Polynomial-degree-robust a posteriori error estimation for the curl-curl problem

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European Research Council

École des Ponts
ParisTech

Outline

The curl-curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{j} = 0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A}) &= \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A} = 0 && \text{in } \Omega, \\ \boldsymbol{A}\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, && \text{on } \Gamma_{\mathrm{D}}, \\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0 && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak formulation (consequence)

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Nédélec finite element discretization (consequence)

$\boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

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Bibliography

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- Monk (1998)
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Main results

Guaranteed upper bound (Chaumont-Frelet & V. (2021))

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}} \le \underbrace{\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h\|}_{\text{computable estimator}}$$

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Patchwise/broken patchwise flux equilibration

Patchwise flux equilibration

- **globally** *equilibrated* $\boldsymbol{H}(\mathrm{curl})$ *flux* $\boldsymbol{h}_h$
- *Prager–Synge* **constant-free** *upper bound*
- *larger vertex patches* $\mathcal{T}_{\mathbf{a}}$
- *equilibration in several stages, more expensive*
- *additional layer for efficiency*
- p-**robust**

Broken patchwise flux equilibration

- *locally equilibrated* $\boldsymbol{H}(\mathrm{curl})$ *fluxes* $\boldsymbol{h}_h^e$
- $6^{1/2}$, $C_{\mathrm{cont,PF}}$, *and* C_{L} *in the upper bound*; $C_{\mathrm{L}} = 1$ *if* Ω *is convex and no mixed BCs*
- *smaller edge patches* $\mathcal{T}_e$
- *equilibration in a single stage, cheaper, explicit for* $p = 0$
- *both estimator and efficiency on* ω_e
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Lift constant C_{L} such that for all $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$, $\nabla\times\boldsymbol{w} = \nabla\times\boldsymbol{v}$, and

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Equilibration – the bottom line

H^1-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v} = j$?
- When $j \in L^2(\Omega)$ and $(j, 1) = 0$ if $\Gamma_\mathrm{N} = \partial\Omega$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v}_h = j$?
- When $j \in \mathcal{P}_p(\mathcal{T}_h)$ and $(j, 1) = 0$ if $\Gamma_\mathrm{N} = \partial\Omega$.

$\boldsymbol{H}$(curl)-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Equilibration – the bottom line

H^1-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v} = j$?
- When $j \in L^2(\Omega)$ and $(j, 1) = 0$ if $\Gamma_\mathrm{N} = \partial\Omega$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v}_h = j$?
- When $j \in \mathcal{P}_p(\mathcal{T}_h)$ and $(j, 1) = 0$ if $\Gamma_\mathrm{N} = \partial\Omega$.

$\boldsymbol{H}$(curl)-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Equilibration – the bottom line

H^1-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v} = j$?
- When $j \in L^2(\Omega)$ and $(j, 1) = 0$ if $\Gamma_{\mathrm{N}} = \partial\Omega$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v}_h = j$?
- When $j \in \mathcal{P}_p(\mathcal{T}_h)$ and $(j, 1) = 0$ if $\Gamma_{\mathrm{N}} = \partial\Omega$.

$\boldsymbol{H}(\mathrm{curl})$-case

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Equilibration – the bottom line

H^1-case

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- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v} = j$?
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- When there exists $\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{v}_h = j$?
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$\boldsymbol{H}$(curl)-case

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- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
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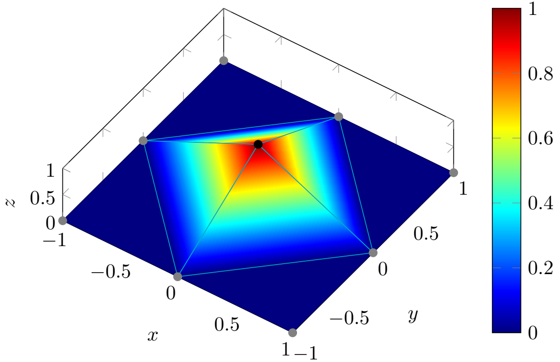
Discrete setting

- When there exists $\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Outline

The hat function and the partition of unity, $\Omega \subset \mathbb{R}^d$



The hat function $\psi^{\boldsymbol{a}}$, $d = 2$

Partition of unity

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi^{\boldsymbol{a}} = 1|_{\Omega}$$

The Laplacian $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Weak solution $u \in H^1_0(\Omega)$ is such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega)$$

Approximation $u_h \in H^1_0(\Omega)$ satisfies

$$(\nabla u_h, \nabla \psi^{a}) = (f, \psi^{a}) \qquad \forall a \in \mathcal{V}_h^{\rm int}$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ is defined by

$$\langle \mathcal{R}(u_h), v\rangle := (f, v) - (\nabla u_h, \nabla v)$$

Norm characterization

$$\|\nabla(u - u_h)\| = \|\mathcal{R}(u_h)\|_{-1} = \sup_{\substack{v \in H^1_0(\Omega)\\ \|\nabla v\| = 1}} \langle \mathcal{R}(u_h), v\rangle$$

$$H^1_*(\omega_a) := \begin{cases} \{v \in H^1(\omega_a);\ (v, 1)_{\omega_a} = 0\} & \text{for interior vertex } a \in \mathcal{V}_h^{\rm int} \\ \{v \in H^1(\omega_a);\ v = 0 \text{ on faces sharing } a\} & \text{for boundary vertex } a \in \mathcal{V}_h^{\rm ext} \end{cases}$$

ψ^{a}-weighted residual on $H^1_*(\omega_a)'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{a \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_a)\\ \|\nabla v\|_{\omega_a} = 1}} \langle \mathcal{R}(u_h), \psi^{a} v\rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_a)'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\rm cont,PF} \left\{ \sum_{a \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_a)\\ \|\nabla v\|_{\omega_a} = 1}} \langle \mathcal{R}(u_h), v\rangle \right]^2 \right\}^{1/2}$$

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$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\mathrm{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\text{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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$\psi^{\mathbf{a}}$-weighted residual on $H^1_*(\omega_{\mathbf{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_*(\omega_{\mathbf{a}})\\ \|\nabla v\|_{\omega_{\mathbf{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\mathbf{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H^1_0(\omega_{\mathbf{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\text{cont,PF}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H^1_0(\omega_{\mathbf{a}})\\ \|\nabla v\|_{\omega_{\mathbf{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

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Weak solution $u \in H_0^1(\Omega)$ is such that

