathcal{R}(u_h^k,i) \|_{W_0^1,^p(\Omega)'}$ on each Newton step $k$ and algebraic step $i$?

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Is it possible to **localize** the dual norm of the residual

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How to **steer adaptively** (adaptive stopping criteria, adaptive mesh refinement) the inexact Newton solver?

How to take into account **nonconforming discretizations**?

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1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace

2. Localization dual norms
   - Local–global equivalence
   - Numerical results

3. Fully adaptive solvers
   - Setting
   - A posteriori guaranteed upper bound
   - Local stopping criteria, efficiency, and robustness
   - Applications
   - Numerical results

4. Conclusions and ongoing work
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Localization dual norms

Setting

- \( V := W^{1,p}_0(\Omega), \ p > 1, \) bounded linear functional \( R \in V' \)
- localized energy space \( V^a := W^{1,p}_0(\omega_a) \) for \( a \in V_h \)
- restriction of \( R \) to \( (V^a)' \) (zero extension of \( v \in V^a \)),

\[
\langle R, v \rangle_{(V^a)',V^a} := \langle R, v \rangle_{V',V} \quad v \in V^a
\]

\[
\|R\|_{(V^a)'} := \sup_{v \in V^a; \|\nabla v\|_{p,\omega_a} = 1} \langle R, v \rangle_{(V^a)',V^a}
\]

Theorem (Localization of \( \|R\|_{V'} \))

There holds

\[
\|R\|_{V'} \leq (d+1)^\frac{1}{p} C_{\text{cont,PF}} \left\{ \sum_{a \in V_h} \|R\|^q_{(V^a)'} \right\}^{\frac{1}{q}} \quad \text{if} \ \langle R, \psi_a \rangle = 0 \ \forall a \in V_h^{\text{int}},
\]

\[
\left\{ \sum_{a \in V_h} \|R\|^q_{(V^a)'} \right\}^{\frac{1}{q}} \leq (d+1)^\frac{1}{q} \|R\|_{V'}.
\]
Setting

- \( V := W_{0}^{1,p}(\Omega), \ p > 1, \) bounded linear functional \( R \in V' \)
- localized energy space \( V^{a} := W_{0}^{1,p}(\omega_a) \) for \( a \in \mathcal{V}_h \)
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### Theorem (Localization of \( \|R\|_{V'} \))

There holds

\[
\|R\|_{V'} \leq (d + 1)\frac{1}{p} C_{\text{cont,PF}} \left\{ \sum_{a \in \mathcal{V}_h} \|R\|_{(V^a)'}^{q} \right\}^{\frac{1}{q}} \quad \text{if} \ \langle R, \psi_a \rangle = 0 \ \forall a \in \mathcal{V}_h^{\text{int}},
\]

\[
\left\{ \sum_{a \in \mathcal{V}_h} \|R\|_{(V^a)'}^{q} \right\}^{\frac{1}{q}} \leq (d + 1)\frac{1}{q} \|R\|_{V'}.
\]
Localization of the dual residual norm

Upper bound (needs vanishing lowest modes).

- partition of unity, the linearity of $\mathcal{R}$, orthogonality wrt $\psi_a$:
  $$\langle \mathcal{R}, v \rangle = \sum_{a \in \mathcal{V}_h} \langle \mathcal{R}, \psi_a v \rangle = \sum_{a \in \mathcal{V}^{\text{int}}_h} \langle \mathcal{R}, \psi_a (v - \Pi_{0,\omega_a} v) \rangle + \sum_{a \in \mathcal{V}^{\text{ext}}_h} \langle \mathcal{R}, \psi_a v \rangle$$

- stability:
  $$\| \nabla (\psi_a (v - \Pi_{0,\omega_a} v)) \|_{p,\omega_a} \leq C_{\text{cont,PF}} \| \nabla v \|_{p,\omega_a}$$

- Hölder inequality:
  $$\langle \mathcal{R}, v \rangle \leq C_{\text{cont,PF}} \left\{ \sum_{a \in \mathcal{V}_h} \| \mathcal{R} \|_{(V^a)'}^q \right\}^{1/q} \left\{ \sum_{a \in \mathcal{V}_h} \| \nabla v \|_{p,\omega_a}^p \right\}^{1/p}$$

