Potential and flux reconstructions for optimal a priori and a posteriori error estimates

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Outline

1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – unconstrained local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - p-stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
5. A posteriori estimates
   - Guaranteed upper bound and polynomial-degree-robust local efficiency
   - Numerical illustration
6. Tools (hp-optimality, p-robustness)
   - Polynomial extension operators
   - p-stable decompositions
7. Conclusions and outlook

A model partial differential equation

Poisson equation

Find $u : \Omega \to \mathbb{R}$

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\begin{aligned}
-\Delta u &= f \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega
\end{aligned}
$$

Setting

- $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, line segment, Lipschitz polygon, or Lipschitz polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

$$u \in H_0^1(\Omega), \quad -\nabla u \in \mathbf{H}(\mathrm{div}, \Omega), \quad \nabla\cdot(-\nabla u) = f$$

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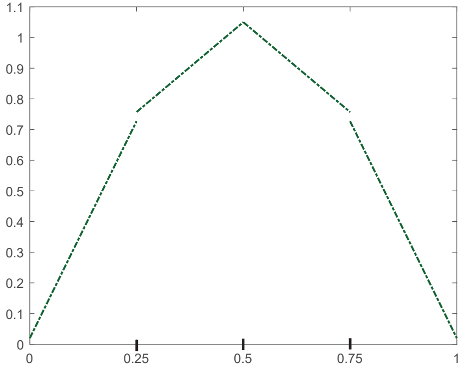
Numerical approximation

- $\mathcal{T}_h$ a simplicial mesh of Ω with characteristic mesh size $h := \max_{K\in\mathcal{T}_h} h_K$
- $\mathcal{P}_p(\mathcal{T}_h)$: piecewise polynomials of total degree $p \geq 0$
- numerical approximation u_h of u

Numerical approximation: $H_0^1(\Omega)$, example in 1D



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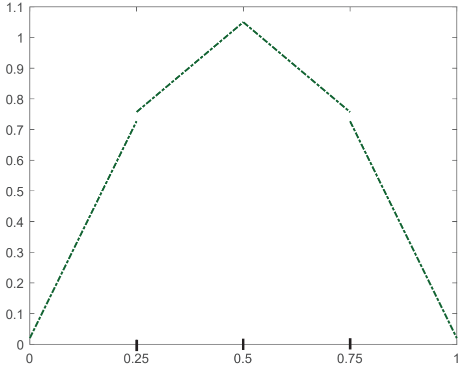

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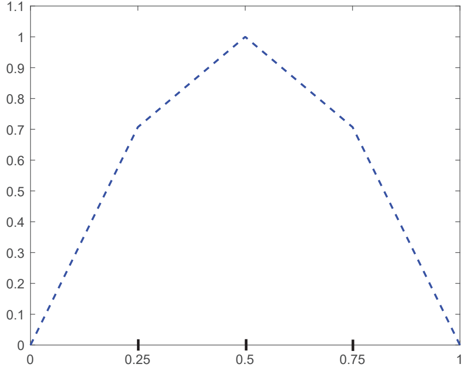
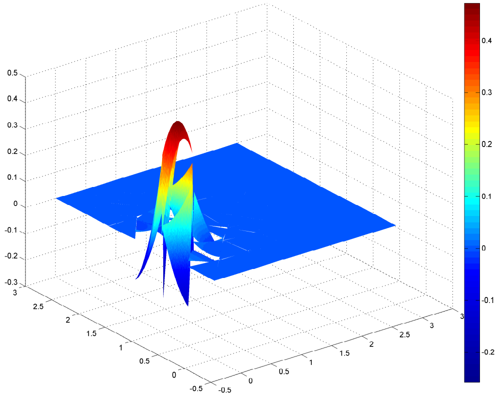
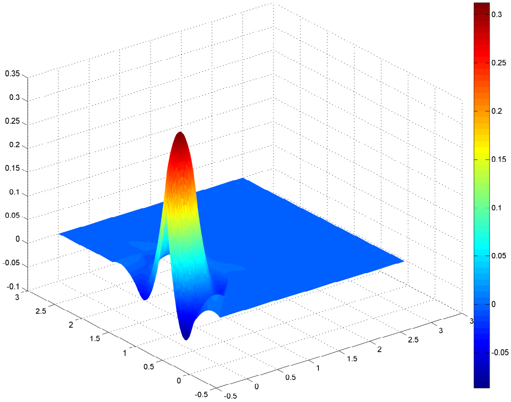


$u_h \in \mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

Numerical approximation: $H_0^1(\Omega)$, example in 2D



$u_h \in \mathcal{P}_2(\mathcal{T}_h)$



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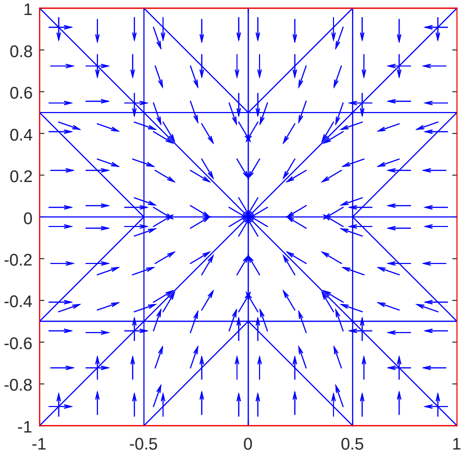
Numerical approximation: $\boldsymbol{H}(\mathrm{div},\Omega)$, example in 2D



$$-\nabla u_h \in [\mathcal{P}_0(\mathcal{T}_h)]^2$$

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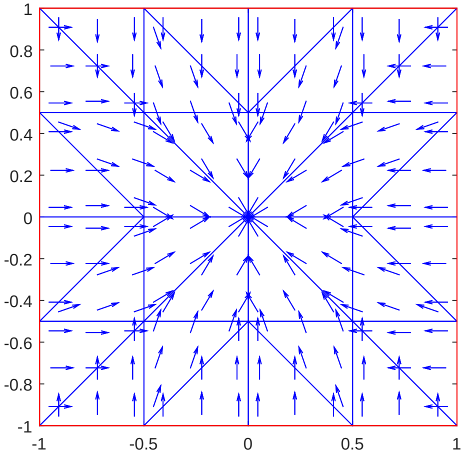

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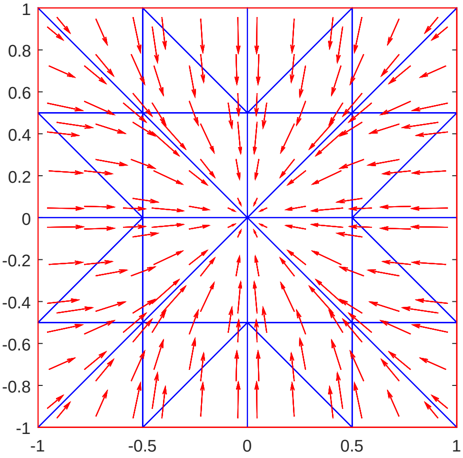

$-\nabla u_h \in \mathcal{RT}_1(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$

Error characterization

Theorem (Error equality)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 0$, be arbitrary. Then

$$\|\nabla_h(u - u_h)\|^2 \quad .$$

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$$= \underbrace{\max_{v \in H_0^1(\Omega),\, \|\nabla v\|=1} \{(f, v) - (\nabla_h u_h, \nabla v)\}}$$

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Optimal a posteriori error estimate, reliability (guaranteed upper bound)

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- **local** construction of **piecewise polynomial** s_h and $\boldsymbol{\sigma}_h$ from u_h

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- **local** construction of **piecewise polynomial** s_h and $\boldsymbol{\sigma}_h$ from u_h
- s_h so good that **no** $v \in H_0^1(\Omega)$ **can do better** (up to a constant)
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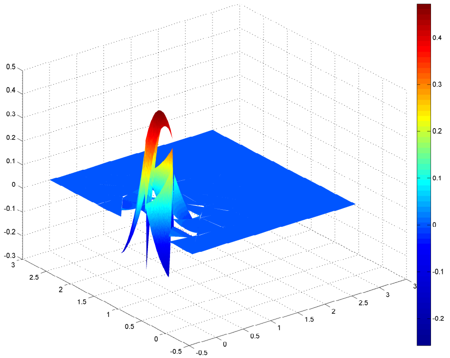
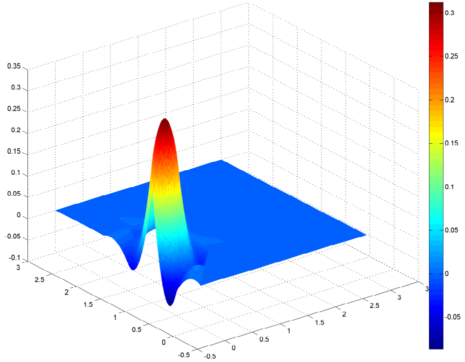


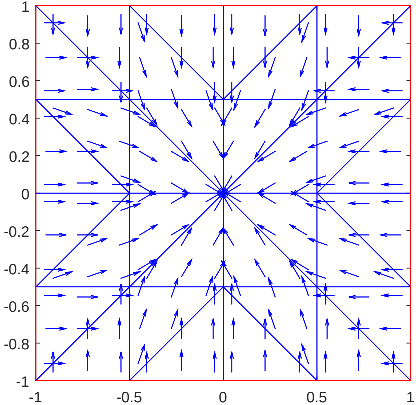
Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

flux reconstruction



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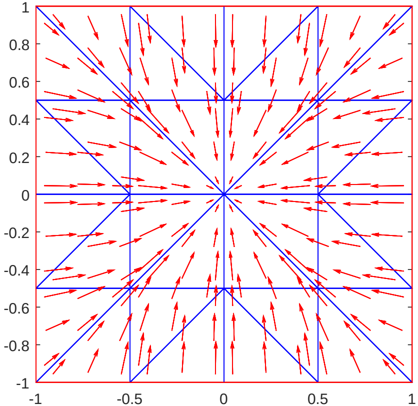
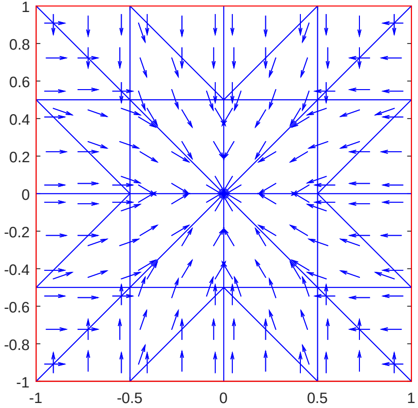


flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \mathbf{H}(\mathrm{div}, \Omega)$$

Equilibrated flux reconstruction



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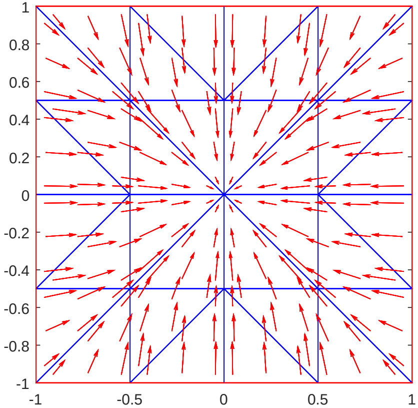

Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+(\xi_h,\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\ \forall \mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

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There holds

$$\underbrace{\|\nabla(u - u_h)\|}_{\displaystyle \min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|}$$