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Approximation $u_h \in H_0^1(\Omega)$ satisfies

$$(\nabla u_h, \nabla \psi^{\boldsymbol{a}}) = (f, \psi^{\boldsymbol{a}}) \qquad \forall \boldsymbol{a} \in \mathcal{V}_h^{\text{int}}$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ is defined by

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v)$$

Norm characterization

$$\|\nabla(u - u_h)\| = \|\mathcal{R}(u_h)\|_{-1} = \sup_{\substack{v \in H_0^1(\Omega) \\ \|\nabla v\| = 1}} \langle \mathcal{R}(u_h), v \rangle$$

$$H_*^1(\omega_{\boldsymbol{a}}) := \begin{cases} \{v \in H^1(\omega_{\boldsymbol{a}});\ (v, 1)_{\omega_{\boldsymbol{a}}} = 0\} \text{ for interior vertex } \boldsymbol{a} \in \mathcal{V}_h^{\text{int}} \\ \{v \in H^1(\omega_{\boldsymbol{a}});\ v = 0 \text{ on faces sharing } \boldsymbol{a}\} \text{ for boundary vertex } \boldsymbol{a} \in \mathcal{V}_h^{\text{ext}} \end{cases}$$

$\psi^{\boldsymbol{a}}$-weighted residual on $H_*^1(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H_*^1(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \right]^2 \right\}^{1/2}$$

Unweighted residual on $H_0^1(\omega_{\boldsymbol{a}})'$

$$\|\nabla(u - u_h)\| \le (d+1)^{1/2} C_{\text{cont,PF}} \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \left[\sup_{\substack{v \in H_0^1(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \right]^2 \right\}^{1/2}$$

Bound by $\psi^{\boldsymbol{a}}$-weighted residuals on $H^1_*(\omega_{\boldsymbol{a}})'$

$v \in H^1_0(\Omega)$, $\|\nabla v\| = 1$:

$$\langle \mathcal{R}(u_h), v\rangle \overset{\mathrm{PU}}{=} \Big\langle \mathcal{R}(u_h), \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\psi^{\boldsymbol{a}} v)\Big\rangle \overset{\mathrm{GO}}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}}(v - \tilde{\Pi}_0 v)\rangle$$

$$= \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}}(v - \tilde{\Pi}_0 v)\rangle}{\|\nabla v\|_{\omega_{\boldsymbol{a}}}} \|\nabla v\|_{\omega_{\boldsymbol{a}}}$$

$$\overset{\mathrm{CS}}{\leq} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left[\sup_{\substack{w\in H^1_*(\omega_{\boldsymbol{a}})\\ \|\nabla w\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} w\rangle \right]^2 \right\}^{1/2} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v\|^2_{\omega_{\boldsymbol{a}}} \right\}^{1/2}$$

$$\overset{\mathrm{overlaps}}{\leq} (d+1)^{1/2} \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left[\sup_{\substack{w\in H^1_*(\omega_{\boldsymbol{a}})\\ \|\nabla w\|_{\omega_{\boldsymbol{a}}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} w\rangle \right]^2 \right\}^{1/2}$$

Bound by unweighted residuals on $H^1_0(\omega_{\boldsymbol a})'$

for $v \in H^1_*(\omega_{\boldsymbol a})$:

$$\|\nabla(\psi^{\boldsymbol a} v)\|_{\omega_{\boldsymbol a}} \leq \|\nabla\psi^{\boldsymbol a} v\|_{\omega_{\boldsymbol a}} + \|\psi^{\boldsymbol a}\nabla v\|_{\omega_{\boldsymbol a}} \leq \underbrace{(1 + C_{\mathrm{PF}} h_{\omega_{\boldsymbol a}} \|\nabla\psi^{\boldsymbol a}\|_{\infty,\omega_{\boldsymbol a}})}_{\leq C_{\mathrm{cont,PF}}} \|\nabla v\|_{\omega_{\boldsymbol a}}$$

bound for the $\psi^{\boldsymbol a}$-weighted residual on $H^1_*(\omega_{\boldsymbol a})'$, since $\psi^{\boldsymbol a} v \in H^1_0(\omega_{\boldsymbol a})$ for $v \in H^1_*(\omega_{\boldsymbol a})$:

$$\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol a})\\ \|\nabla v\|_{\omega_{\boldsymbol a}}=1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol a} v\rangle = \sup_{v \in H^1_*(\omega_{\boldsymbol a})} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol a} v\rangle}{\|\nabla v\|_{\omega_{\boldsymbol a}}} = \sup_{v \in H^1_*(\omega_{\boldsymbol a})} \frac{\langle \mathcal{R}(u_h), \psi^{\boldsymbol a} v\rangle}{\|\nabla(\psi^{\boldsymbol a} v)\|_{\omega_{\boldsymbol a}}} \frac{\|\nabla(\psi^{\boldsymbol a} v)\|_{\omega_{\boldsymbol a}}}{\|\nabla v\|_{\omega_{\boldsymbol a}}}$$

$$\leq C_{\mathrm{cont,PF}} \sup_{\substack{v \in H^1_0(\omega_{\boldsymbol a})\\ \|\nabla v\|_{\omega_{\boldsymbol a}}=1}} \langle \mathcal{R}(u_h), v\rangle$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ on $\partial\omega_{\boldsymbol{a}}$ and $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}$,

$$
\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \leq \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ with $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

$$
\begin{aligned}
\langle \mathcal{R}(u_h), v \rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}
$$

$$
\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \leq \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and*
$\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ *on* $\partial\omega_{\boldsymbol{a}}$
and $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}$,

$$\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}$$

$$\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \leq \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and*
$\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

$$\begin{aligned}
\langle \mathcal{R}(u_h), v\rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}$$

$$\sup_{\substack{v \in H^1_0(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), v\rangle \leq \|\nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

Patchwise bounds by equilibrated fluxes

$\psi^{\mathbf{a}}$-weighted residual on $H^1_*(\omega_{\mathbf{a}})'$

for $v \in H^1_*(\omega_{\mathbf{a}})$ *with* $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$ *and*

$\boldsymbol{\sigma}_h^{\mathbf{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with* $\boldsymbol{\sigma}_h^{\mathbf{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\mathbf{a}}} = 0$ *on* $\partial\omega_{\mathbf{a}}$

and $\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}} = f\psi^{\mathbf{a}} - \nabla u_h \cdot \nabla \psi^{\mathbf{a}}$,

$$\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\mathbf{a}} v \rangle \\
&= (f, \psi^{\mathbf{a}} v)_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(\psi^{\mathbf{a}} v))_{\omega_{\mathbf{a}}} \\
&= (f\psi^{\mathbf{a}} - \nabla u_h \cdot \nabla \psi^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} - (\psi^{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} - (\psi^{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\mathbf{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\mathbf{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}
\end{aligned}$$

$$\sup_{\substack{v \in H^1_*(\omega_{\mathbf{a}}) \\ \|\nabla v\|_{\omega_{\mathbf{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\mathbf{a}} v \rangle \leq \|\psi^{\mathbf{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