- overlapping of the patches:
  $$\sum_{a \in \mathcal{V}_h} \| \nabla v \|_{p,\omega_a}^p = \sum_{K \in \mathcal{T}_h} \sum_{a \in \mathcal{V}_K} \| \nabla v \|_{p,K}^p \leq (d + 1) \sum_{K \in \mathcal{T}_h} \| \nabla v \|_{p,K}^p$$
Localization of the dual residual norm

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\langle \mathcal{R}, v \rangle \leq C_{\text{cont,PF}} \left\{ \sum_{a \in \mathcal{V}_h} \| \mathcal{R} \|_{(V_a)^q} \right\}^{1/q} \left\{ \sum_{a \in \mathcal{V}_h} \| \nabla v \|_{p, \omega_a}^p \right\}^{1/p}
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Localization of the dual residual norm

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  \]

- Hölder inequality:
  \[
  \langle \mathcal{R}, v \rangle \leq C_{\text{cont,PF}} \left\{ \sum_{a \in \mathcal{V}_h} \|\mathcal{R}\|_{(V_a)' \vphantom{\mathcal{V}_h}}^q \right\}^{\frac{1}{q}} \left\{ \sum_{a \in \mathcal{V}_h} \|\nabla v\|_{p,\omega_a}^p \right\}^{\frac{1}{p}}
  \]

- overlapping of the patches:
  \[
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  \]
Localization of the dual residual norm

Lower bound (unconditioned).

$p$-Laplacian lifting of the residual on the patch $\omega_a$:

$\hat{\varphi}^a \in V^a = \mathcal{W}^{1,p}_0(\omega_a)$ such that

$$
(|\nabla \hat{\varphi}^a|^{p-2} \nabla \hat{\varphi}^a, \nabla v)_{\omega_a} = \langle R, v \rangle \quad \forall v \in V^a
$$

- energy equality:

$$
\|\nabla \hat{\varphi}^a\|_{p,\omega_a}^p = (|\nabla \hat{\varphi}^a|^{p-2} \nabla \hat{\varphi}^a, \nabla \hat{\varphi}^a)_{\omega_a} = \langle R, \hat{\varphi}^a \rangle = \|R\|_{(V^a)'}^q
$$

- setting $\hat{\varphi} := \sum_{a \in \mathcal{V}_h} \hat{\varphi}^a \in V$:

$$
\sum_{a \in \mathcal{V}_h} \|R\|_{(V^a)'}^q = \sum_{a \in \mathcal{V}_h} \langle R, \hat{\varphi}^a \rangle = \langle R, \hat{\varphi} \rangle \leq \|R\|_V \|\nabla \hat{\varphi}\|_p
$$

- overlapping of the patches:

$$
\|\nabla \hat{\varphi}\|_p^p \leq (d + 1)^{p-1} \sum_{a \in \mathcal{V}_h} \|\nabla \hat{\varphi}^a\|_{p,\omega_a}^p
$$
Localization of the dual residual norm

**Lower bound (unconditioned).**

- \( p \)-Laplacian lifting of the residual on the patch \( \omega_a \):
  \[ \tilde{r}^a \in V^a = W_0^{1,p}(\omega_a) \text{ such that} \]
  \[ (|\nabla \tilde{r}^a|^{p-2}\nabla \tilde{r}^a, \nabla v)_{\omega_a} = \langle R, v \rangle \quad \forall v \in V^a \]

- Energy equality:
  \[ \|\nabla \tilde{r}^a\|_{p,\omega_a}^p = (|\nabla \tilde{r}^a|^{p-2}\nabla \tilde{r}^a, \nabla \tilde{r}^a)_{\omega_a} = \langle R, \tilde{r}^a \rangle = \|R\|_{(V^a)^\prime}^q \]