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- $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
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a priori error estimate

Conforming finite element approximation

Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$$

Theorem (Optimal a priori error estimate)

There holds

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Optimal a priori error estimate elementwise, both h and p

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Find $u_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$, $p \geq 1$, such that

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$$= \left\{ \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K^2}_{\text{local-best approximation of } u \text{ on each } K} \right\}^{1/2}$$

local-best approximation of u on each K
no interface constraints
regularity only in K counts

- $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u-v_h)\|_K$: $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$ but $\xi_h \notin H_0^1(\Omega)$
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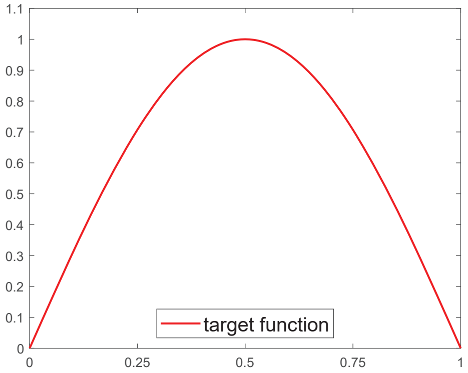
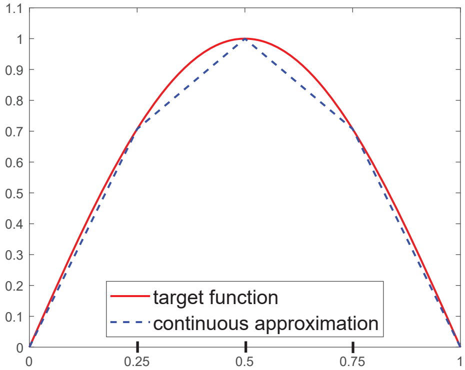
Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



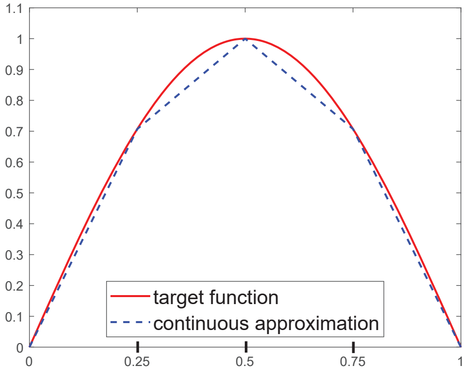
Target function in $H_0^1(\Omega)$

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**



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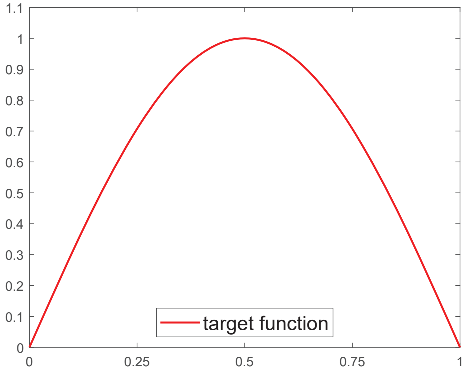

Target function in $H_0^1(\Omega)$

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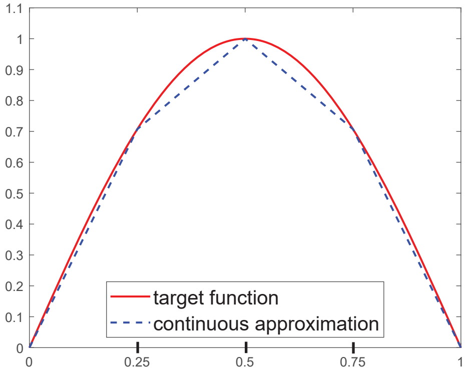

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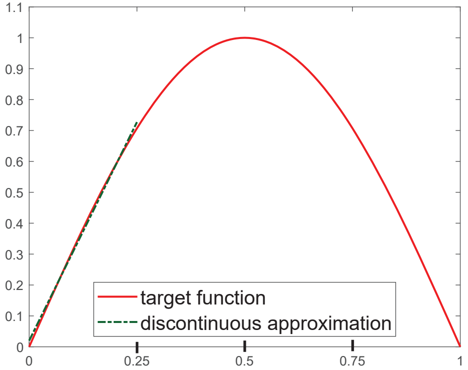
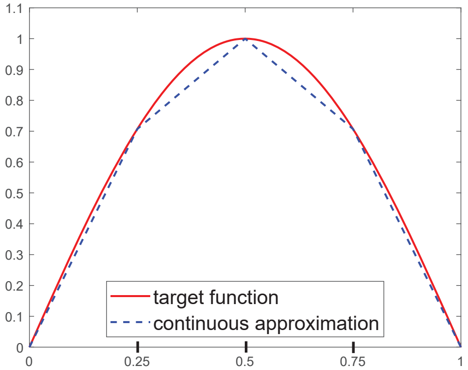
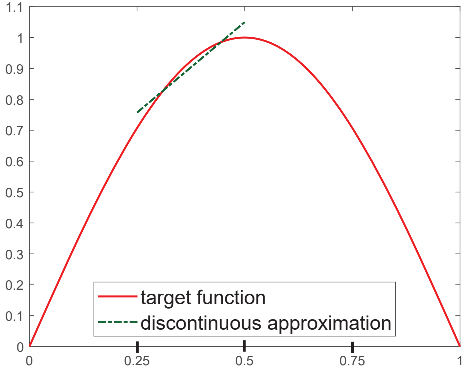


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

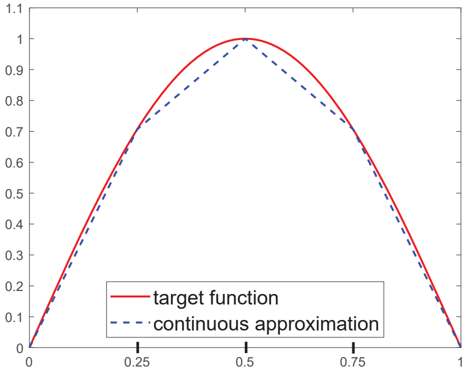


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

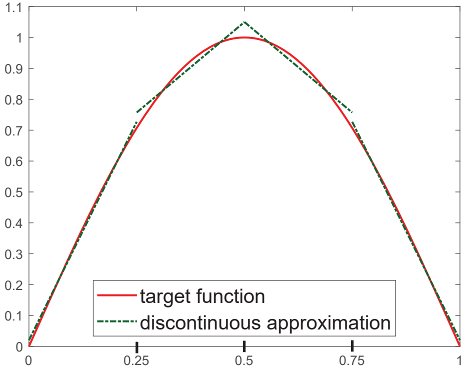


Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: **1D**

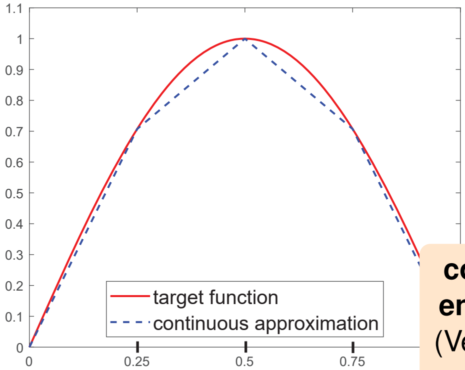


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem



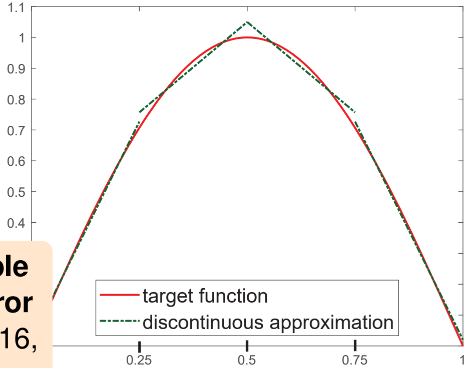
Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of global- and local-best approximations in $H_0^1(\Omega)$: 1D



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

comparable energy error (Veeser 2016, p-robustness V. 2024)



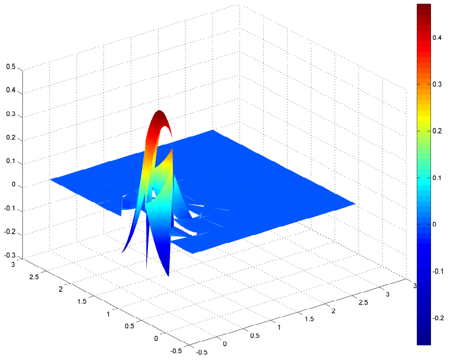
approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Outline

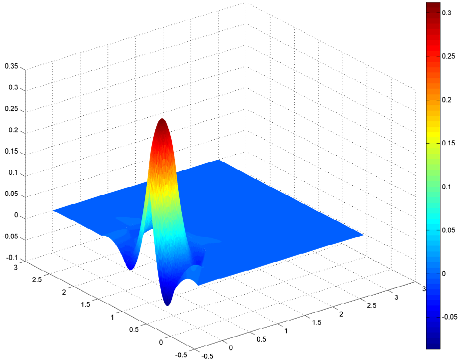
1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – unconstrained local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - p-stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
5. A posteriori estimates
   - Guaranteed upper bound and polynomial-degree-robust local efficiency
   - Numerical illustration
6. Tools (hp-optimality, p-robustness)
   - Polynomial extension operators
   - p-stable decompositions
7. Conclusions and outlook

Outline

Potential reconstruction



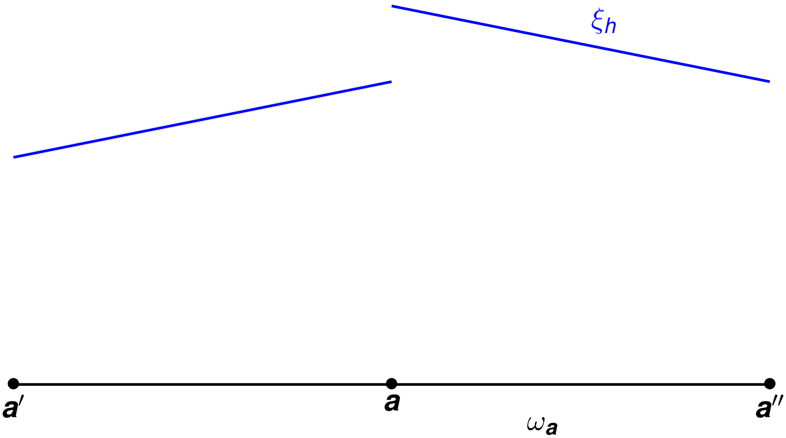
Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Potential reconstruction in 1D, $p = 1$, $p' = 2$



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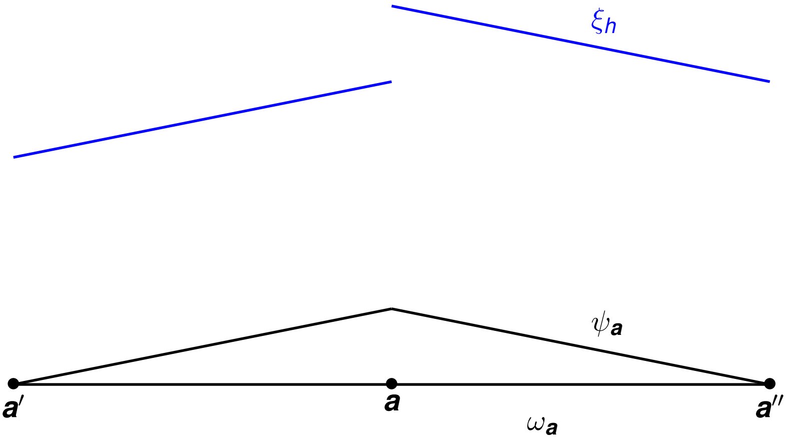

