Adaptive regularization, linearization, and numerical solution of unsteady nonlinear problems

Martin Vohralík

INRIA Paris-Rocquencourt

joint work with C. Cancès, D. A. Di Pietro, A. Ern, I. S. Pop, M. F. Wheeler, and S. Yousef

Banff, September 18, 2012

Outline

Outline

Numerical approximation of a nonlinear, unsteady PDE

Exact and approximate solution

- let u be the weak solution of $A(u) = f$, A **nonlinear**, **unsteady**, posed on $\Omega \times (0, T)$
- let $u_{h\tau}$ be its approximate numerical solution, $A_{h\tau}(u_{h\tau}) = f_{h\tau}$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by the time steps t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n, solve a nonlinear algebraic problem $A_h^n(u_h^n) = f_h^n$

Numerical approximation of a nonlinear, unsteady PDE

Exact and approximate solution

- let u be the weak solution of $A(u) = f$, A **nonlinear**, **unsteady**, posed on $\Omega \times (0, T)$
- let $u_{h\tau}$ be its approximate numerical solution, $A_{h\tau}(u_{h\tau}) = f_{h\tau}$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by the time steps t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n, solve a nonlinear algebraic problem $A_h^n(u_h^n) = f_h^n$

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Regularization, linearization, and algebraic solution

Regularization

- regularize the nonlinear operator A_h^n by $A_h^{n,\epsilon}$
- **choice of ϵ?**

Iterative linearization

- $A_h^{n,\epsilon,k-1} u_h^{n,\epsilon,k} = f_h^{n,k-1}$: $A_h^{n,\epsilon,k-1}$ linear, linearization step k
- **when do we stop?**

Iterative algebraic solution

- iterative algebraic solver employed: step i approximation $u_h^{n,\epsilon,k,i}$
- **when do we stop?**

Approximate solution

- the approximate solution $u_h^{n,\epsilon,k,i}$ that we have as an outcome does not solve $A_h^n(u_h^n) = f_h^n$
- how big is the **overall error** $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$**?**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Aims and benefits of this work

Aims of this work

- give a **guaranteed**, **robust**, and **tight** upper bound on the overall error $\|u - u_{h\tau}^{\epsilon,k,i}\|_{\Omega\times(0,T)}$
- distinguish the different error components (algebraic, linearization, regularization, spatial, temporal)
- **stop** the **iterative solvers** whenever algebraic/linearization errors do not affect the overall error significantly
- **adjust** the **regularization parameter** so that it does not to affect the overall error significantly
- **equilibrate** the space and time **error components**

Benefits

- **optimal computable overall error bound**
- **improvement of approximation precision**
- **important computational savings**

Previous results

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid solver
- Maday and Patera (2000), linear functional errors
- Arioli (2000's), general algebraic solvers

Inexact Newton method

- Eisenstat and Walker (1990's)
- Moret (1989)

Modeling error

- Ladevèze (since 1980's)
- Braack and Ern (2000's)
- Babuška and Oden (2000's)

Previous results

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid solver
- Maday and Patera (2000), linear functional errors
- Arioli (2000's), general algebraic solvers

Inexact Newton method

- Eisenstat and Walker (1990's)
- Moret (1989)

Modeling error

- Ladevèze (since 1980's)
- Braack and Ern (2000's)
- Babuška and Oden (2000's)

Previous results

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid solver
- Maday and Patera (2000), linear functional errors
- Arioli (2000's), general algebraic solvers

Inexact Newton method

- Eisenstat and Walker (1990's)
- Moret (1989)

Modeling error

- Ladevèze (since 1980's)
- Braack and Ern (2000's)
- Babuška and Oden (2000's)

Previous results – a posteriori error estimates

Nonlinear steady problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), linearization errors
- Kim (2007), guaranteed estimates, loc. cons. methods

Linear unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness wrt the final time

Nonlinear unsteady problems

- Verfürth (1998), framework for energy norm control
- Nochetto, Schmidt, Verdi (2000), degenerate problems
- Ohlberger (2001), non energy-norm estimates

Previous results – a posteriori error estimates

Nonlinear steady problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), linearization errors
- Kim (2007), guaranteed estimates, loc. cons. methods