- Setting \( \tilde{r} := \sum_{a \in \mathcal{V}_h} \tilde{r}^a \in V \):
  \[ \sum_{a \in \mathcal{V}_h} \|R\|_{(V^a)^\prime}^q = \sum_{a \in \mathcal{V}_h} \langle R, \tilde{r}^a \rangle = \langle R, \tilde{r} \rangle \leq \|R\|_{V^\prime} \|\nabla \tilde{r}\|_p \]

- Overlapping of the patches:
  \[ \|\nabla \tilde{r}\|_p^p \leq (d + 1)^{p-1} \sum_{a \in \mathcal{V}_h} \|\nabla \tilde{r}^a\|_{p,\omega_a}^p \]
Localization of the dual residual norm

Lower bound (unconditioned).

- $p$-Laplacian lifting of the residual on the patch $\omega_a$:
  \[ \hat{\varphi}^a \in V^a = W_0^{1,p}(\omega_a) \text{ such that} \]
  \[ (|\nabla \hat{\varphi}^a|^{p-2} \nabla \hat{\varphi}^a, \nabla v)_{\omega_a} = \langle R, v \rangle \quad \forall v \in V^a \]

- Energy equality:
  \[ \|\nabla \hat{\varphi}^a\|_{p,\omega_a}^p = (|\nabla \hat{\varphi}^a|^{p-2} \nabla \hat{\varphi}^a, \nabla \hat{\varphi}^a)_{\omega_a} = \langle R, \hat{\varphi}^a \rangle = \|R\|_{(V^a)^{q'}} \]

- Setting $\varphi := \sum_{a \in \mathcal{V}_h} \hat{\varphi}^a \in V$:
  \[ \sum_{a \in \mathcal{V}_h} \|R\|_{(V^a)^{q'}} = \sum_{a \in \mathcal{V}_h} \langle R, \hat{\varphi}^a \rangle = \langle R, \varphi \rangle \leq \|R\|_{V'} \|\nabla \varphi\|_p \]

- Overlapping of the patches:
  \[ \|\nabla \varphi\|_p^p \leq (d + 1)^{p-1} \sum_{a \in \mathcal{V}_h} \|\nabla \hat{\varphi}^a\|_{p,\omega_a}^p \]
Localization of the dual residual norm

Lower bound (unconditioned).

- \( p \)-Laplacian lifting of the residual on the patch \( \omega_a \):
  \[ \tilde{r}^a \in V^a = W_0^{1,p}(\omega_a) \] such that
  \[ (|\nabla \tilde{r}^a|^{p-2} \nabla \tilde{r}^a, \nabla \nu)_{\omega_a} = \langle R, \nu \rangle \quad \forall \nu \in V^a \]

- Energy equality:
  \[ \| \nabla \tilde{r}^a \|_{p, \omega_a}^p = (|\nabla \tilde{r}^a|^{p-2} \nabla \tilde{r}^a, \nabla \tilde{r}^a)_{\omega_a} = \langle R, \tilde{r}^a \rangle = \| R \|_{(V^a)^{'}}^q \]

- Setting \( r := \sum_{a \in \mathcal{V}_h} \tilde{r}^a \in V \):
  \[ \sum_{a \in \mathcal{V}_h} \| R \|_{(V^a)^{'}}^q = \sum_{a \in \mathcal{V}_h} \langle R, \tilde{r}^a \rangle = \langle R, r \rangle \leq \| R \|_{V'} \| \nabla \tilde{r} \|_p \]

- Overlapping of the patches:
  \[ \| \nabla \tilde{r} \|_p \leq (d + 1)^{p-1} \sum_{a \in \mathcal{V}_h} \| \nabla \tilde{r}^a \|_{p, \omega_a}^p \]
Outline

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   • Nonlinear Laplace

2 Localization dual norms
   • Local–global equivalence
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   • Numerical results

4 Conclusions and ongoing work
Numerical results

Model problems

- $p$-Laplacian

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega,$$
$$u = u_D \quad \text{on } \partial \Omega$$