Unweighted residual on $H^1_0(\omega_{\mathbf{a}})'$

for $v \in H^1_0(\omega_{\mathbf{a}})$ *with* $\|\nabla v\|_{\omega_{\mathbf{a}}} = 1$ *and*

$\boldsymbol{\sigma}_h^{\mathbf{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\mathbf{a}})$ *with* $\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}} = f$,

$$\begin{aligned}
\langle \mathcal{R}(u_h), v \rangle &= (f, v)_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}, v)_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} \\
&\overset{\text{Green}}{=} -(\nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} \\
&\overset{\text{CS}}{\leq} \|\nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}
\end{aligned}$$

$$\sup_{\substack{v \in H^1_0(\omega_{\mathbf{a}}) \\ \|\nabla v\|_{\omega_{\mathbf{a}}} = 1}} \langle \mathcal{R}(u_h), v \rangle \leq \|\nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

Patchwise bounds by equilibrated fluxes

$\psi^{\boldsymbol{a}}$-weighted residual on $H^1_*(\omega_{\boldsymbol{a}})'$

for $v \in H^1_*(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and*
$\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ *on* $\partial\omega_{\boldsymbol{a}}$
and $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}$,

$$\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{Green}}{=} -(\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&\overset{\text{CS}}{\leq} \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}
\end{aligned}$$

$$\sup_{\substack{v \in H^1_*(\omega_{\boldsymbol{a}}) \\ \|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1}} \langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v \rangle \leq \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$$

Unweighted residual on $H^1_0(\omega_{\boldsymbol{a}})'$

for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and*
$\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = f$,

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\langle \mathcal{R}(u_h), v \rangle &= (f, v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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\end{aligned}$$

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Patchwise bounds by equilibrated fluxes

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$\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ *with* $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \cdot \boldsymbol{n}|_{\partial\omega_{\boldsymbol{a}}} = 0$ *on* $\partial\omega_{\boldsymbol{a}}$
and $\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi^{\boldsymbol{a}}$,

$$\begin{aligned}
&\langle \mathcal{R}(u_h), \psi^{\boldsymbol{a}} v\rangle \\
&= (f, \psi^{\boldsymbol{a}} v)_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(\psi^{\boldsymbol{a}} v))_{\omega_{\boldsymbol{a}}} \\
&= (f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
&= (\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}} \nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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for $v \in H^1_0(\omega_{\boldsymbol{a}})$ *with* $\|\nabla v\|_{\omega_{\boldsymbol{a}}} = 1$ *and*
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Patchwise bounds by equilibrated fluxes

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&= (f\psi^{\boldsymbol{a}} - \nabla u_h\cdot\nabla\psi^{\boldsymbol{a}}, v)_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\nabla u_h, \nabla v)_{\omega_{\boldsymbol{a}}} \\
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\end{aligned}$$

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Patchwise bounds by equilibrated fluxes

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\end{aligned}$$

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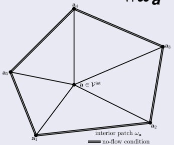
Discrete (broken) patchwise equilibrated fluxes ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$)

Definition (Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_p(f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^{\boldsymbol{a}})}} \|\psi^{\boldsymbol{a}} \nabla u_h + \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$



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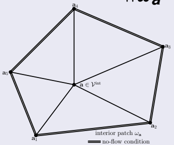
and combine $\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}$.



Key points

- **homogeneous normal BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{\sigma}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$
- **global equilibrium** $\nabla\cdot\boldsymbol{\sigma}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \Pi_p(f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi^{\boldsymbol{a}}) = \Pi_p f$

Definition ($\approx$ Chaumont-Frelet, Ern, Vohralík (2021))

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Key points

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Discrete (broken) patchwise equilibrated fluxes ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$)

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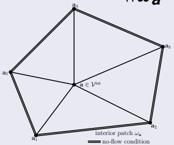

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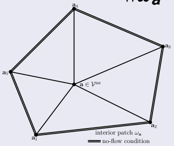

Key points

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For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{\sigma}_h^{\boldsymbol a} := \arg \min_{\substack{\boldsymbol v_h \in [\text{illegible}]\, \boldsymbol H(\mathrm{div}, \omega_{\boldsymbol a}) \\ \nabla\cdot \boldsymbol v_h = [\text{illegible}]}} \|\nabla u_h + \boldsymbol v_h\|_{\omega_{\boldsymbol a}}^2$$

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Discrete (broken) patchwise equilibrated fluxes ($u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$)

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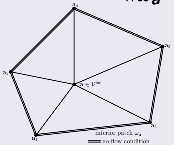

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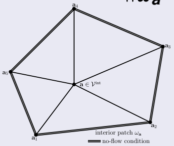

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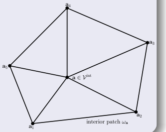

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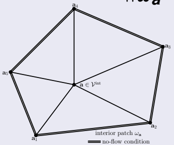

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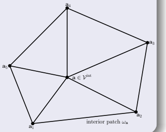

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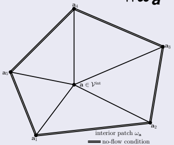

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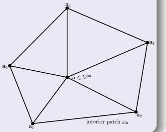

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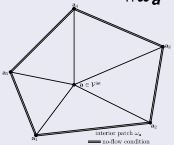

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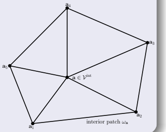

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The Laplacian $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Guaranteed upper bound

$$\underbrace{\|\nabla(u-u_h)\|}_{\text{unknown error}} \le \underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|}_{\text{computable estimator}}$$

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Outline

The curl-curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{j} = 0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A} : \Omega \to \mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A}) &= \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A} = 0 && \text{in } \Omega,\\ \boldsymbol{A}\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, && \text{on } \Gamma_{\mathrm{D}},\\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0 && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

Weak formulation (consequence)

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Nédélec finite element discretization (consequence)

$\boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

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Discrete (broken) patchwise equilibrated fluxes