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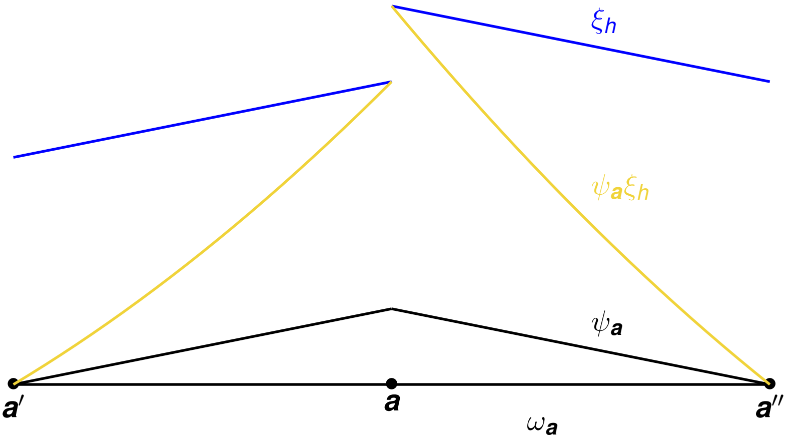

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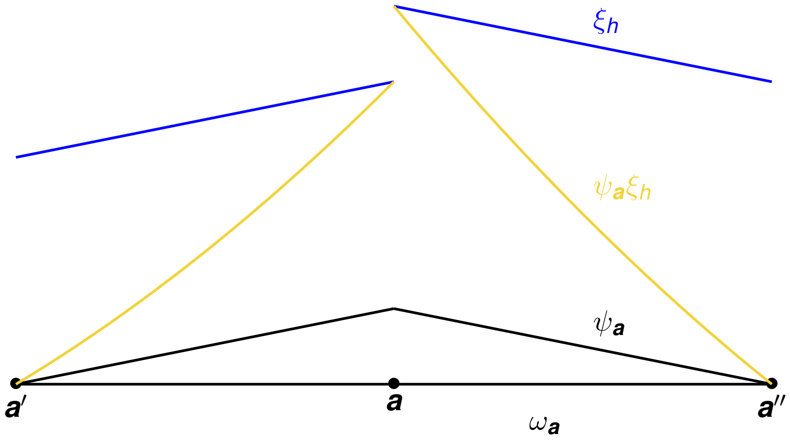

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

Definition (Construction of s_h Ern & V. (2015), ≈ Carstensen and Merdon (2013)**)**

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathcal{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h (\psi_{\boldsymbol{a}} \xi_h) - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}.$$

Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

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Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathcal{P}_{p'}(\mathcal{T}_h) \cap H_0^1(\Omega)$
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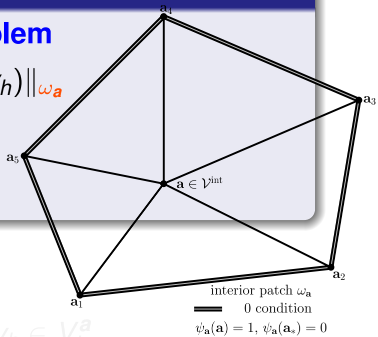

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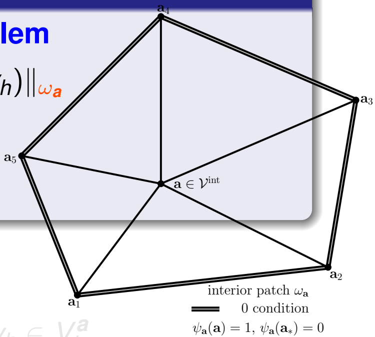

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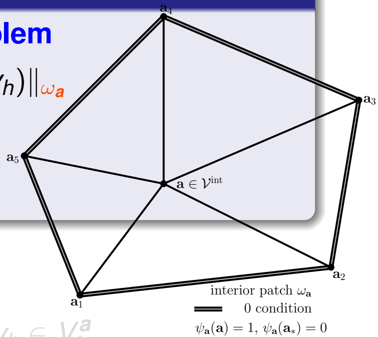

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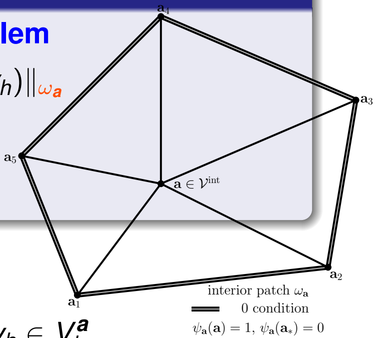

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- $p' = p + 1$ or $p' = p$

Potential reconstruction: datum $\xi_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$

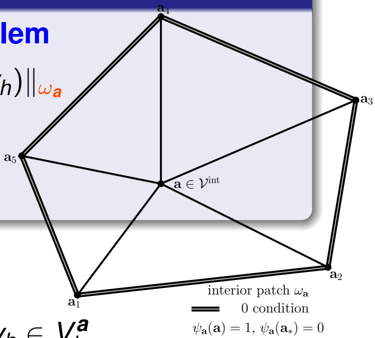
Definition (Construction of s_h Ern & V. (2015), $\approx$ Carstensen and Merdon (2013))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local minimization problem**

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in V_h^{\boldsymbol{a}} := \mathcal{P}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla_h(I_{p'}(\psi_{\boldsymbol{a}} \xi_h) - v_h)\|_{\omega_{\boldsymbol{a}}}$$

and combine

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}}.$$



Equivalent form: conforming FEs

Find $s_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that

$$(\nabla s_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (\nabla_h I_{p'}(\psi_{\boldsymbol{a}} \xi_h), \nabla v_h)_{\omega_{\boldsymbol{a}}} \qquad \forall v_h \in V_h^{\boldsymbol{a}}.$$

Key points

- localization to patches $\mathcal{T}_{\boldsymbol{a}}$
- cut-off by hat basis functions $\psi_{\boldsymbol{a}}$
- projection of the discontinuous $\psi_{\boldsymbol{a}} \xi_h$ to conforming space
- homogeneous Dirichlet BC on $\partial\omega_{\boldsymbol{a}}$: $s_h \in \mathcal{P}_{p'}(\mathcal{T}_h) \cap H_0^1(\Omega)$
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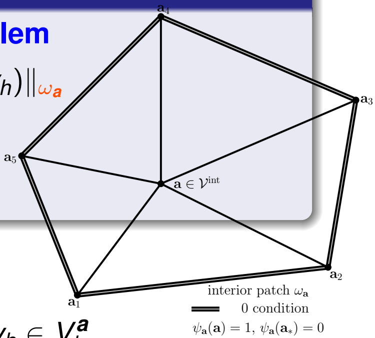

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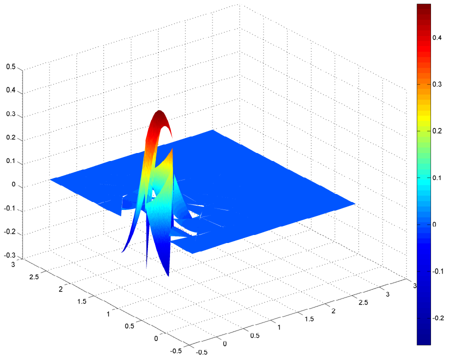
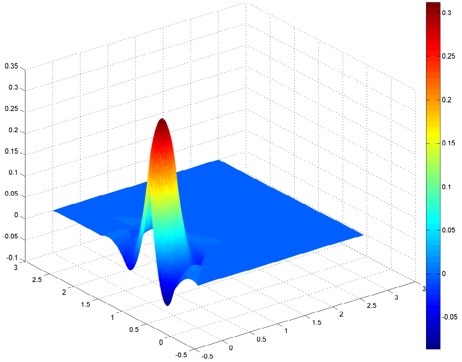


Potential ξ_h



Potential reconstruction s_h

$$\xi_h \in \mathcal{P}_p(\mathcal{T}_h) \to s_h \in \underbrace{\mathcal{P}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap H_0^1(\Omega)$$

Stability of the potential reconstruction

Theorem (Local stability Ern & V. (2015, 2020), using ▸ Tools**)**

There holds

$$\min_{v_h \in \mathcal{P}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v_h)\|_{\omega_{\mathbf{a}}} \lesssim \min_{v \in H^1_0(\omega_{\mathbf{a}})} \|\nabla_h (I_{p'}(\psi_{\mathbf{a}} \xi_h) - v)\|_{\omega_{\mathbf{a}}}.$$

Stability of the potential reconstruction

Corollary (Global stability; $p' = p + 1$)

Up to a jump term, s_h is closer to ξ_h than any $u \in H_0^1(\Omega)$:

$$\|\nabla_h(\xi_h - s_h)\| \lesssim \|\nabla_h(\xi_h - u)\| + \left\{ \sum_{F\in\mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![\xi_h]\!]\|_F^2 \right\}^{1/2}.$$

s_h so good that no $u \in H_0^1(\Omega)$ can do better

Stability of the potential reconstruction

Corollary (Global stability; $p' = p$ after a p-robust correction)

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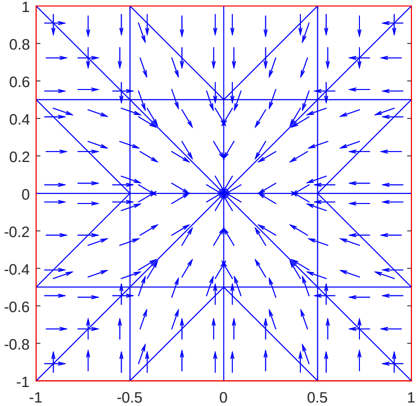
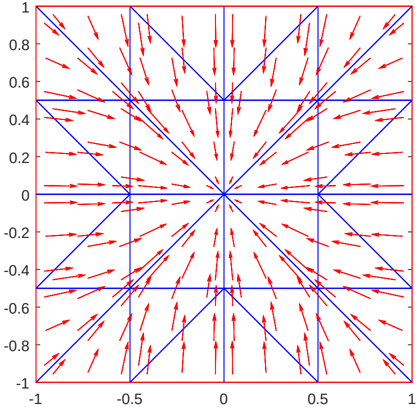
Outline

Equilibrated flux reconstruction



Flux ξ_h



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{\sigma}_h \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega), \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Flux reconstruction: $\xi_h \in \mathcal{RT}_p(\mathcal{T}_h)$, $p \geq 0$, $f \in L^2(\Omega)$

Assumption (Orthogonality wrt hat functions)

There holds

$$(f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (\xi_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

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$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}} := \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = \Pi_{p'}(f\psi_{\mathbf{a}} + \xi_h \cdot \nabla \psi_{\mathbf{a}})}} \|(\psi_{\mathbf{a}} \xi_h) - \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

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$$\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}.$$