- $\Omega = (0, 1) \times (0, 1)$ and, for $p = 1.5$ and $10$,

$$u(x, y) = -\frac{p-1}{p} \left((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p} \left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$ and, for $p = 4$,

$$u(r, \theta) = r^{\frac{7}{8}} \sin(\frac{7}{8} \theta)$$

- three successive uniformly refined meshes
Numerical results

Model problems

- $p$-Laplacian

\[ \nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \]
\[ u = u_D \quad \text{on } \partial \Omega \]

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- three successive uniformly refined meshes
Numerical results

Model problems

- **$p$-Laplacian**

  \[
  \nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \\
  u = u_D \quad \text{on } \partial \Omega
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  \]

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Model problems

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$$u = u_D \quad \text{on } \partial \Omega$$

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- $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$ and, for $p = 4$,

$$u(r, \theta) = r^{\frac{7}{8}} \sin(\theta^{\frac{7}{8}})$$

- three successive uniformly refined meshes
Effectivity indices of the localization bounds

- Upper bound, $p = 4$
- Lower bound, $p = 4$
- Upper bound, $p = 1.5$
- Lower bound, $p = 1.5$
- Upper bound, $p = 10$
- Lower bound, $p = 10$
Local and global residual distributions, $p = 1.5$
Local and global residual distributions, $p = 10$
Local and global residual distributions, $p = 4$
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4. Conclusions and ongoing work
Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step $k$, algebraic step $i$
- $u_{h}^{k,i} \in V(\mathcal{T}_h) := \{ v \in L^p(\Omega), \ v|_K \in W^{1,p}(K) \ \forall K \in \mathcal{T}_h \} \not\subset V$

Assumption A (Total flux reconstruction)

There exists $\sigma_{h}^{k,i} \in H^q(\text{div}, \Omega)$ and $\rho_{h}^{k,i} \in L^q(\Omega)$ such that
\[ \nabla \cdot \sigma_{h}^{k,i} = f_h - \rho_{h}^{k,i}. \]

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $d_{h}^{k,i}, l_{h}^{k,i}, a_{h}^{k,i} \in [L^q(\Omega)]^d$ such that

(i) $\sigma_{h}^{k,i} = d_{h}^{k,i} + l_{h}^{k,i} + a_{h}^{k,i}$;

(ii) as the linear solver converges, $\|a_{h}^{k,i}\|_{q} \to 0$;

(iii) as the nonlinear solver converges, $\|l_{h}^{k,i}\|_{q} \to 0$. 

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Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step $k$, algebraic step $i$
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{ v \in L^p(\Omega), \, v|_K \in \mathcal{W}^{1,p}(K) \, \forall K \in \mathcal{T}_h \} \not\subset V$

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Abstract assumptions

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$$\nabla \cdot \sigma^{k,i}_h = f_h - \underbrace{\rho^{k,i}_h}_{\text{algebraic remainder}}.$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $d^{k,i}_h, l^{k,i}_h, a^{k,i}_h \in [L^q(\Omega)]^d$ such that

(i) $\sigma^{k,i}_h = d^{k,i}_h + l^{k,i}_h + a^{k,i}_h$

(ii) as the linear solver converges, $\|a^{k,i}_h\|_q \to 0$

(iii) as the nonlinear solver converges, $\|l^{k,i}_h\|_q \to 0$
Abstract assumptions

Numerical approximation

- simplicial mesh \( T_h \), linearization step \( k \), algebraic step \( i \)
- \( u_h^{k,i} \in V(T_h) := \{ v \in L^p(\Omega), v|_K \in W^{1,p}(K) \quad \forall K \in T_h \} \notin V \)

Assumption A (Total flux reconstruction)

There exists \( \sigma_h^{k,i} \in H^q(\text{div}, \Omega) \) and \( \rho_h^{k,i} \in L^q(\Omega) \) such that

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There exist fluxes \( d_h^{k,i}, l_h^{k,i}, a_h^{k,i} \in [L^q(\Omega)]^d \) such that

(i) \( \sigma_h^{k,i} = d_h^{k,i} + l_h^{k,i} + a_h^{k,i} \);

(ii) as the linear solver converges, \( \|a_h^{k,i}\|_q \to 0 \);

(iii) as the nonlinear solver converges, \( \|l_h^{k,i}\|_q \to 0 \).
Outline

1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace

2. Localization dual norms
   - Local–global equivalence
   - Numerical results

3. Fully adaptive solvers
   - Setting
   - A posteriori guaranteed upper bound
   - Local stopping criteria, efficiency, and robustness
   - Applications
   - Numerical results

4. Conclusions and ongoing work
Theorem (Estimate distinguishing different error components)

Let

- \( u \in V \) be the weak solution,
- \( u_h^{k,i} \in V(T_h) \) be arbitrary,
- Assumptions A and B hold.