Linear unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness wrt the final time

Nonlinear unsteady problems

- Verfürth (1998), framework for energy norm control
- Nochetto, Schmidt, Verdi (2000), degenerate problems
- Ohlberger (2001), non energy-norm estimates

Previous results – a posteriori error estimates

Nonlinear steady problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
- Carstensen and Klose (2003), guaranteed estimates
- Chaillou and Suri (2006, 2007), linearization errors
- Kim (2007), guaranteed estimates, loc. cons. methods

Linear unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness wrt the final time

Nonlinear unsteady problems

- Verfürth (1998), framework for energy norm control
- Nochetto, Schmidt, Verdi (2000), degenerate problems
- Ohlberger (2001), non energy-norm estimates

Outline

Outline

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u, \nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v, \boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v, \boldsymbol{\xi}) \in \mathbb{R} \times \mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v, \boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u, \nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(v)\xi \qquad \forall (v,\xi) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(\xi)\xi \qquad \forall \xi \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\xi) = |\xi|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u, \nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v, \xi) = \underline{\mathbf{A}}(v)\xi \qquad \forall (v, \xi) \in \mathbb{R} \times \mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v, \xi) = \underline{\mathbf{A}}(\xi)\xi \qquad \forall \xi \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\xi) = |\xi|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v \rangle_{V',V} := (\boldsymbol{\sigma}(u, \nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i},\nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\sigma(u,\nabla u) - \sigma(u_h^{k,i},\nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\, v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K\in\mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Outline

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla\cdot\mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \ \ \forall K \in \mathcal{T}_h$.

Theorem (A posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \ \ \forall K \in \mathcal{T}_h$.

Theorem (A posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \ \forall K \in \mathcal{T}_h$.

Theorem (A posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization error flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic error flux reconstruction*

Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization error flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic error flux reconstruction*

Estimate distinguishing different error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}.$$

Estimate distinguishing different error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}.$$

Estimators

- *discretization estimator*

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{1/p}\left(\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{\sum_{e\in\mathcal{E}_K} h_e^{1-q}\|[\![u_h^{k,i}]\!]\|_{q,e}^q\right\}^{\frac{1}{q}}\right)$$

- *linearization estimator*

$$\eta_{\mathrm{lin},K}^{k,i} := \|\mathbf{l}_h^{k,i}\|_{q,K}$$

- *algebraic estimator*

$$\eta_{\mathrm{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_{q,K}$$

- *algebraic remainder estimator*

$$\eta_{\mathrm{rem},K}^{k,i} := h_\Omega\|\rho_h^{k,i}\|_{q,K}$$

- *quadrature estimator*

$$\eta_{\mathrm{quad},K}^{k,i} := \|\boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}) - \overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K}$$

- *data oscillation estimator*

$$\eta_{\mathrm{osc},K}^{k,i} := C_{\mathrm{P},p} h_K\|f - f_h\|_{q,K}$$

- $\eta_{\cdot}^{k,i} := \left\{\sum_{K\in\mathcal{T}_h}\left(\eta_{\cdot,K}^{k,i}\right)^q\right\}^{1/q}$

Outline

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{k,i} \le \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},$$
$$\eta_{\mathrm{alg}}^{k,i} \le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem},K}^{k,i} \le \gamma_{\mathrm{rem},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}, \eta_{\mathrm{alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{alg},K}^{k,i} \le \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \le \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{k,i} \leq \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},$$
$$\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem},K}^{k,i} \leq \gamma_{\mathrm{rem},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}, \eta_{\mathrm{alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{alg},K}^{k,i} \leq \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \leq \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Assumption for efficiency

Assumption C (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} \lesssim \eta_{\sharp,\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i},$$

where

$$\eta_{\sharp,\mathfrak{T}_K}^{k,i} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f_h + \nabla \cdot \overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K'}^q + \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\overline{\boldsymbol{\sigma}}_h^{k,i} \cdot \mathbf{n}_e]\!]\|_{q,e}^q + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\rm quad}^{k,i} + \eta_{\rm osc}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \le \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Outline