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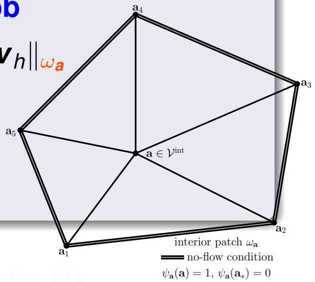

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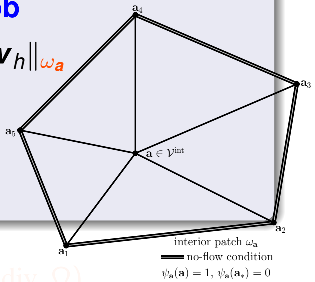

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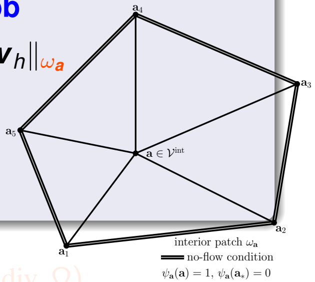

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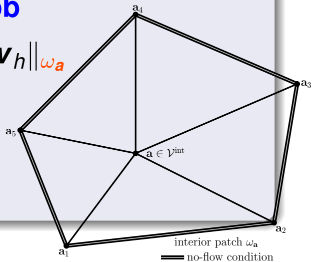

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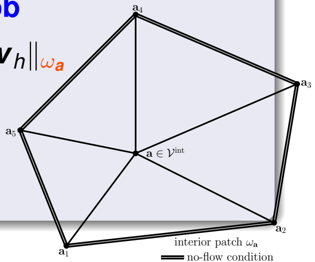

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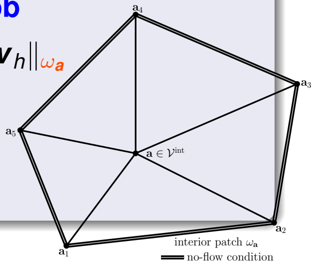

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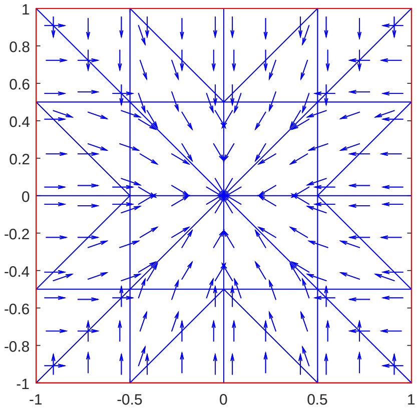
Equilibrated flux reconstruction



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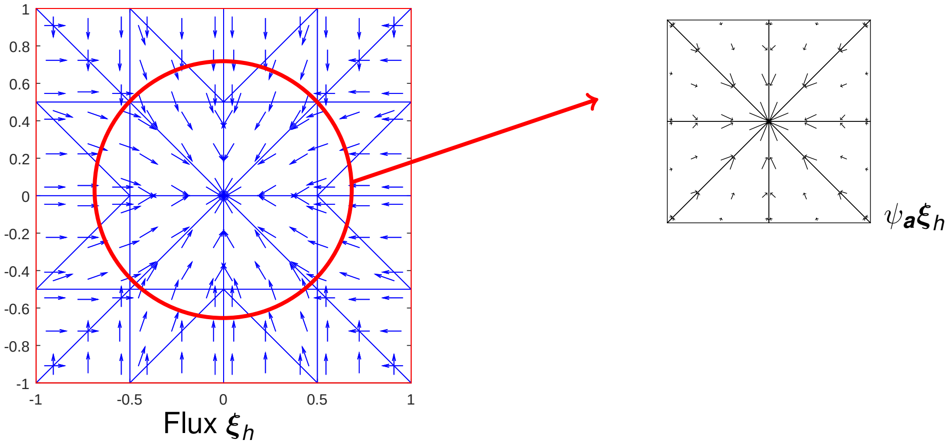

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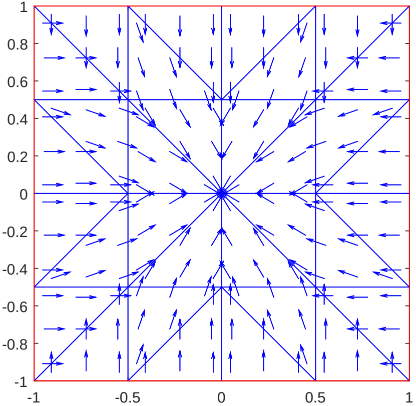

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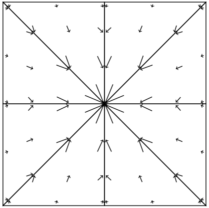
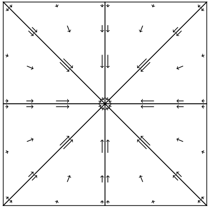


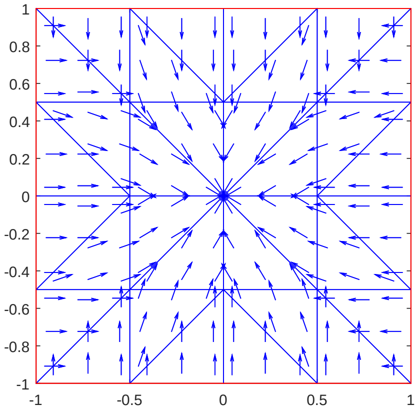
$\psi_{\mathbf{a}} \xi_h$



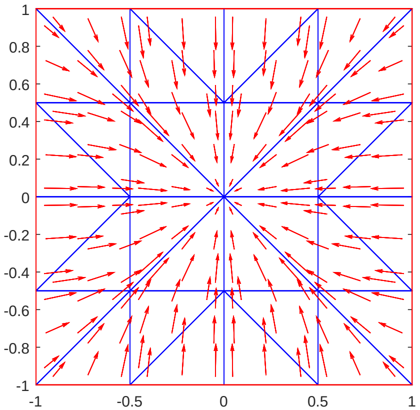
$\boldsymbol{\sigma}_h^{\mathbf{a}}$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in L^2(\Omega)}_{(f,\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}+(\xi_h,\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}}=0\ \forall \mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \qquad \boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{V}_h^{\mathbf{a}} := \mathcal{RT}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\mathbf{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_{p'}(f\psi_{\mathbf{a}} + \xi_h\cdot\nabla\psi_{\mathbf{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\mathbf{a}}\xi_h) - \boldsymbol{v}_h\|_{\omega_{\mathbf{a}}}$$

Equilibrated flux reconstruction



Flux ξ_h



Equilibrated flux reconstruction $\boldsymbol{\sigma}_h$

$$\underbrace{\xi_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in L^2(\Omega)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\xi_h,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\rm int}} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \underbrace{\mathcal{RT}_{p'}(\mathcal{T}_h)}_{p'=p \text{ or } p'=p+1} \cap \boldsymbol{H}(\mathrm{div},\Omega),\ \nabla\cdot\boldsymbol{\sigma}_h = \Pi_{p'} f$$

Stability of the flux reconstruction

Theorem (Local stability Braess, Pillwein, Schöberl (2009; 2D), Ern & V. (2020; 3D), using ▸ Tools**)**

There holds

$$\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \Pi_{p'}(f\psi_{\boldsymbol{a}} + \boldsymbol{\xi}_h \cdot \nabla \psi_{\boldsymbol{a}})}} \|\boldsymbol{I}_{p'}(\psi_{\boldsymbol{a}} \boldsymbol{\xi}_h) - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Stability of the flux reconstruction

Corollary (Global stability; $p' = p + 1$)

$\boldsymbol{\sigma}_h$ *is closer to* ξ_h *than any* $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *such that* $\nabla \cdot \boldsymbol{\sigma} = f$*:*

$$\|\xi_h - \boldsymbol{\sigma}_h\| \lesssim \|\xi_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\sigma} = f$ can do better

Stability of the flux reconstruction

Corollary (Global stability; $p' = p$ after a p-robust correction)

$\boldsymbol{\sigma}_h$ is closer to $\boldsymbol{\xi}_h$ than any $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\nabla \cdot \boldsymbol{\sigma} = f$:

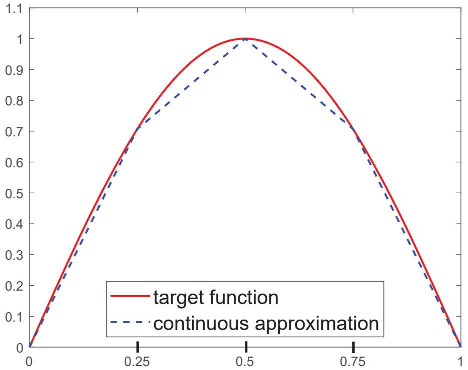
$$\|\boldsymbol{\xi}_h - \boldsymbol{\sigma}_h\| \lesssim \|\boldsymbol{\xi}_h - \boldsymbol{\sigma}\| + \left\{ \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|f - \Pi_p f\|_K^2 \right\}^{1/2}.$$

$\boldsymbol{\sigma}_h$ so good that no $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\sigma} = f$ can do better

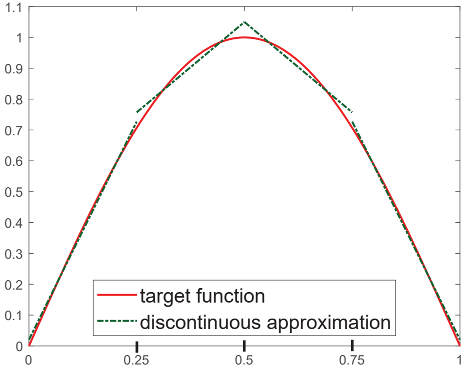
Outline

Outline

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$: 1D

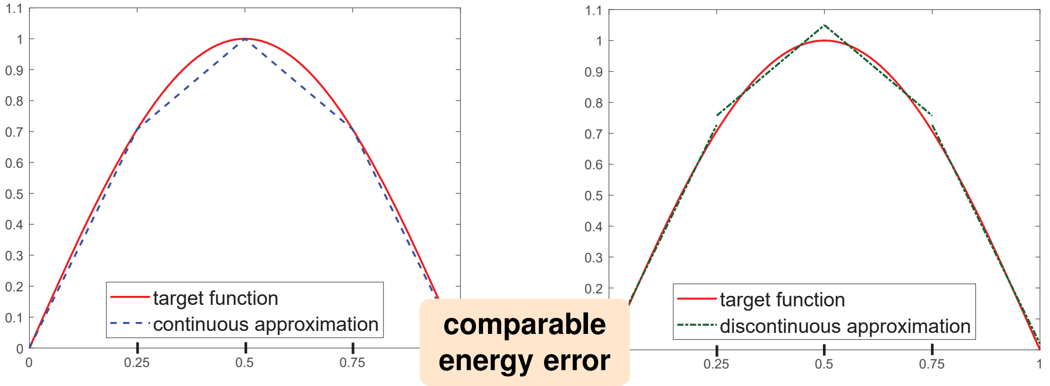


Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$, **global** problem



Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: 1D



Best approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$, **global** problem

Best approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$, **local** problems

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

$$\textit{bigger} \approx_p \textit{smaller}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

$$\min_{\textit{smaller space}} \approx_p \min_{\textit{bigger space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016))

$$\min_{\textit{CG space}} \approx_p \min_{\textit{DG space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), V. (2024))

Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

- $\approx_p$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$, and polynomial degree p
- proof taking $\xi_h|_K := \arg\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K$ with $(\xi_h, 1)_K = (u, 1)_K$ for all $K \in \mathcal{T}_h$, applying with $p' = p$, and using its

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Theorem (Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), V. (2024))

Let $u \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint} \\ \text{CG space (much smaller)}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint} \\ \text{DG space (much bigger)}}}.$$

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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), V. (2024))

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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

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- $\approx_{\not p}$: up to a generic constant that only depends on space dimension d and shape-regularity of the mesh $\mathcal{T}_h$, ~~and polynomial degree p~~
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Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Theorem (Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016), V. (2024))

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Optimal a priori error estimate

Theorem (Local hp-optimal approximation under minimal Sobolev regularity)

Let $v \in H_0^1(\Omega)$ with

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$.