Then there holds

\[
\|R(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} + NC \leq \eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_{\text{quad}}^{k,i} + \eta_{\text{osc}}^{k,i},
\]

with \( \eta_{\cdot}^{k,i} := \left\{ \sum_{K \in T_h}(\eta_{\cdot}^{k,i},_{K})_{q} \right\}^{1/q} \) and

\[
\eta_{\text{disc},K}^{k,i} := 2\frac{1}{p}\left(\|\Delta(u_h^{k,i}, \nabla u_h^{k,i}) + d_h^{k,i}\|_{q,K} + \left\{ \sum_{e \in \mathcal{E}_K} h_e^{1-q} \left\| u_h^{k,i} \right\|_{q,e} \right\}^{\frac{1}{q}} \right).
\]
Theorem (Estimate distinguishing different error components)

Let

- \( u \in V \) be the weak solution,
- \( u_h^{k,i} \in V(\mathcal{T}_h) \) be arbitrary,
- Assumptions A and B hold.

Then there holds

\[
\| \mathcal{R}(u_h^{k,i}) \|_{W_0^{1,p}(\Omega)} + NC \leq \eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_{\text{quad}}^{k,i} + \eta_{\text{osc}}^{k,i},
\]

with \( \eta^{k,i} := \left\{ \sum_{K \in \mathcal{T}_h} \eta^{k,i}_{\text{disc},K} \right\}^{1/q} \) and

\[
\eta_{\text{disc},K}^{k,i} := 2^{\frac{1}{p}} \left( \| \sigma(u_h^{k,i}, \nabla u_h^{k,i}) + d_h^{k,i} \|_{q,K} + \left\{ \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| \| u_h^{k,i} \|_{q,e} \right\}^{\frac{1}{q}} \right).
\]
Stopping criteria and efficiency

Global stopping criteria \(\approx\) Becker, Johnson, and Rannacher (1995), Arioli (2000’s)

\[
\eta_{\text{rem},i} \leq \gamma_{\text{rem}} \max\{\eta_{\text{disc},i}, \eta_{\text{lin},i}, \eta_{\text{alg},i}\},
\]

\[
\eta_{\text{alg},i} \leq \gamma_{\text{alg}} \max\{\eta_{\text{disc},i}, \eta_{\text{lin},i}\}, \quad \gamma_{\text{rem}}, \gamma_{\text{alg}}, \gamma_{\text{lin}} \approx 0.1
\]

\[
\eta_{\text{lin},i} \leq \gamma_{\text{lin}} \eta_{\text{disc},i}
\]

Local stopping criteria

- stop whenever:

\[
\eta_{\text{rem},i} \leq \gamma_{\text{rem},i} \max\{\eta_{\text{disc},i}, \eta_{\text{lin},i}, \eta_{\text{alg},i}\} \quad \forall K \in \mathcal{T}_h,
\]

\[
\eta_{\text{alg},i} \leq \gamma_{\text{alg},i} \max\{\eta_{\text{disc},i}, \eta_{\text{lin},i}\} \quad \forall K \in \mathcal{T}_h,
\]

\[
\eta_{\text{lin},i} \leq \gamma_{\text{lin},i} \eta_{\text{disc},i} \quad \forall K \in \mathcal{T}_h
\]

\[
\gamma_{\text{rem},i}, \gamma_{\text{alg},i}, \gamma_{\text{lin},i} \approx 0.1
\]
Global stopping criteria \((\approx \text{Becker, Johnson, and Rannacher (1995), Arioli (2000's)})\)