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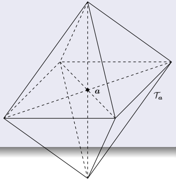

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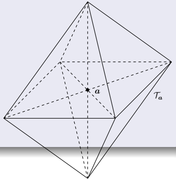

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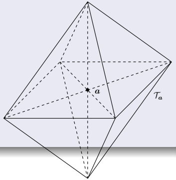

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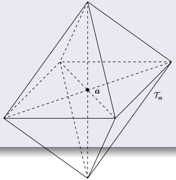

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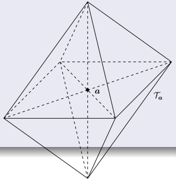

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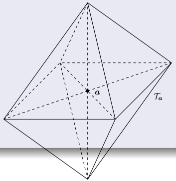

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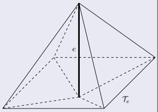

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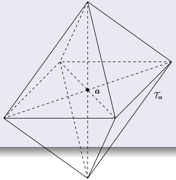

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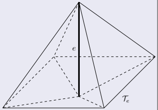

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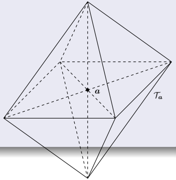

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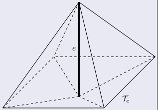

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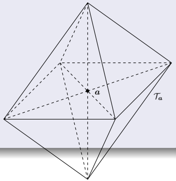

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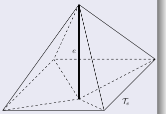

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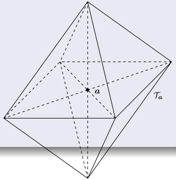

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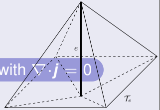
▸ **well-posed** for $\boldsymbol{j}|_{\omega_e} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_e) \cap \boldsymbol{H}(\mathrm{div}, \omega_e)$ with $\nabla \cdot \boldsymbol{j} = 0$



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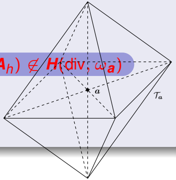
$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{j} + \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)}} \|\psi^{\boldsymbol{a}} (\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

▸ $\psi^{\boldsymbol{a}} \boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ but $\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h) \notin \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



Key points

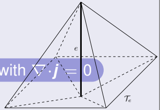
- **homogeneous tangential BC** on $\partial \omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
- **global equilibrium** $\nabla \times \boldsymbol{h}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \times \boldsymbol{h}_h^{\boldsymbol{a}}$
$$= \sum_{\boldsymbol{a} \in \mathcal{V}_h} (\psi^{\boldsymbol{a}} \boldsymbol{j} + \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)) = \boldsymbol{j}$$

Definition (Chaumont-Frelet, Ern, Vohralík (2021))

For each **edge** $e \in \mathcal{E}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{h}_h^{e} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p}(\mathcal{T}_{e}) \cap \boldsymbol{H}(\mathrm{curl}, \omega_{e}) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}}} \|\nabla \times \boldsymbol{A}_h - \boldsymbol{v}_h\|_{\omega_{e}}^2.$$

▸ **well-posed** for $\boldsymbol{j}|_{\omega_e} \in \boldsymbol{\mathcal{RT}}_{p}(\mathcal{T}_{e}) \cap \boldsymbol{H}(\mathrm{div}, \omega_{e})$ with $\nabla \cdot \boldsymbol{j} = 0$



Key points

- **no BC** on $\partial \omega_e$: $\boldsymbol{h}_h = \sum_{e \in \mathcal{E}_h} \boldsymbol{h}_h^e \notin \boldsymbol{H}(\mathrm{curl}, \Omega)$
- only **local equilibrium** $\nabla \times \boldsymbol{h}_h^e = \boldsymbol{j}$

Discrete (broken) patchwise equilibrated fluxes

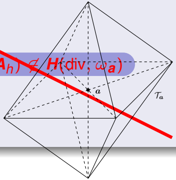
Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \psi^{\boldsymbol{a}} \boldsymbol{j} + \nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)}} \|\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

- $\psi^{\boldsymbol{a}} \boldsymbol{j} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ but $\nabla \psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h) \notin \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



Key points

- **homogeneous tangential BC** on $\partial \omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
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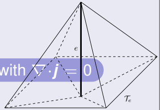
Definition (Chaumont-Frelet, Ern, Vohralík (2021))

For each **edge** $e \in \mathcal{E}_h$, solve the **local constrained minimization pb**

$$\boldsymbol{h}_h^{e} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p}(\mathcal{T}_e) \cap \boldsymbol{H}(\mathrm{curl}, \omega_e) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}}} \|\nabla \times \boldsymbol{A}_h - \boldsymbol{v}_h\|_{\omega_e}^2.$$

- **well-posed** for $\boldsymbol{j}|_{\omega_e} \in \mathcal{RT}_p(\mathcal{T}_e) \cap \boldsymbol{H}(\mathrm{div}, \omega_e)$ with $\nabla \cdot \boldsymbol{j} = 0$



Key points

- **no BC** on $\partial \omega_e$: $\boldsymbol{h}_h = \sum_{e \in \mathcal{E}_h} \boldsymbol{h}_h^e \notin \boldsymbol{H}(\mathrm{curl}, \Omega)$
- only **local equilibrium** $\nabla \times \boldsymbol{h}_h^e = \boldsymbol{j}$

Equilibration – the bottom line

Continuous setting

- When there exists $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v} = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

Discrete setting

- When there exists $\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}$?
- When $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$.

We suppose

- $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$
- $\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)$ (no data oscillation, simplicity of presentation)

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- $\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h)$ (no data oscillation, simplicity of presentation)

The curl–curl case

Guaranteed upper bound (Chaumont-Frelet & V. (2021))

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}} \le \underbrace{\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h\|}_{\text{computable estimator}}$$

$$\le 2\left\{\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)-\boldsymbol{h}_h^{\boldsymbol{a}}\|^2_{\omega_{\boldsymbol{a}}}\right\}^{1/2}$$

p-robust local efficiency (Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009, 2012); Chaumont-Frelet & V. (2021))

$$\|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)-\boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}\le C_{\mathrm{st}}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}^*(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}}\langle\mathcal{R}(\boldsymbol{A}_h),\psi^{\boldsymbol{a}}\boldsymbol{v}\rangle$$

$$\le C_{\mathrm{st}}\,C_{\mathrm{cont,PFW}}\,\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}$$

Guaranteed upper bound (Ern, Ch.-Frelet, & V. (2021))