- $P_h^p : H_0^1(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$: a locally defined projector
- $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Optimal a priori error estimate

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $v \in H_0^1(\Omega)$ with

$$v|_K \in H^{s_K}(K) \qquad \forall K \in \mathcal{T}_h$$

for $s_K \geq 1$. Then

$$\|\nabla(v - P_h^p v)\|_K^2 \leq C(\kappa_{\mathcal{T}_h}, \kappa_p, d, s) \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(\underline{p}_K, s_L - 1)}}{\underline{p}_K^{s_L - 1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h.$$

- $P_h^p : H_0^1(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$: a locally defined projector
- $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

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- $\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\}$: smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$

Outline

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

bigger $\approx_p$ *smaller*

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

$$\min_{\textit{smaller space with constraints}} \approx_p \min_{\textit{bigger space without constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021))

$$\min_{\text{MFE space with constraints}} \approx_p \min_{\text{broken MFE space without constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021) Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021) Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

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$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p
- proof using ... with $p' = p$ & ...

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2}$$

global-best on Ω
normal trace-continuity constraint
divergence constraint
MFE space (much smaller)

$$\approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p \nabla\cdot\boldsymbol{v}\|_K^2 \right]}.$$

local-best on each K
no normal trace-continuity constraint
no divergence constraint
broken MFE space (much bigger)

- $\approx_p$: only depends on d, shape-regularity of $\mathcal{T}_h$, and p
- proof using flux reconstruction with $p' = p$ & $\boldsymbol{H}(\mathrm{div})$ stability

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024)**)**

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_p\nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2}\|\nabla\cdot\boldsymbol{v} - \Pi_p\nabla\cdot\boldsymbol{v}\|_K^2\right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_{\not p}$: only depends on d, shape-regularity of $\mathcal{T}_h$, ~~and p~~ p
- proof using ▸ flux reconstruction with $p' = p$ & ▸ $\boldsymbol{H}(\mathrm{div})$ stability

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{div})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{div})$, Ern, Gudi, Smears, & V. (2021), Demkowicz & V. (2024))

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p(\nabla\cdot\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p\nabla\cdot\boldsymbol{v}\|_K^2}_{\substack{\text{global-best on } \Omega \\ \text{normal trace-continuity constraint} \\ \text{divergence constraint} \\ \text{MFE space (much smaller)}}}$$

$$\approx \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{RT}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \frac{h_K^2}{(p+1)^2} \|\nabla\cdot\boldsymbol{v} - \Pi_p\nabla\cdot\boldsymbol{v}\|_K^2 \right]}_{\substack{\text{local-best on each } K \\ \text{no normal trace-continuity constraint} \\ \text{no divergence constraint} \\ \text{broken MFE space (much bigger)}}}.$$

- $\approx_{\not p}$: only depends on d, shape-regularity of $\mathcal{T}_h$, ~~and~~ p
- proof using ‣ flux reconstruction with $p' = p$ & ‣ $\boldsymbol{H}(\mathrm{div})$ stability

Outline

Commuting de Rham diagram with operator $\boldsymbol{P}_h^{p,\mathrm{div}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
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Properties of $\boldsymbol{P}_h^{p,\mathrm{div}}$

1. is defined over the **entire $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$** (**minimal regularity**)
2. is defined **locally** (in neighborhood of mesh elements)
3. is defined simply (starting from the **elementwise L^2 orthogonal projection**)
4. has optimal *hp* approximation properties, that of **elementwise div-unconstrained L^2-orthogonal projector** (global–local equivalence)
5. is **stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfies the **commuting properties** expressed by the arrows
7. is **projector**, i.e., leaves intact piecewise polynomials

p-table local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not L^2-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on part of $\partial\Omega$
- Arnold and Guzmán (2021): L^2-stable
- Ern, Gudi, Smears, and V. (2022): all the properties in $\boldsymbol{H}(\mathrm{div})$ but not p-robust
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Outline

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Laplace model problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

Theorem (A guaranteed a posteriori error estimate Prager and Synge (1947), Ladevèze (1975), Dari, Durán, Padra, & Vampa (1996), Ainsworth (2005), Kim (2007), V. (2007), ...**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in \mathcal{P}_p(\mathcal{T}_h)$, $p \geq 1$, *be arbitrary subject to*

$$(\nabla_h u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}};$$

- $\xi_h := u_h$: $s_h \in \mathcal{P}_{p+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$ potential reconstruction;
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Then

$$\|\nabla_h(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} \Big(\underbrace{\|\nabla_h u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \Pi_p f\|_K}_{\text{equilibrium/data osc.}} \Big)^2 + \sum_{K \in \mathcal{T}_h} \underbrace{\|\nabla_h(u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency; $f \in \mathcal{P}_{p-1}(\mathcal{T}_h)$ for simplicity Braess, Pillwein, and Schöberl (2009), Ern & V. (2015, 2020)**)**

Let $u \in H_0^1(\Omega)$ be the weak solution. Then

$$\|\nabla_h(u_h - s_h)\| \lesssim \|\nabla_h(u - u_h)\| + \left\{ \sum_{F \in \mathcal{F}_h} h_F^{-1} \|\Pi_0^F [\![u_h]\!]\|_F^2 \right\}^{1/2},$$

$$\|\nabla_h u_h + \boldsymbol{\sigma}_h\| \lesssim \|\nabla_h(u - u_h)\|.$$

Remarks

- immediate consequence of H^1 stability and $\boldsymbol{H}(\mathrm{div})$ stability with $p' = p + 1$
- p-robustness
- local efficiency on patches
- maximal overestimation guaranteed (computable bounds on the constants)

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Outline

How **large** is the error? (numerical simulation, known solution)

$h \approx 1/\lvert\mathcal{T}_\ell\rvert^{\frac{1}{2}}$	p	relative error estimate $\frac{\eta(u_h)}{\lVert\nabla u_h\rVert}$	relative error $\frac{\lVert\nabla(u-u_h)\rVert}{\lVert\nabla u_h\rVert}$	effectivity index $\frac{\eta(u_h)}{\lVert\nabla(u-u_h)\rVert}$
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

How **large** is the error? (numerical simulation, known solution)

$h \approx 1/\lvert\mathcal{T}_\ell\rvert^{\frac{1}{2}}$	p	relative error estimate $\frac{\eta(u_h)}{\lVert\nabla u_h\rVert}$	relative error $\frac{\lVert\nabla(u-u_h)\rVert}{\lVert\nabla u_h\rVert}$	effectivity index $\frac{\eta(u_h)}{\lVert\nabla(u-u_h)\rVert}$
h_0	1	$2.8 \times 10^{1}\%$	$2.4 \times 10^{1}\%$	**1.17**
$\approx h_0/2$		$1.4 \times 10^{1}\%$	$1.3 \times 10^{1}\%$	**1.09**
$\approx h_0/4$		7.0%	6.6%	**1.06**
$\approx h_0/8$		3.3%	3.1%	**1.04**
$\approx h_0/2$	2	$9.5 \times 10^{-1}\%$	$9.2 \times 10^{-1}\%$	**1.04**
$\approx h_0/4$	3	$5.9 \times 10^{-3}\%$	$5.9 \times 10^{-3}\%$	**1.01**
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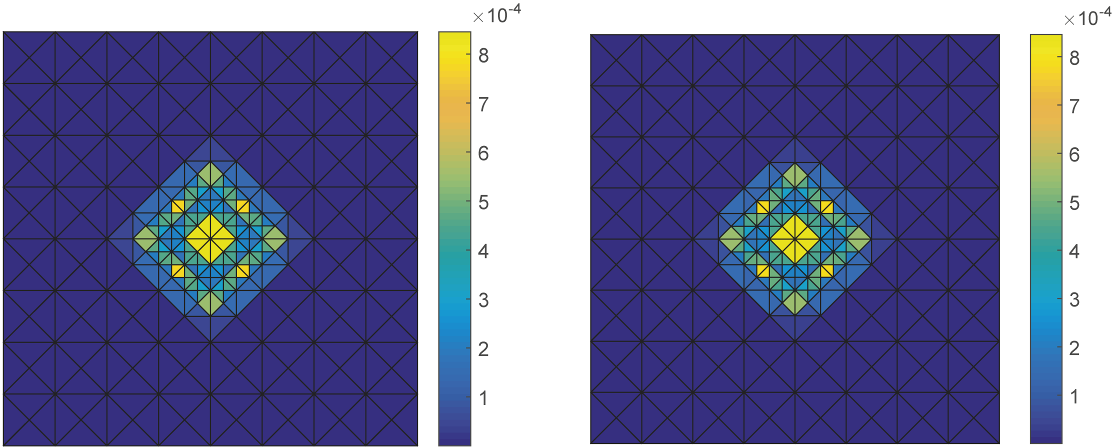
A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

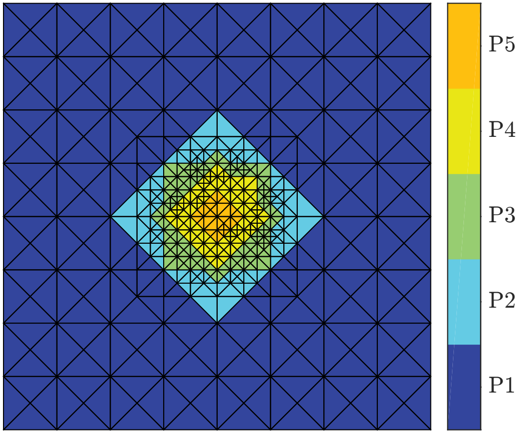
Where (in space) is the error **localized**?