\[
\eta_{\text{rem}}^{k,i} \leq \gamma_{\text{rem}} \max\{\eta_{\text{disc}}^{k,i}, \eta_{\text{lin}}^{k,i}, \eta_{\text{alg}}^{k,i}\}, \\
\eta_{\text{alg}}^{k,i} \leq \gamma_{\text{alg}} \max\{\eta_{\text{disc}}^{k,i}, \eta_{\text{lin}}^{k,i}\}, \\
\gamma_{\text{rem}}, \gamma_{\text{alg}}, \gamma_{\text{lin}} \approx 0.1
\]

Local stopping criteria

- stop whenever:

\[
\eta_{\text{rem},K}^{k,i} \leq \gamma_{\text{rem},K} \max\{\eta_{\text{disc},K}^{k,i}, \eta_{\text{lin},K}^{k,i}, \eta_{\text{alg},K}^{k,i}\}, \quad \forall K \in \mathcal{T}_h, \\
\eta_{\text{alg},K}^{k,i} \leq \gamma_{\text{alg},K} \max\{\eta_{\text{disc},K}^{k,i}, \eta_{\text{lin},K}^{k,i}\}, \quad \forall K \in \mathcal{T}_h, \\
\eta_{\text{lin},K}^{k,i} \leq \gamma_{\text{lin},K} \eta_{\text{disc},K}^{k,i}, \quad \forall K \in \mathcal{T}_h
\]

- \(\gamma_{\text{rem},K}, \gamma_{\text{alg},K}, \gamma_{\text{lin},K} \approx 0.1\)
Assumptions for efficiency

Assumption C (Piecewise polynomials, meshes, quadrature)

The approximation $u_h^{k,i}$ is piecewise polynomial. The meshes $\mathcal{T}_h$ are shape-regular. The quadrature error is negligible.

Assumption D (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$
\| \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + d_h^{k,i} \|_{q,K} \leq C \left\{ \sum_{K' \in \mathcal{T}_K} h_{K'}^q \| f + \nabla \cdot \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \|_{q,K'}^q 
+ \sum_{e \in \mathcal{E}^{\text{int}}_K} h_e \| [\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \cdot n_e] \|_{q,e}^q 
+ \sum_{e \in \mathcal{E}^{\text{ext}}_K} h_e^{1-q} \| [u_h^{k,i}] \|_{q,e}^{1/q} \right\}.
$$
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\| \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + d_h^{k,i} \|_{q,K} \leq C \left\{ \sum_{K' \in 2K} h_{K'}^q \| f + \nabla \cdot \overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \|_{q,K'} 
+ \sum_{e \in \Gamma_{int}} h_e \| [\overline{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) \cdot n_e] \|_{q,e} 
+ \sum_{e \in \Gamma_K} h_e^{1-q} \| [u_h^{k,i}] \|_{q,e} \right\}^{1/q}.
$$
### Global efficiency

#### Theorem (Global efficiency)

Let the Assumptions C and D be satisfied. Let the **global** stopping criteria hold. Then,  

\[
\eta_{\text{disc}}^k,i + \eta_{\text{lin}}^k,i + \eta_{\text{alg}}^k,i + \eta_{\text{rem}}^k,i \leq C \left( \| R(u_h^k,i) \|_{W_{0,0}^1\Omega} + NC \right),
\]

where \( C \) is independent of \( \sigma \) and \( q \).

#### Theorem (Local efficiency)

Let the Assumptions C and D be satisfied. Let the **local** stopping criteria hold. Then, for all \( K \in T_h \),  

\[
\eta_{\text{disc},K}^k,i + \eta_{\text{lin},K}^k,i + \eta_{\text{alg},K}^k,i + \eta_{\text{rem},K}^k,i \leq C \sum_{a \in V_K} \left( \| R(u_h^k,i) \|_{W_{0,0}^1\omega_a} + NC \right).
\]

- **robustness** with respect to the nonlinearity
- \( \| R(u_h^k,i) \|_{W_{0,0}^1\Omega} + NC \) is **localizable**
Global efficiency

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\[ \eta_{\text{disc}}^k,i + \eta_{\text{lin}}^k,i + \eta_{\text{alg}}^k,i + \eta_{\text{rem}}^k,i \leq C \left( \| \mathcal{R}(u_h^k,i) \|_{W_0^1,p(\Omega)'} + NC \right), \]

where \( C \) is independent of \( \sigma \) and \( q \).