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Example: nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f_h, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- $f_h := \Pi_0 f$
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Example: nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f_h, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- $f_h := \Pi_0 f$
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f_h, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\rm int}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2} \boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2} \boldsymbol{\xi} + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^k U^k = F^k$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f_h, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\text{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi} + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^k U^k = F^k$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\sigma^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f_h, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\text{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\text{int}}}$
- discrete system

$$\mathbb{A}^k U^k = F^k - R^{k,i}$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f_h, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\mathrm{int}}}$
- discrete system

$$\mathbb{A}^k U^k = F^k - R^{k,i}$$

Flux reconstructions

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e}.$$

Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{\mathrm{quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Flux reconstructions

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e}.$$

Definition (Construction of $\overline{\sigma}_h^{k,i}$)

Set $\overline{\sigma}_h^{k,i} := \sigma(\nabla u_h^{k,i})$. Consequently, $\eta_{\mathrm{quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Flux reconstructions

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla \psi_e) \qquad \forall e \in \mathcal{E}_h^{\rm int}$.

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e}.$$

Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{{\rm quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Verification of the assumptions – upper bound

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_0(\mathcal{S}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$.

Verification of the assumptions – upper bound

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_0(\mathcal{S}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$.

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\sigma(\nabla u_h^{k,i})$
- approximation properties of Raviart–Thomas–Nédélec spaces

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\boldsymbol{\sigma}(\nabla u_h^{k,i})$
- approximation properties of Raviart–Thomas–Nédélec spaces

Summary

Discretization methods

- nonconforming finite elements
- discontinuous Galerkin
- finite elements
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- nonconforming finite elements
- discontinuous Galerkin
- finite elements
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- nonconforming finite elements
- discontinuous Galerkin
- finite elements
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- nonconforming finite elements
- discontinuous Galerkin
- finite elements
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Outline

Numerical experiment I

Model problem

- p-Laplacian

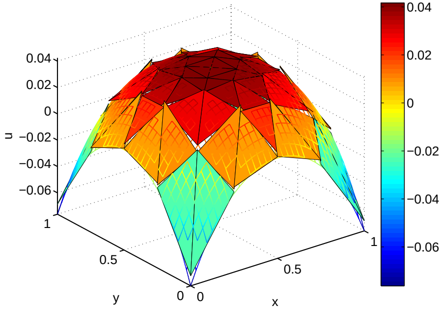
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

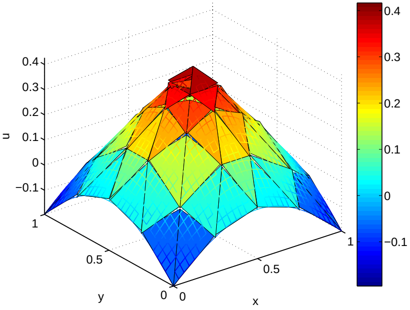
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- nonconforming finite elements