Estimated error distribution $\eta_K(u_h)$ Exact error distribution $\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**smooth** solution)



Mesh $\mathcal{T}_\ell$ and pol. degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

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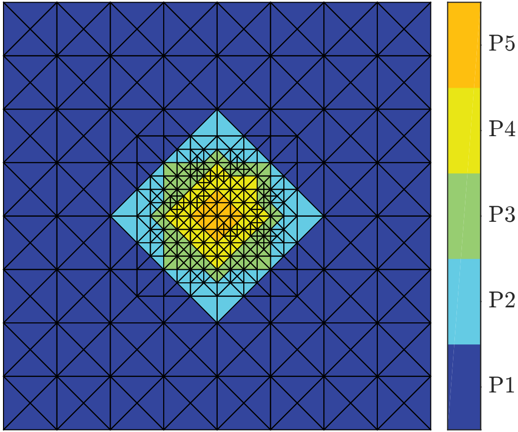

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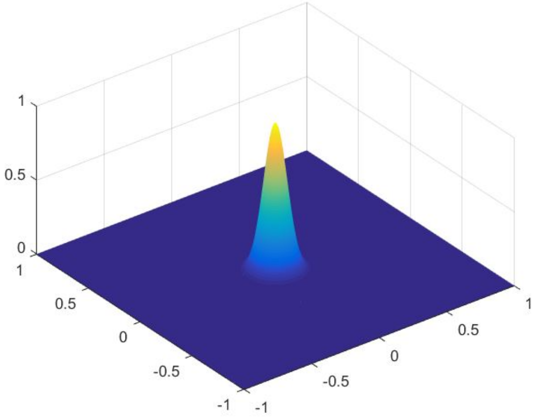
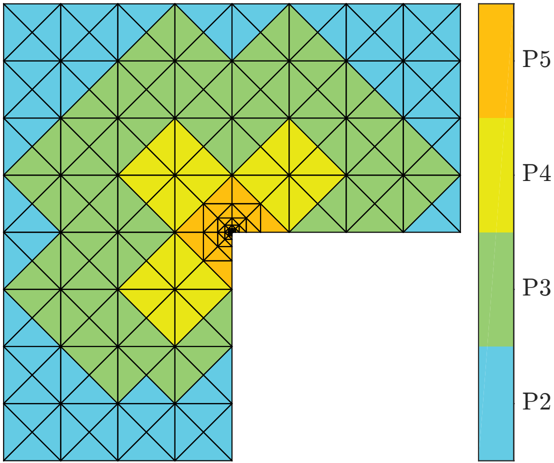


Exact solution

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

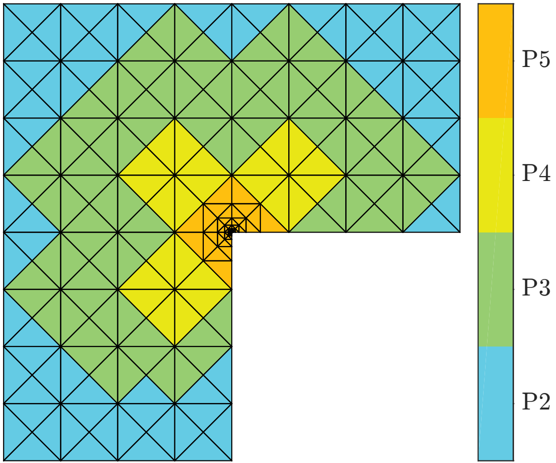
Can we decrease the error efficiently? *hp* adaptivity, (**singular** solution)



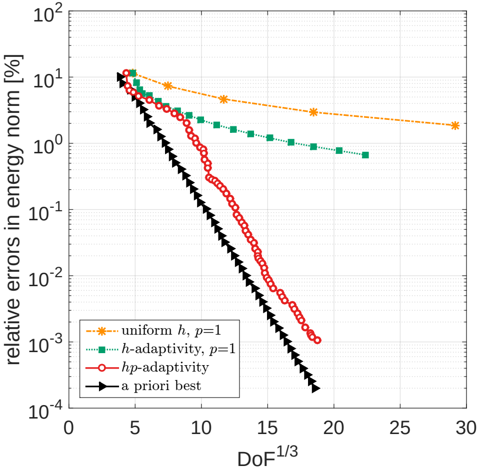
Mesh $\mathcal{T}_\ell$ and polynomial degrees p_K

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**singular** solution)



Mesh $\mathcal{T}_\ell$ and polynomial degrees p_K



Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Outline

Outline

1. Introduction
2. Potential reconstruction
3. Flux reconstruction
4. A priori estimates
   - Global-best – local-best equivalence in H^1
   - Constrained global-best – unconstrained local-best equivalence in $\boldsymbol{H}(\mathrm{div})$
   - p-stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$
5. A posteriori estimates
   - Guaranteed upper bound and polynomial-degree-robust local efficiency
   - Numerical illustration
6. Tools (hp-optimality, p-robustness)
   - Polynomial extension operators
   - p-stable decompositions
7. Conclusions and outlook

Potentials: one element

Lemma (H^1 polynomial extension on a tetrahedron Babuška, Suri (1987; 2D), Muñoz-Sola (1997), Demkowicz, Gopalakrishnan, & Schöberl (2009)**)**

Let $p \geq 1$, $K \in \mathcal{T}_h$, and $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then

$$\min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}}.$$

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Context

$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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$$\|\nabla \zeta_{h,K}\|_K \overset{FEs}{=} \min_{\substack{v_h \in \mathcal{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \lesssim \underbrace{\min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K}_{\|r\|_{H^{1/2}(\partial K)}} = \|\nabla \zeta_K\|_K.$$

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Potentials: vertex patch

Theorem (Broken H^1 polynomial extension on a vertex patch Ern & V. (2015, 2020))

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}_h^{\rm int}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}^{\rm int})$. Suppose the compatibility

$$r_F|_{F \cap \partial\omega_{\boldsymbol{a}}} = 0 \qquad \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int},$$
$$\sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e = 0 \qquad \forall e \in \mathcal{E}_{\boldsymbol{a}}.$$

Then

$$\min_{\substack{v_h \in \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}}) \\ v_h = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext} \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v_h\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{v \in H^1(\mathcal{T}_{\boldsymbol{a}}) \\ v = 0\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm ext} \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\rm int}}} \|\nabla_h v\|_{\omega_{\boldsymbol{a}}}.$$

Fluxes: one element

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron Costabel & Mc-Intosh (2010); Ainsworth & Demkowicz (2009; 2D), Demkowicz, Gopalakrishnan, & Schöberl (2012); Ern & V. (2020)**)**

Let $p \geq 0$, $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathcal{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathcal{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v}_h = r_K}} \|\boldsymbol{v}_h\|_K \lesssim \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, K) \\ \boldsymbol{v} \cdot \boldsymbol{n}_K = r_F \ \forall F \in \mathcal{F}_K^{\mathrm{N}} \\ \nabla \cdot \boldsymbol{v} = r_K}} \|\boldsymbol{v}\|_K \quad .$$

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$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\varphi_K := -\nabla \zeta_K$.

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Context

$$\begin{aligned} -\Delta \zeta_K &= r_K && \text{in } K, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{N}}, \\ \zeta_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{N}}. \end{aligned}$$

Set $\boldsymbol{\varphi}_K := -\nabla \zeta_K$.

Fluxes: vertex patch

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension on a vertex patch Braess, Pillwein, & Schöberl (2009; 2D), Ern & V. (2020; 3D)**)**

For $p \geq 0$ and $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, let $r \in \mathcal{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then

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Outline

$\boldsymbol{H}(\mathrm{div})$ stable decomposition

Theorem ($\boldsymbol{H}(\mathrm{div})$ stable decomposition in 2D; in extension of Schöberl, Melenk, Pechstein, & Zaglmayr (2008))

Let $d = 2$ and let $\overline{\Omega}$ be contractible. Let

$$\boldsymbol{\delta}_p \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p = 0, \qquad \text{div-free}$$

$$(\boldsymbol{\delta}_p, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^d, \ \forall K \in \mathcal{T}_h. \qquad \text{vanishing means}$$

Then there exists a decomposition of $\boldsymbol{\delta}_p$ as

$$\boldsymbol{\delta}_p = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_p^{\boldsymbol{a}}, \qquad \text{decomposition}$$

where

$\boldsymbol{\delta}_p^{\boldsymbol{a}}$ are supported on the vertex patch subdomains $\omega_{\boldsymbol{a}}$, linearly depend on $\boldsymbol{\delta}_p$ on the extended vertex patch subdomains $\widetilde{\omega}_{\boldsymbol{a}}$,

and satisfy

$$\boldsymbol{\delta}_p^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_p^{\boldsymbol{a}} = 0, \qquad \text{local}$$

$$\|\boldsymbol{\delta}_p^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\delta}_p\|_{\widetilde{\omega}_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \qquad p\text{-stable}$$

Outline

Conclusions and outlook

Conclusions

- p-stable local commuting projectors
- p-robust global-best – local-best equivalence in H^1
- p-robust global-best – local-best equivalence in $\boldsymbol{H}(\mathrm{div})$, removing constraints
- optimal hp localized a priori error estimates under minimal elementwise regularity
- p-robust a posteriori error estimates (unified framework for all classical numerical schemes)
- extensions to nonmatching meshes (robust wrt number of hanging nodes), mixed parallelepipedal–simplicial meshes, varying polynomial degree, general BCs, H^{-1} source terms, splines and IGA, and others carried out

Ongoing work

- extensions to other settings

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Thank you for your attention!

Outline

- Potential and flux reconstructions

8 Application to IGA
- The Poisson model problem and its IGA approximation
- Equilibration in IGA: a first idea
- Equilibration: breaking the large patch problems

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw polynomial ▸ potential reconstruction

Potential and flux reconstructions

Potential reconstruction

- discontinuous pw polynomial → continuous pw polynomial ‣ potential reconstruction
- a posteriori analysis of mixed and nonconforming FEs:

  estimate ≈ error

- a priori analysis of conforming FEs:

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  approximation continuous pw pols ≈ discontinuous pw pols

Equilibrated flux reconstruction

- pw vector-valued polynomial with discontinuous normal trace and no equilibrium → continuous normal trace & equilibrium ‣ flux reconstruction
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  global-best–local-best equivalence in $\mathbf{H}(\mathrm{div})$ Ern, Gudi, Smears, V. (2022)

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8 Application to IGA

- The Poisson model problem and its IGA approximation
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The Poisson model problem and its Galerkin approximation

The Poisson problem

Find $u : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, *such that*

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Weak formulation

Find $u \in H_0^1(\Omega)$ *such that*

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \quad \text{for all } v \in H_0^1(\Omega).$$

Galerkin approximation

Find $u_h \in V_h \subset H_0^1(\Omega)$ *such that*

$$V_h = Q^p(\mathcal{T}_h) \cap C^{p-m}(\Omega)$$

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Partition of unity, $V_h = \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega)$

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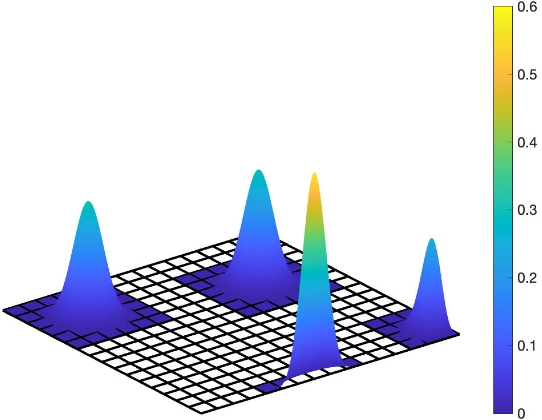

Spline basis functions $\psi_{\boldsymbol{a}} \in \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega) \subset V_h$