Theorem (Local efficiency)

Let the Assumptions C and D be satisfied. Let the local stopping criteria hold. Then, for all \( K \in \mathcal{T}_h \),

\[ \eta_{\text{disc},K}^k,i + \eta_{\text{lin},K}^k,i + \eta_{\text{alg},K}^k,i + \eta_{\text{rem},K}^k,i \leq C \sum_{a \in \mathcal{V}_K} \left( \| \mathcal{R}(u_h^k,i) \|_{W_0^1,p(\omega_a)' + NC} \right). \]

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Global efficiency

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Let the Assumptions C and D be satisfied. Let the global stopping criteria hold. Then,

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Let the Assumptions C and D be satisfied. Let the local stopping criteria hold. Then, for all \( K \in \mathcal{T}_{h} \),

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4. Conclusions and ongoing work
Nonconforming finite elements for the $p$-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\sigma(\nabla u_h), \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h.$$ 

- $\sigma(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- $V_h \not\subset V$ the Crouzeix–Raviart space
- leads to the system of nonlinear algebraic equations

$$A(U) = F$$
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- leads to the system of **nonlinear algebraic equations**

\[
\mathcal{A}(U) = F
\]
Linearization

Find $u_h^k \in V_h$ such that

$$(\sigma^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f, \psi_e) \quad \forall e \in \mathcal{E}_h^{\text{int}}.$$ 

- $u_h^0 \in V_h$ yields the initial vector $U^0$
- fixed-point linearization
  $$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi$$
- Newton linearization
  $$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi + (p - 2)|\nabla u_h^{k-1}|^{p-4}$$
  $$\left(\nabla u_h^{k-1} \otimes \nabla u_h^{k-1}\right)(\xi - \nabla u_h^{k-1})$$

leads to the system of linear algebraic equations

$$\bigwedge^{k-1} U^k = F^{k-1}.$$
Linearization

Find $u_h^k \in V_h$ such that

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$$ \sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi $$

- Newton linearization

$$ \sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2} \xi + (p-2)|\nabla u_h^{k-1}|^{p-4} $$

$$ (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\xi - \nabla u_h^{k-1}) $$

- leads to the system of linear algebraic equations

$$ A^{k-1} U^k = F^{k-1} $$
Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\sigma^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f, \psi_e) - R_{e}^{k,i} \quad \forall e \in \mathcal{E}_h^{\text{int}}.$$ 

- algebraic residual vector $R_{e}^{k,i} = \{R_{e}^{k,i}\}_{e \in \mathcal{E}_h^{\text{int}}}$
- discrete system

$$A^{k-1} U^k = F^{k-1} - R_{e}^{k,i}$$
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$$\Delta^{k-1} U^k = F^{k-1} - R^{k,i}$$
Flux reconstructions

Definition (Construction of \((d_h^{k,i} + I_h^{k,i})\))

For all \(K \in \mathcal{T}_h\),
\[
(d_h^{k,i} + I_h^{k,i})|_K := -\sigma^{k-1}(\nabla u_h^{k,i})|_K + \frac{f|_K}{d}(x - x_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(x - x_K)|_{K_e},
\]
where \(R_e^{k,i} = (f, \psi_e) - (\sigma^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e)\) \(\forall e \in \mathcal{E}_h^{\text{int}}\).