Analytical and approximate solutions

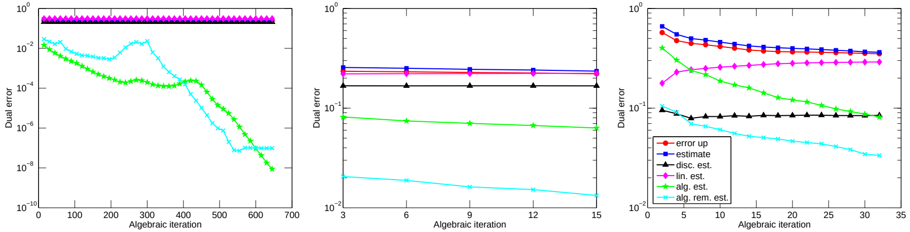


Case $p = 1.5$



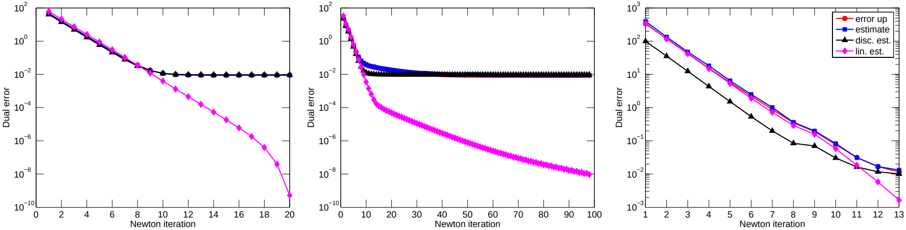
Case $p = 10$

Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



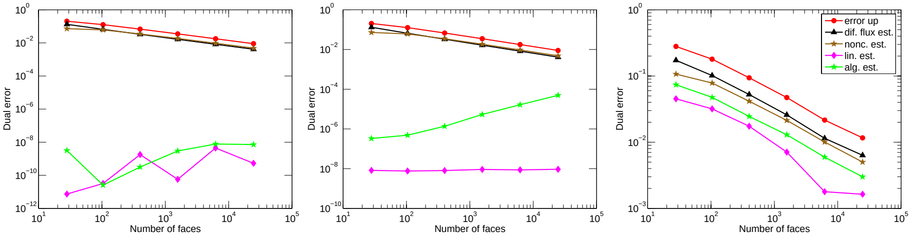
Newton | inexact Newton | ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



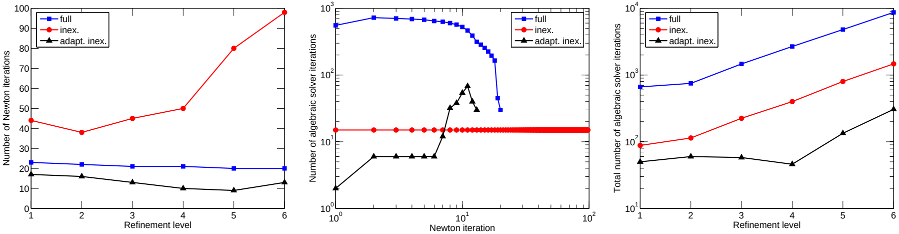
Newton inexact Newton ad. inexact Newton

Error and estimators, $p = 10$



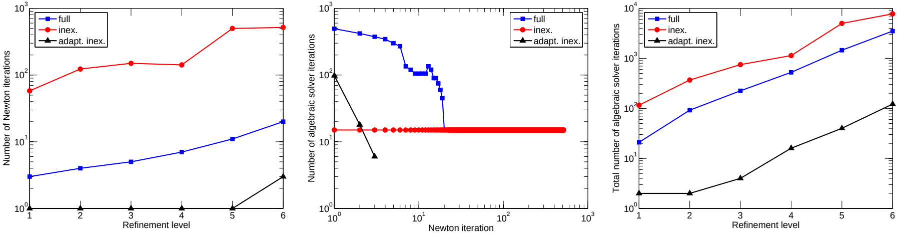
Newton inexact Newton ad. inexact Newton

Newton and algebraic iterations, $p = 10$



Newton it. / refinement   alg. it. / Newton step   alg. it. / refinement

Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement alg. it. / Newton step alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

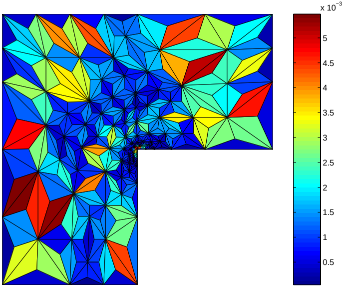
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

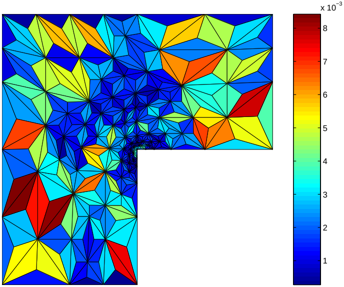
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