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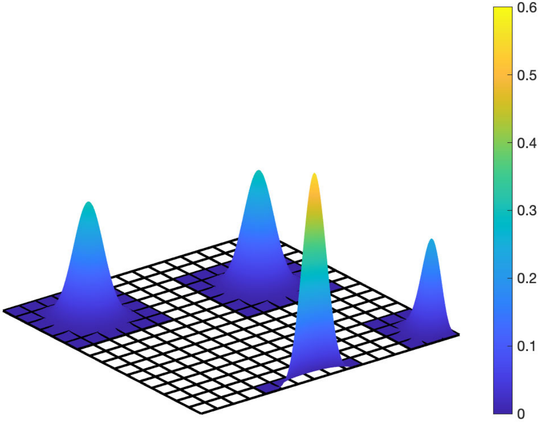
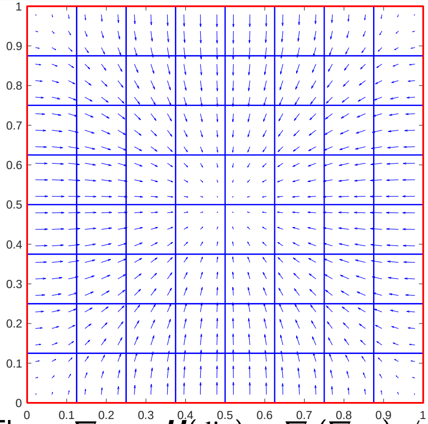


$$\sum_{\mathbf{a} \in \mathcal{V}_h} \psi_{\mathbf{a}} = 1$$

Spline basis functions $\psi_{\mathbf{a}} \in \mathcal{Q}^p(\mathcal{T}_h) \cap C^{p-1}(\Omega) \subset V_h$

Equilibrated flux reconstruction in IGA (a first idea)



Flux $-\nabla u_h \in \boldsymbol{H}(\mathrm{div})$, $-\nabla{\cdot}(\nabla u_h) \neq f$

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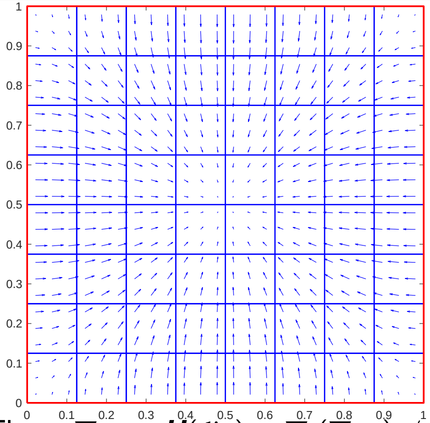

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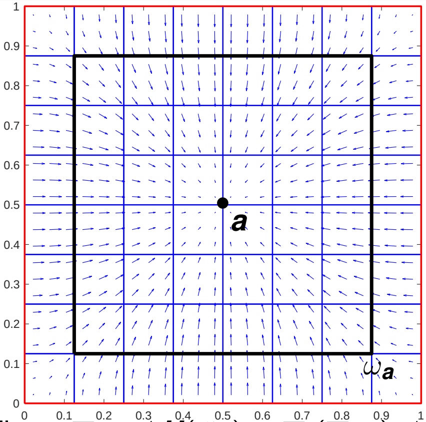

Flux $-\nabla u_h \notin \boldsymbol{H}(\mathrm{div})$, $-\nabla\cdot(\nabla u_h) \neq f$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h),\, f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)}_{(f,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}}$$

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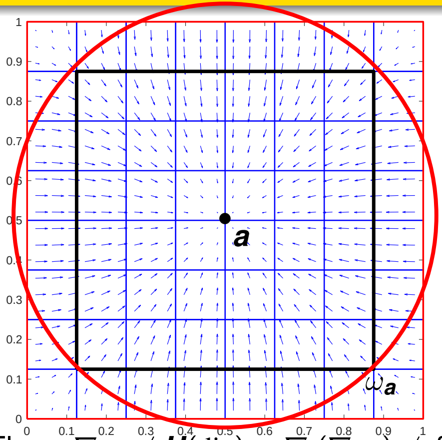

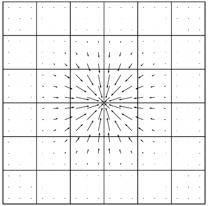

$-\psi_{a}\nabla u_h$

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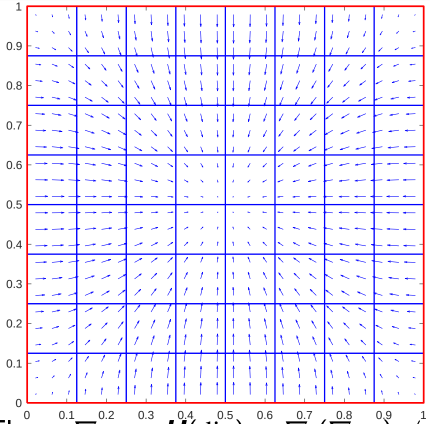

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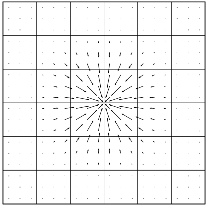

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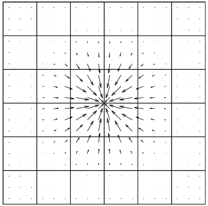

$\boldsymbol{\sigma}_h^{\boldsymbol{a}}$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)} \qquad \boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{2p}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = f\psi_{\boldsymbol{a}} - \nabla u_h \cdot \nabla\psi_{\boldsymbol{a}}}} \|\psi_{\boldsymbol{a}} \nabla u_h + \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}$$

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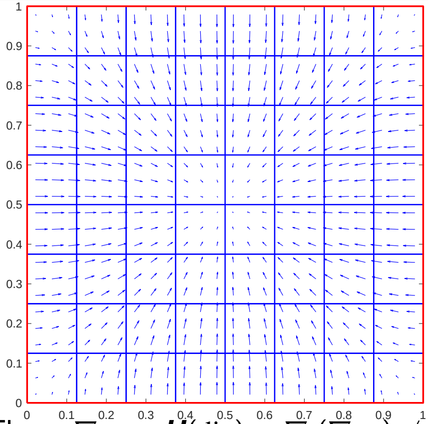

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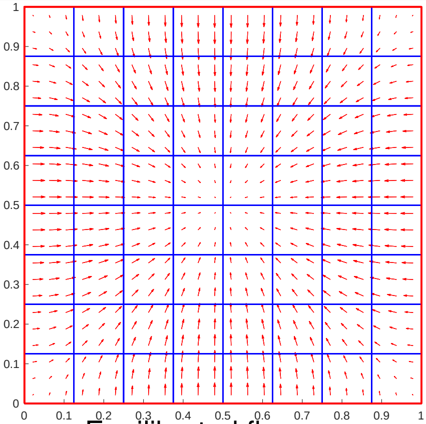

Equilibrated flux $\boldsymbol{\sigma}_h$

$$\underbrace{\nabla u_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{Q}^{p-1}(\mathcal{T}_h)} \to \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \mathcal{RT}_{2p}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{\sigma}_h = f$$

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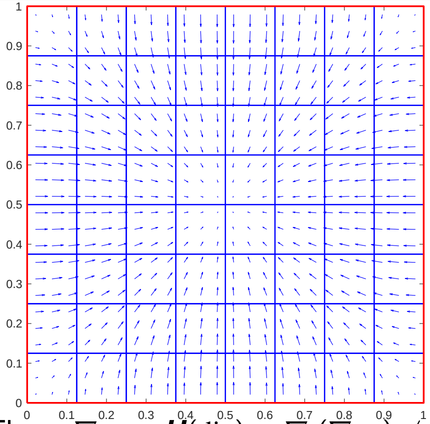

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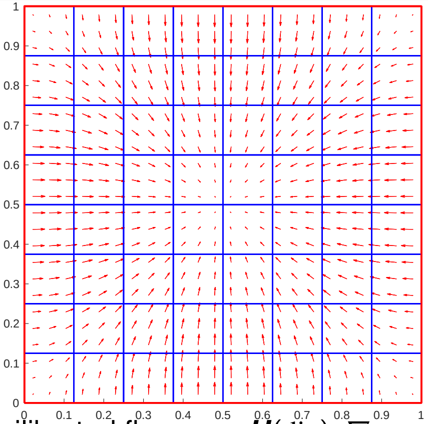

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Equilibrated flux reconstruction in IGA (a first idea)

Observations

✓ works in principle

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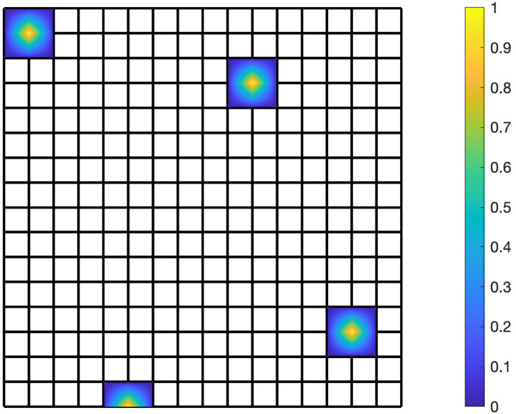
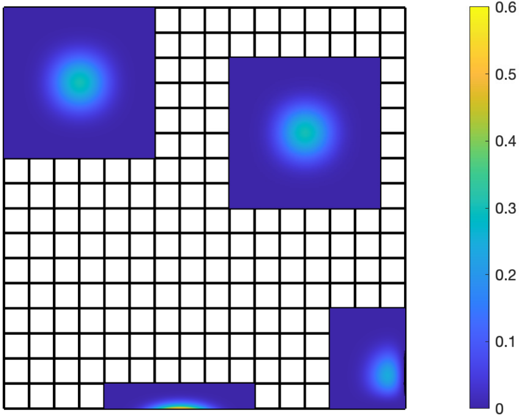
- ✗ p-robustness possibly upon extension of available tools to the large patches

Equilibration patches and partition of unity functions $\psi_{\boldsymbol{a}}$



$\psi_{\boldsymbol{a}} \in \mathcal{Q}^{1}(\mathcal{T}_h) \cap C^{0}(\Omega)$, p arbitrary



$\psi_{\boldsymbol{a}} \in \mathcal{Q}^{p}(\mathcal{T}_h) \cap C^{p-1}(\Omega)$, $p = 5$

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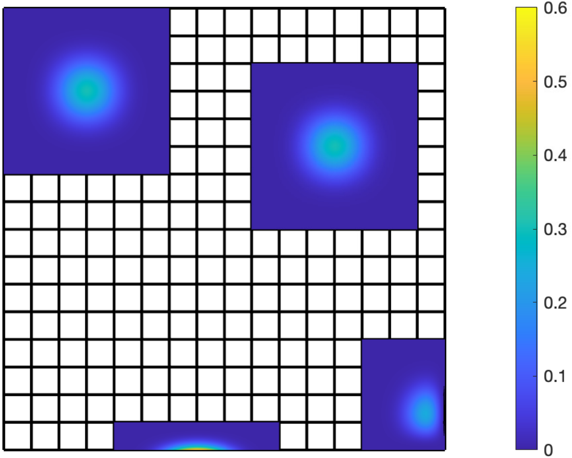
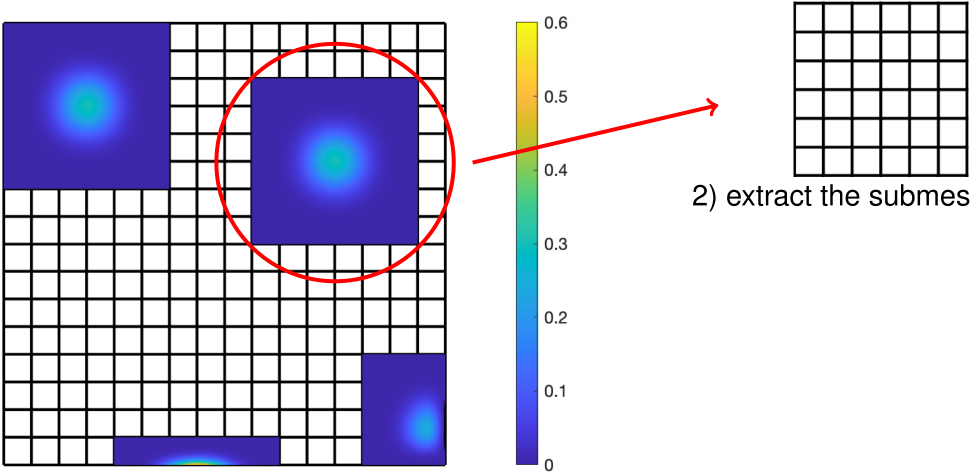
Breaking the large patch problems