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}}$$

$$\underbrace{\le C_{\mathrm{L}}\,6^{1/2}\,C_{\mathrm{cont,PF}}\left\{\sum_{e\in\mathcal{E}_h}\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h^{e}\|^2_{\omega_e}\right\}^{1/2}}_{\text{computable estimator}}$$

p-robust local efficiency (Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Ern, Chaumont-Frelet, & V. (2021))

$$\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h^{e}\|_{\omega_e}\le C_{\mathrm{st}}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_e)\\ \|\nabla\times\boldsymbol{v}\|_{\omega_e}=1}}\langle\mathcal{R}(\boldsymbol{A}_h),\boldsymbol{v}\rangle$$

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The curl–curl case

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$$\le C_{\mathrm{st}}\, C_{\mathrm{cont,PFW}}\, \|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|_{\omega_{\boldsymbol{a}}}$$

Guaranteed upper bound (Ern, Ch.-Frelet, & V. (2021))

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}}$$

$$\le \underbrace{C_{\mathrm{L}}\, 6^{1/2}\, C_{\mathrm{cont,PF}}\left\{\sum_{e\in\mathcal{E}_h}\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h^e\|_{\omega_e}^2\right\}^{1/2}}_{\text{computable estimator}}$$

p-robust local efficiency (Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Ern, Chaumont-Frelet, & V. (2021))

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The curl–curl case

Guaranteed upper bound (Chaumont-Frelet & V. (2021))

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}} \leq \underbrace{\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h\|}_{\text{computable estimator}}$$

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$$\|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)-\boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}\leq C_{\mathrm{st}}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}^*(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}=1}}\langle\mathcal{R}(\boldsymbol{A}_h),\psi^{\boldsymbol{a}}\boldsymbol{v}\rangle$$

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$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\text{unknown error}}$$

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$$\|\nabla\times\boldsymbol{A}_h-\boldsymbol{h}_h^e\|_{\omega_e}\leq C_{\mathrm{st}}\sup_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_e)\\ \|\nabla\times\boldsymbol{v}\|_{\omega_e}=1}}\langle\mathcal{R}(\boldsymbol{A}_h),\boldsymbol{v}\rangle$$

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The curl–curl case

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The curl–curl case

Guaranteed upper bound (Chaumont-Frelet & V. (2021))

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Outline

1. Introduction
2. Reminder on the H^1-case
3. The **H**(curl)-case
4. **H**(curl) patchwise equilibration
5. Stable (broken) **H**(curl) polynomial extensions
6. Numerical experiments
7. Conclusions

Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h)$ to a Raviart–Thomas space

For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, *consider* $p' := \min\{p, 1\}$*-degree patchwise minimizations:*

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = -\nabla \psi^{\boldsymbol{a}} \cdot \boldsymbol{j} \\ (\boldsymbol{v}_h, \boldsymbol{r}_h)_K = (\nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h), \boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2.$$

Comments

- $\nabla\psi^{\boldsymbol{a}} \times (\nabla \times \boldsymbol{A}_h) \notin \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$
- additional orthogonality constraint
  - crucial for stage 2
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Stage 2: **divergence-free decomposition** of the given divergence-free Raviart-Thomas piecewise polynomial $\boldsymbol{\delta}_h$

Divergence-free decomposition of $\boldsymbol{\delta}_h$

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$-degree elementwise minimizations:

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_1(K) \\ \nabla\cdot\boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K = \boldsymbol{I}_1^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}_1^{\mathcal{RT}}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\|_K^2 \quad \forall \boldsymbol{a} \in \mathcal{V}_K \ \textit{when } p = 0,$$

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Comments

- patchwise contributions

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla\cdot\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h$$

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- $\delta_h^{\boldsymbol{a}}$ form a **divergence-free decomposition** of δ_h, $\delta_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \delta_h^{\boldsymbol{a}}$

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Stage 2: divergence-free decomposition of the given divergence-free current density $\boldsymbol{j}$

Divergence-free decomposition of the current density $\boldsymbol{j}$

Set

$$\boldsymbol{j}_h^{\boldsymbol{a}} := \psi^{\boldsymbol{a}} \boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}.$$

Then

$$\boldsymbol{j}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}),$$

$$\nabla\cdot \boldsymbol{j}_h^{\boldsymbol{a}} = 0,$$

$$\sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}.$$

Stage 3: discrete patchwise equilibrated fluxes

Definition (Chaumont-Frelet, Vohralík (2021))

For each **vertex** $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

and combine

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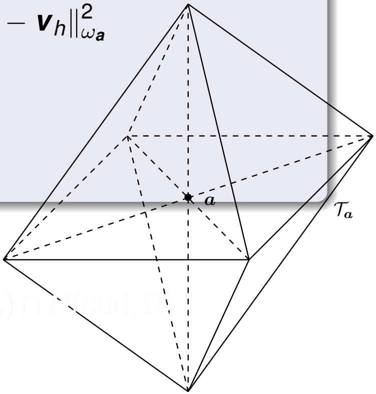

Key points

- **homogeneous tangential BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
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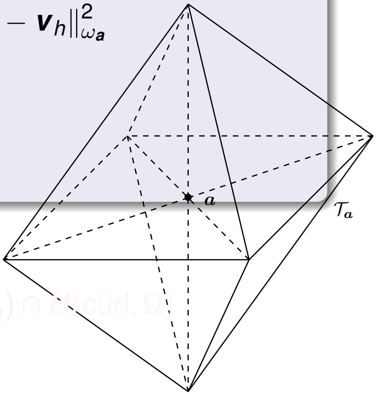

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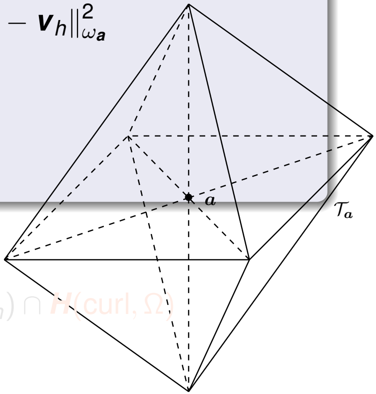

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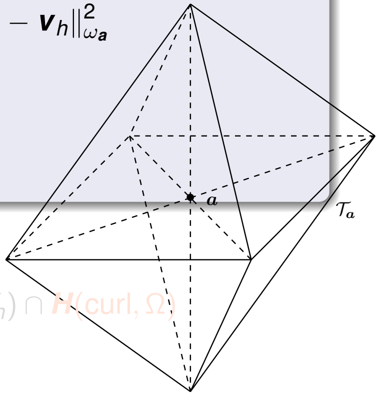