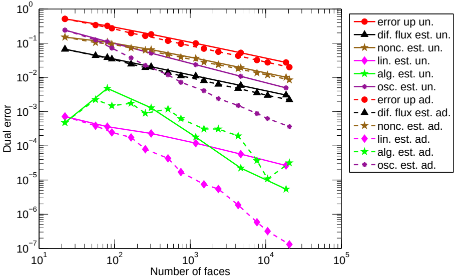


Estimated error distribution

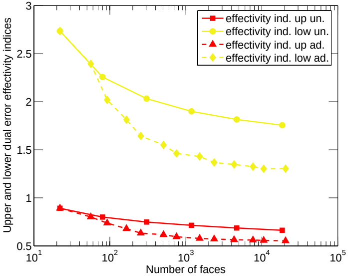


Exact error distribution

Estimated and actual errors and the effectivity index

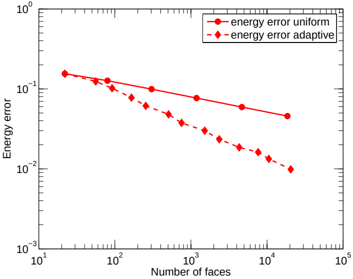


Estimated and actual errors

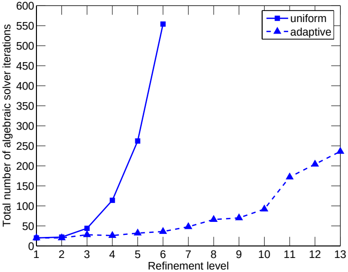


Effectivity index

Energy error and overall performance



Energy error



Overall performance

Outline

The Stefan problem

The Stefan problem

$$\begin{aligned} \partial_t u - \Delta\beta(u) &= f && \text{in } \Omega \times (0,T), \\ u(\cdot,0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0,1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$

The Stefan problem

The Stefan problem

$$\begin{aligned} \partial_t u - \Delta\beta(u) &= f && \text{in } \Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0, T) \end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0, 1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$

Outline

Residual and its dual norm

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z \qquad$ with $\beta(u) \in X$

$u(\cdot,0) = u_0 \qquad$ in Ω

$$\langle \partial_t u, \varphi\rangle(t) + (\nabla\beta(u), \nabla\varphi)(t) = (f,\varphi)(t) \quad \forall \varphi \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T)$$

Residual for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}),\varphi\rangle_{X',X} = \int_0^T \{\langle \partial_t(u-u_{h\tau}),\varphi\rangle + (\nabla\beta(u)-\nabla\beta(u_{h\tau}),\nabla\varphi)\}(t)\,\mathrm{d}t,$$

$$\varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi\in X,\, \|\varphi\|_X=1} \langle \mathcal{R}(u_{h\tau}),\varphi\rangle_{X',X}$$

Residual and its dual norm

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z$ with $\beta(u) \in X$

$u(\cdot,0) = u_0$ in Ω

$$\langle \partial_t u, \varphi\rangle(t) + (\nabla\beta(u), \nabla\varphi)(t) = (f,\varphi)(t) \quad \forall \varphi \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T)$$

Residual for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X} = \int_0^T \{\langle \partial_t(u - u_{h\tau}), \varphi\rangle + (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla\varphi)\}(t)\,\mathrm{d}t,$$

$$\varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi \in X,\, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X}$$

Residual and its dual norm

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z \qquad$ with $\beta(u) \in X$

$u(\cdot,0) = u_0 \qquad$ in Ω

$$\langle \partial_t u, \varphi\rangle(t) + (\nabla\beta(u), \nabla\varphi)(t) = (f,\varphi)(t) \quad \forall \varphi \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T)$$

Residual for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X} = \int_0^T \{\langle \partial_t(u - u_{h\tau}), \varphi\rangle + (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla\varphi)\}(t)\,\mathrm{d}t,$$

$$\varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi\in X,\, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X}$$

Residual and its dual norm

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z$ with $\beta(u) \in X$

$u(\cdot,0) = u_0$ in Ω

$$\langle \partial_t u, \varphi\rangle(t) + (\nabla\beta(u), \nabla\varphi)(t) = (f,\varphi)(t) \quad \forall \varphi \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T)$$

Residual for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X} = \int_0^T \{\langle \partial_t(u - u_{h\tau}), \varphi\rangle + (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla\varphi)\}(t)\,\mathrm{d}t,$$

$$\varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi\in X,\, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X}$$