1) consider the large patches (supports of $\psi_{\boldsymbol{a}}$)

Breaking the large patch problems



2) extract the submeshes $\mathcal{T}_{\boldsymbol{a}}$

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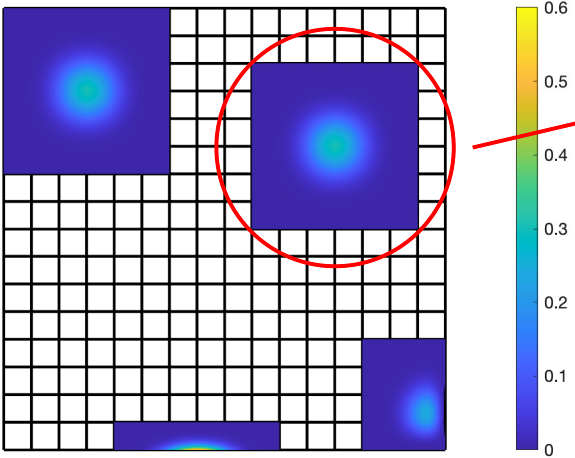

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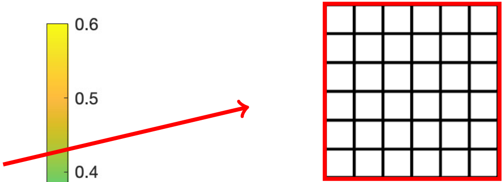
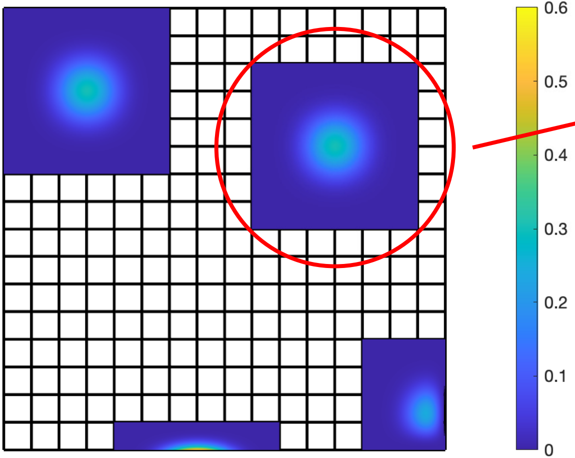


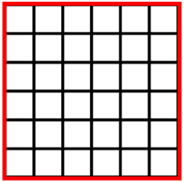
2) extract the submeshes $\mathcal{T}_{\boldsymbol{a}}$

3) **solve** a $\mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})$ **problem** on $\omega_{\boldsymbol{a}}$

Breaking the large patch problems

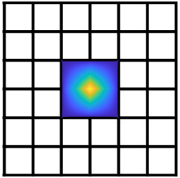


1) consider the large patches (supports of $\psi_{\mathbf{a}}$)



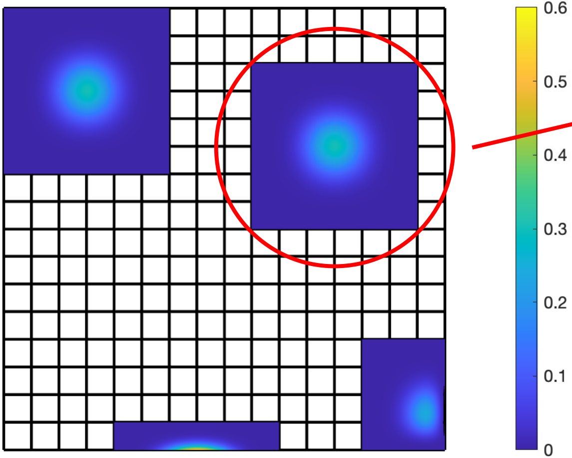
2) extract the submeshes $\mathcal{T}_{\mathbf{a}}$

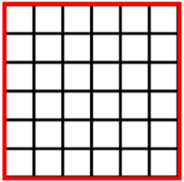
3) **solve** a $\mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})$ **problem** on $\omega_{\mathbf{a}}$



4) consider the hat b.f. $\psi_{\mathbf{a}'} \in \mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})$

Breaking the large patch problems





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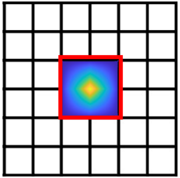
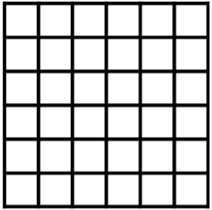


1) consider the large patches (supports of $\psi_{\boldsymbol{a}}$)

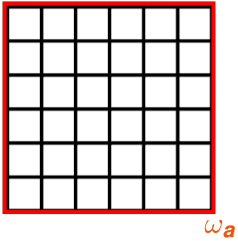
4) consider the hat b.f. $\psi_{\boldsymbol{a}'} \in \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})$

5) **perform equilibration** on $\omega_{\boldsymbol{a}'}$

Breaking the large patch problems

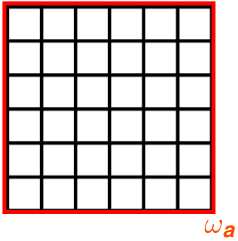


Breaking the large patch problems



3) **solve** the $\boxed{V_h^{\boldsymbol{a}} := \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})}$ **problem**:

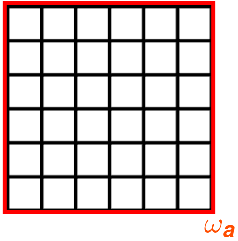
Breaking the large patch problems



3) **solve** the $\boxed{V_h^{\boldsymbol{a}} := \mathcal{Q}^1(\mathcal{T}_{\boldsymbol{a}}) \cap C^0(\omega_{\boldsymbol{a}})}$ **problem**: find $r_h^{\boldsymbol{a}} \in V_h^{\boldsymbol{a}}$ such that, for all $v_h \in V_h^{\boldsymbol{a}}$,

$$\boxed{(\nabla r_h^{\boldsymbol{a}}, \nabla v_h)_{\omega_{\boldsymbol{a}}} = (f, v_h \psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - (\nabla u_h, \nabla(v_h \psi_{\boldsymbol{a}}))_{\omega_{\boldsymbol{a}}}}$$

Breaking the large patch problems



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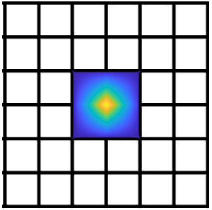
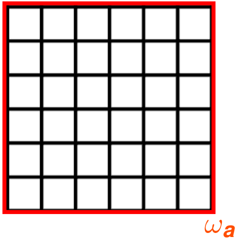
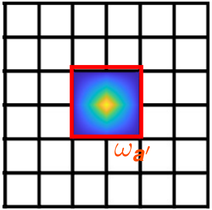


Breaking the large patch problems



3) **solve** the $V_h^{\mathbf{a}} := \mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})$ **problem**: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that, for all $v_h \in V_h^{\mathbf{a}}$,

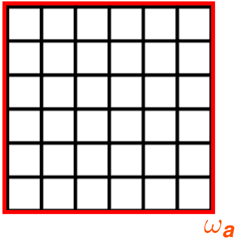
$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (f, v_h \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(v_h \psi_{\mathbf{a}}))_{\omega_{\mathbf{a}}}$$



5) **perform equilibration** on $\omega_{\mathbf{a}'}$:

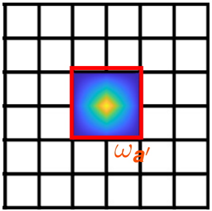
$$\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{a}'} := \arg \min_{\substack{\mathbf{v}_h \in \mathcal{RT}_{2p+1}(\mathcal{T}_{\mathbf{a}}) \cap \mathbf{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \mathbf{v}_h = \Upsilon_{Q_h^{\mathbf{a},\mathbf{a}'}} (f \psi_{\mathbf{a}} \psi_{\mathbf{a}'} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}} \psi_{\mathbf{a}'}) - \nabla r_h^{\mathbf{a}} \cdot \nabla \psi_{\mathbf{a}'})}} \|\psi_{\mathbf{a}'}(\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}}) + \mathbf{v}_h\|_{\omega_{\mathbf{a}'}}$$

Breaking the large patch problems



3) **solve** the $\boxed{V_h^{\mathbf{a}} := \mathcal{Q}^1(\mathcal{T}_{\mathbf{a}}) \cap C^0(\omega_{\mathbf{a}})}$ **problem**: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ such that, for all $v_h \in V_h^{\mathbf{a}}$,

$$\boxed{(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (f, v_h \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla(v_h \psi_{\mathbf{a}}))_{\omega_{\mathbf{a}}}}$$



5) **perform equilibration** on $\omega_{\mathbf{a}'}$:

$$\boxed{\boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{a}'} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{2p+1}(\mathcal{T}_{\mathbf{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\mathbf{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \Upsilon_{Q_h^{\mathbf{a},\mathbf{a}'}}(f \psi_{\mathbf{a}} \psi_{\mathbf{a}'} - \nabla u_h \cdot \nabla(\psi_{\mathbf{a}} \psi_{\mathbf{a}'}) - \nabla r_h^{\mathbf{a}} \cdot \nabla \psi_{\mathbf{a}'})}} \| \psi_{\mathbf{a}'} (\psi_{\mathbf{a}} \nabla u_h + \nabla r_h^{\mathbf{a}}) + \boldsymbol{v}_h \|_{\omega_{\mathbf{a}'}}}$$

6) **combine**:

$$\boxed{\boldsymbol{\sigma}_h^{\mathbf{a}} := \sum_{\mathbf{a}' \in \mathcal{V}_h^{\mathbf{a}}} \boldsymbol{\sigma}_h^{\mathbf{a},\mathbf{a}'}, \qquad \boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}}$$

Breaking the large patch problems

Same building principles

Additive Schwarz smoother/preconditiner Schöberl, Melenk, Pechstein, & Zaglmayr (2008): only $\mathcal{P}_1$ global problem, then high-order patch remainders

H^{-1} problems and parabolic time stepping Ern, Smears, & Vohralík (2017): arbitrary coarsening

Details

GANTNER G., VOHRALÍK M. Inexpensive polynomial-degree- and number-of-hanging-nodes-robust equilibrated flux a posteriori estimates for isogeometric analysis. *Math. Models Methods Appl. Sci.* **34** (2024), 477–522.

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