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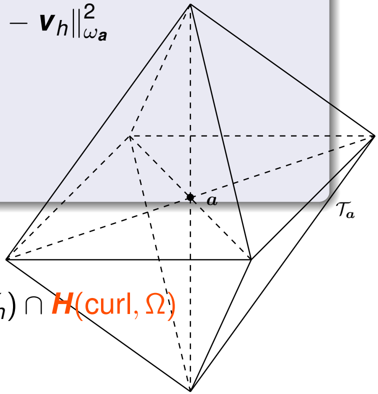

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Outline

$\boldsymbol{H}$(curl) polynomial extension on a tetrahedron

Theorem ($\boldsymbol{H}$(curl) polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Chaumont-Frelet, Ern, & V. (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $\boldsymbol{r}_K \in \boldsymbol{\mathcal{RT}}_p(K)$ such that $\nabla\cdot\boldsymbol{r}_K = 0$, and for all $\boldsymbol{r}_\mathcal{F} \in \boldsymbol{\mathcal{N}}_p^\tau(\Gamma_\mathcal{F})$ such that $\boldsymbol{r}_K\cdot\boldsymbol{n}_F = \mathrm{curl}_F(\boldsymbol{r}_F)$ for all $F \in \mathcal{F}$, there holds

$$\min_{\substack{\boldsymbol{v}_p \in \boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_p = \boldsymbol{r}_K\\ \boldsymbol{v}_p|^\tau_\mathcal{F} = \boldsymbol{r}_\mathcal{F}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{v} = \boldsymbol{r}_K\\ \boldsymbol{v}|^\tau_\mathcal{F} = \boldsymbol{r}_\mathcal{F}}} \|\boldsymbol{v}\|_K.$$

Comments

- C_{st} only depends on the shape-regularity of K
- for (pw) p-polynomial data $\boldsymbol{r}_K$, $\boldsymbol{r}_\mathcal{F}$, minimization over $\boldsymbol{\mathcal{N}}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{curl}, K)$
- extension to an edge patch: Chaumont-Frelet, Ern, & V. (2021)
- extension to a vertex patch: Chaumont-Frelet & V. (to come)

H(curl) polynomial extension on a tetrahedron

Theorem (H(curl) polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Chaumont-Frelet, Ern, & V. (2020))

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $\boldsymbol{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla\cdot\boldsymbol{r}_K = 0$, and for all $\boldsymbol{r}_{\mathcal{F}} \in \mathcal{N}_p^{\boldsymbol{\tau}}(\Gamma_{\mathcal{F}})$ such that $\boldsymbol{r}_K\cdot\boldsymbol{n}_F = \mathrm{curl}_F(\boldsymbol{r}_F)$ for all $F \in \mathcal{F}$, there holds

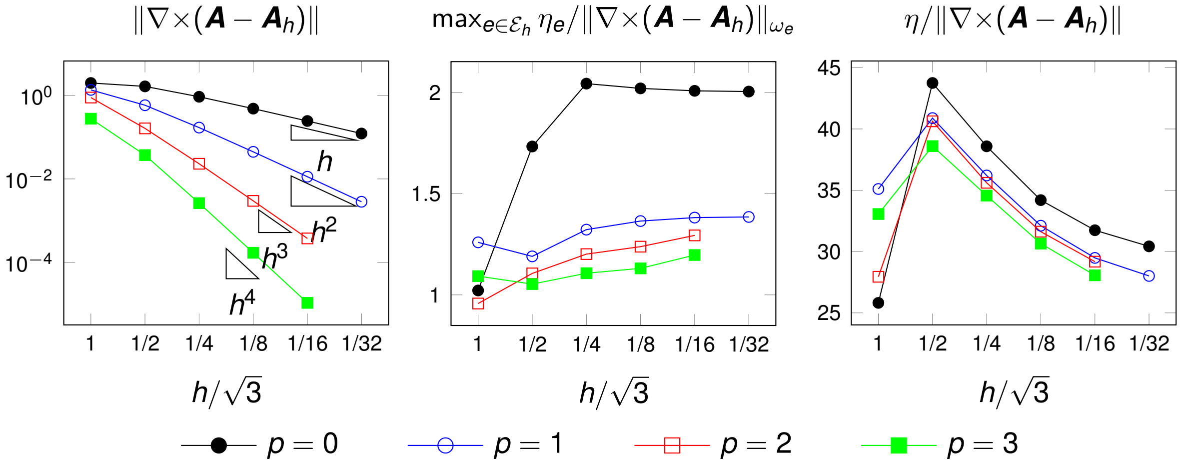
$$\min_{\substack{\boldsymbol{v}_p\in\mathcal{N}_p(K)\\ \nabla\times\boldsymbol{v}_p=\boldsymbol{r}_K\\ \boldsymbol{v}_p|^{\boldsymbol{\tau}}_{\mathcal{F}}=\boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{v}=\boldsymbol{r}_K\\ \boldsymbol{v}|^{\boldsymbol{\tau}}_{\mathcal{F}}=\boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}\|_K.$$

Comments

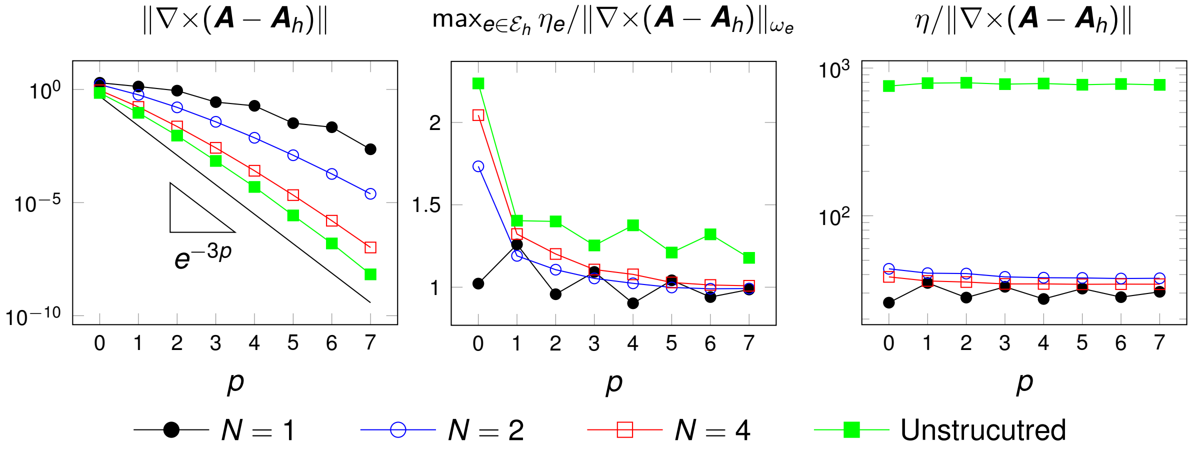
- C_{st} only depends on the shape-regularity of K
- for (pw) p-polynomial data $\boldsymbol{r}_K$, $\boldsymbol{r}_{\mathcal{F}}$, minimization over $\mathcal{N}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{curl}, K)$
- extension to an edge patch: Chaumont-Frelet, Ern, & V. (2021)
- extension to a vertex patch: Chaumont-Frelet & V. (to come)