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Set \(a_h^{k,i} := (d_h^{k,i} + I_h^{k,i}) - (d_h^{k,i} + I_h^{k,i})\) for (adaptively chosen) \(\nu > 0\) additional algebraic solvers steps; \(R^{k,i+\nu} \rightsquigarrow \rho_h^{k,i}\).
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**Verification of the assumptions**

**Lemma (Assumptions A and B)**

**Assumptions A and B hold.**

**Comments**

- $\|a_{h}^{k,i}\|_{q,K} \rightarrow 0$ as the linear solver converges by definition
- $\|\hat{l}_{h}^{k,i}\|_{q,K} \rightarrow 0$ as the nonlinear solver converges by the construction of $\hat{l}_{h}^{k,i}$

**Lemma (Assumptions C and D)**

**Assumptions C and D hold.**

**Comments**

- Quadrature error is zero
- $d_{h}^{k,i}$ is close to $\sigma(\nabla u_{h}^{k,i})$: approximation properties of the Raviart–Thomas–Nédélec spaces
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Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

... all Assumptions A to D verified
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Model problem

- $p$-Laplacian

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega,$$

$$u = u_D \quad \text{on } \partial \Omega$$

- weak solution (used to impose the Dirichlet BC)

$$u(x, y) = -\frac{p-1}{p} \left( (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 \right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p} \left( \frac{1}{2} \right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10

- Crouzeix–Raviart nonconforming finite elements
Analytical and approximate solutions

Case $p = 1.5$

Case $p = 10$
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.
Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh

Newton

inexact Newton

ad. inexact Newton
Error and estimators, $p = 10$

- Newton
- Inexact Newton
- Ad. inexact Newton

Graphs showing the number of faces vs dual error for different methods.
Effectivity indices, $p = 10$

![Graphs showing effectivity indices for Newton, inexact Newton, and ad. inexact Newton methods.](image-url)

- Newton
- inexact Newton
- ad. inexact Newton
Error distribution, $p = 10$

Estimated error distribution

Exact error distribution
Newton and algebraic iterations, $p = 10$

Newton it. / refinement  alg. it. / Newton step  alg. it. / refinement

- Newton iteration vs. Refinement level
- Number of algebraic solver iterations vs. Newton iteration
- Total number of algebraic solver iterations vs. Refinement level
Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.
Error and estimators as a function of Newton iterations, $\rho = 1.5$, 6th level mesh

Newton

inexact Newton

ad. inexact Newton
Error and estimators, $p = 1.5$

Newton  

inexact Newton  

ad. inexact Newton
Effectivity indices, $p = 1.5$

- Newton
- inexact Newton
- ad. inexact Newton
Newton and algebraic iterations, $p = 1.5$

- Newton it. / refinement
- alg. it. / Newton step
- alg. it. / refinement
Numerical experiment II

Model problem

- $p$-Laplacian

\[ \nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \]
\[ u = u_D \quad \text{on } \partial \Omega \]

- weak solution (used to impose the Dirichlet BC)

\[ u(r, \theta) = r^{\frac{7}{8}} \sin(\theta^{\frac{7}{8}}) \]

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))

- Crouzeix–Raviart nonconforming finite elements
Error distribution on an adaptively refined mesh

Estimated error distribution

Exact error distribution
Estimated and actual errors and the effectivity index

Estimated and actual errors

Effectivity index

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Fully adaptive solvers via localization 42 / 45
Energy error and overall performance

![Energy error graph](image1)

![Overall performance graph](image2)

Energy error

Overall performance
Outline

1. Residuals and their dual norms
   - Laplace
   - Nonlinear Laplace

2. Localization dual norms
   - Local–global equivalence
   - Numerical results

3. Fully adaptive solvers
   - Setting
   - A posteriori guaranteed upper bound
   - Local stopping criteria, efficiency, and robustness
   - Applications
   - Numerical results

4. Conclusions and ongoing work
Conclusions

- dual residual norms are localizable
- local stopping criteria then lead to local efficiency
- concept of full adaptivity (linear solver, nonlinear solver, mesh)

Ongoing work

- multigrid as a linear solver
- convergence and optimality
Conclusions

- **dual residual norms** are localizable
- **local stopping criteria** then lead to **local efficiency**
- concept of **full adaptivity** (linear solver, nonlinear solver, mesh)

Ongoing work

- multigrid as a linear solver
- convergence and optimality
Bibliography

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Thank you for your attention!