Outline

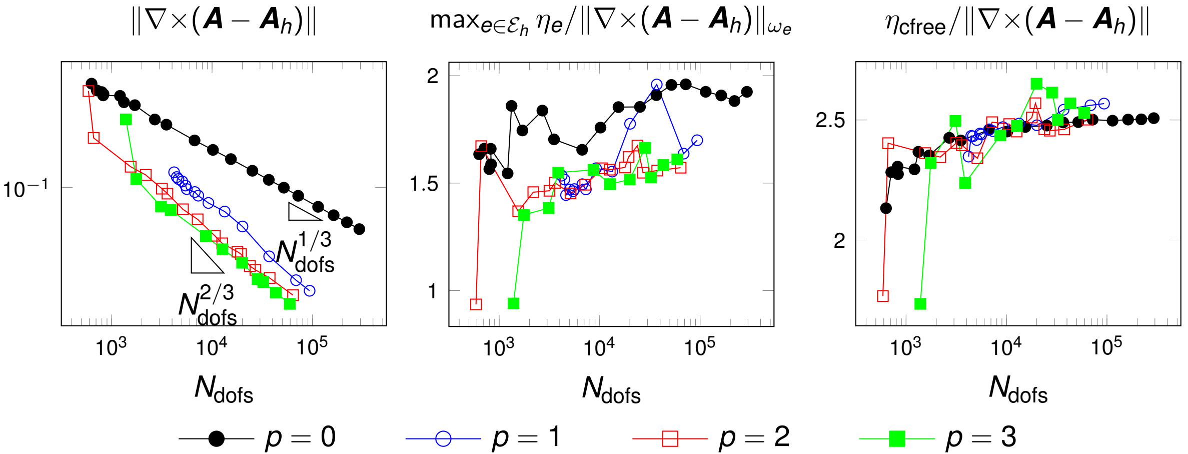
Broken patchwise equilibration, **smooth solution**, *h*-refinement



Broken patchwise equilibration, smooth solution, p-refinement



Broken patchwise equilibration, **singular solution**, **adap.** refinement



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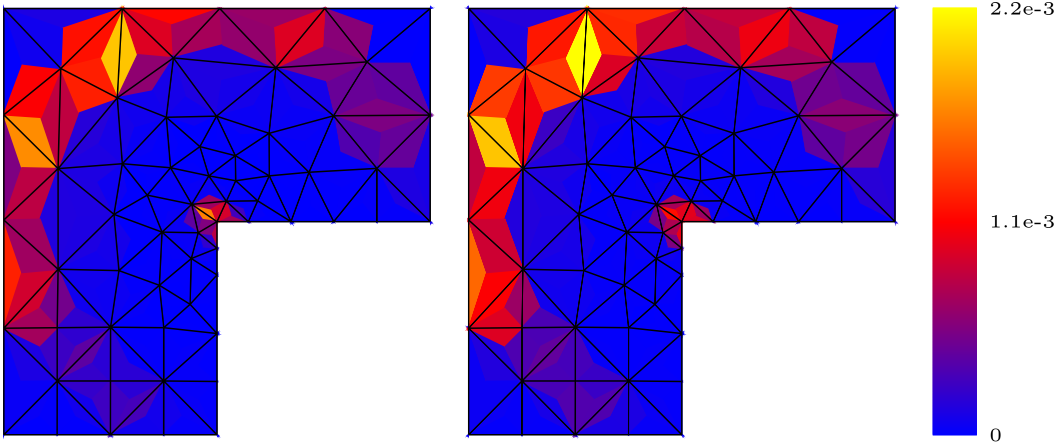
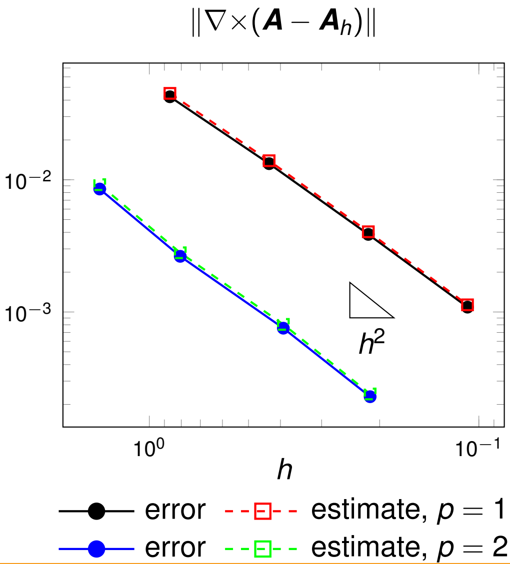
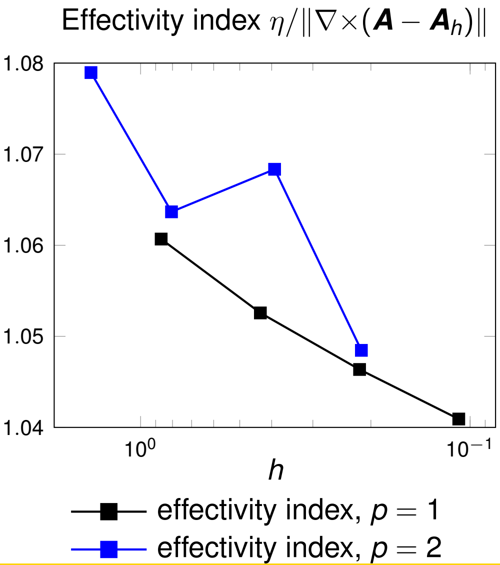


Estimators (left) and actual error (right), $p = 3$

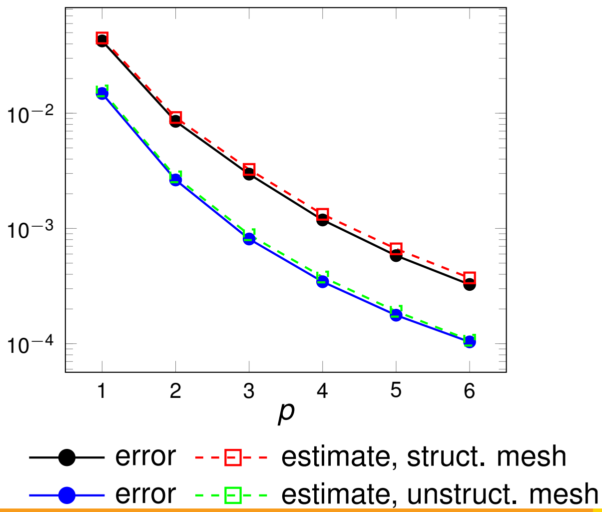
Patchwise equilibration, $\boldsymbol{H}^3$ solution, h-refinement



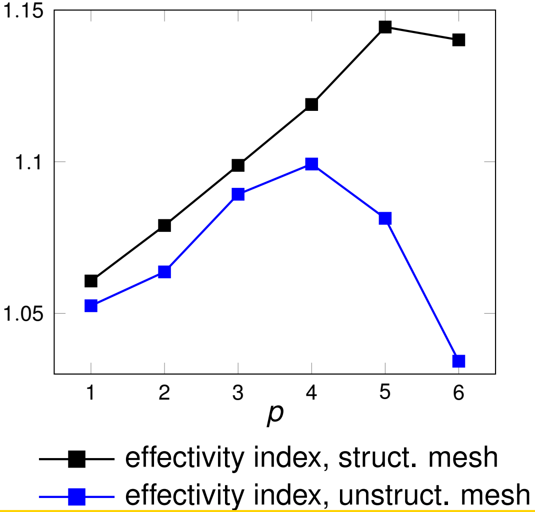


Patchwise equilibration, $\boldsymbol{H}^3$ solution, p-refinement

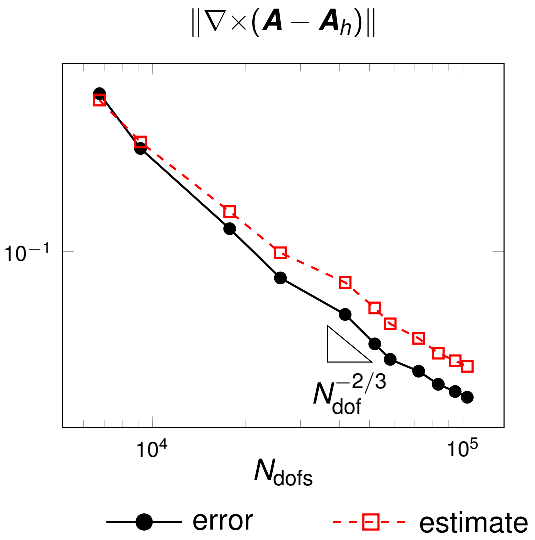
$\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$

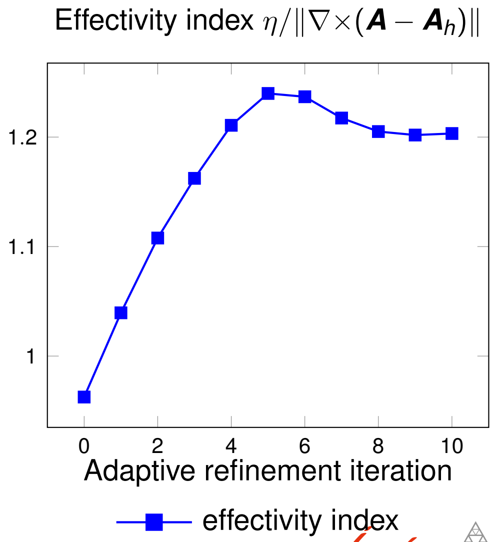


Effectivity index $\eta/\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|$

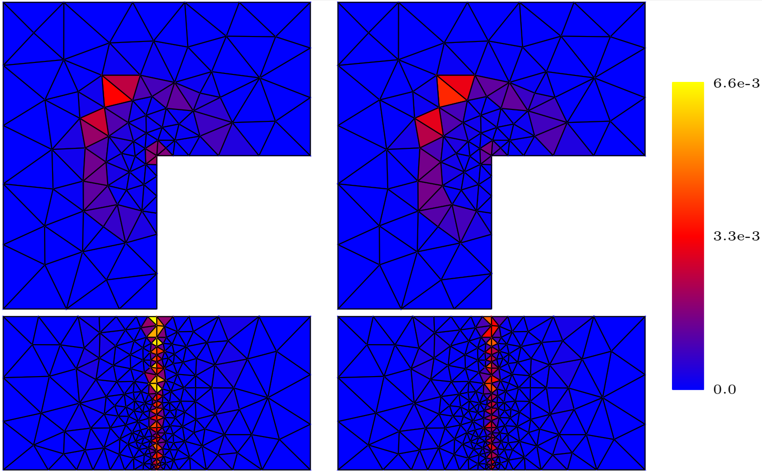


Patchwise equilibration, **singular solution**, **adap.** refinement ($p = 2$)





Patchwise equilibration, **singular solution**, **adap.** refinement ($p = 2$)



Estimators (left) and actual error (right), adaptive mesh refinement iteration #10.
Top view (top) and side view (bottom)

Outline

Conclusions

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- reliable, locally efficient, p-robust, and possibly constant-free estimates
- divergence-free decompositions of Raviart–Thomas piecewise polynomials

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Chaumont-Frelet T., Ern A., Vohralík M., Polynomial-degree-robust ***H***(curl)-stability of discrete minimization in a tetrahedron, *C. R. Math. Acad. Sci. Paris* **358** (2020), 1101–1110.

Chaumont-Frelet T., Ern A., Vohralík M., Stable broken ***H***(curl) polynomial extensions and *p*-robust a posteriori error estimates by broken patchwise equilibration for the curl–curl problem, *Math. Comp.* (2021), DOI 10.1090/mcom/3673.

Chaumont-Frelet T., Vohralík M., *p*-robust equilibrated flux reconstruction in ***H***(curl) based on local minimizations. Application to a posteriori analysis of the curl–curl problem, HAL Preprint 03227570, submitted for publication.

Thank you for your attention!

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