Time-localization of the dual norm of the residual

Time interval I_n

$$X_n := L^2(I_n; H_0^1(\Omega))$$

$$\|\mathcal{R}(u_{h\tau})\|_{X_n'} := \sup_{\varphi \in X_n,\, \|\varphi\|_{X_n}=1} \int_{I_n} \{\langle \partial_t(u - u_{h\tau}), \varphi\rangle + (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla\varphi)\}(t)\,\mathrm{d}t$$

L^2 in time ...

$$\|\mathcal{R}(u_{h\tau})\|_{X'}^2 = \sum_{1\le n\le N} \|\mathcal{R}(u_{h\tau})\|_{X_n'}^2$$

Time-localization of the dual norm of the residual

Time interval I_n

$$X_n := L^2(I_n; H_0^1(\Omega))$$

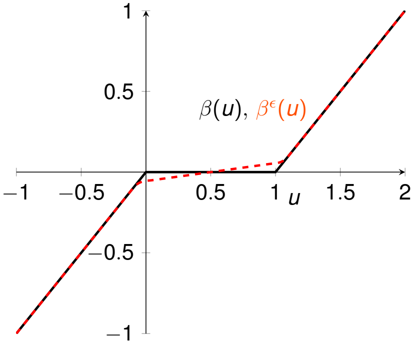
$$\|\mathcal{R}(u_{h\tau})\|_{X_n'} := \sup_{\varphi \in X_n, \|\varphi\|_{X_n}=1} \int_{I_n} \{\langle \partial_t (u - u_{h\tau}), \varphi \rangle + (\nabla \beta(u) - \nabla \beta(u_{h\tau}), \nabla \varphi)\}(t)\, \mathrm{d}t$$

L^2 in time . . .

$$\|\mathcal{R}(u_{h\tau})\|_{X'}^2 = \sum_{1 \le n \le N} \|\mathcal{R}(u_{h\tau})\|_{X_n'}^2$$

Practice: regularization

Regularization with a parameter ϵ



Practice: questions

Discretization

- ...

Question (Stopping and balancing criteria)

- *What is a good choice of the*
  - *regularization parameter ϵ?*
  - *time step?*
  - *space mesh?*
- *What is a good stopping criterion for the*
  - *nonlinear solver?*
  - *linear solver?*

Question (Error)

- *How big is the error $\|u|_{I_n} - u_{h\tau}^{n,\epsilon,k,i}\|$ on time step n, space mesh $\mathcal{T}_h^n$, for the regularization parameter ϵ, Newton step k, and algebraic solver step i? How big are the individual components? How is error distributed in time and space?*

Practice: questions

Discretization

- . . .

Question (Stopping and balancing criteria)

- *What is a good choice of the*
  - *regularization parameter* ϵ*?*
  - *time step?*
  - *space mesh?*
- *What is a good stopping criterion for the*
  - *nonlinear solver?*
  - *linear solver?*

Question (Error)

- *How big is the error* $\|u|_{I_n} - u_{h\tau}^{n,\epsilon,k,i}\|$ *on time step* n*, space mesh* $\mathcal{T}_h^n$*, for the regularization parameter* ϵ*, Newton step* k*, and algebraic solver step* i*? How big are the individual components? How is error distributed in time and space?*

Practice: questions

Discretization

- ...

Question (Stopping and balancing criteria)

- *What is a good choice of the*
  - *regularization parameter ϵ?*
  - *time step?*
  - *space mesh?*
- *What is a good stopping criterion for the*
  - *nonlinear solver?*
  - *linear solver?*

Question (Error)

- *How big is the error $\|u|_{I_n} - u_{h\tau}^{n,\epsilon,k,i}\|$ on time step n, space mesh $\mathcal{T}_h^n$, for the regularization parameter ϵ, Newton step k, and algebraic solver step i? How big are the individual components? How is error distributed in time and space?*

Outline

A posteriori estimate and its efficiency

Theorem (Estimate and its efficiency)

There holds

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}$$

$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n\right)^2 \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}}$$

$$\lesssim \|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_h^n\|_K,$$

$$\eta_{\mathrm{F},K}^n(t) := \|\nabla \beta(u_{h\tau}(t)) + \mathbf{t}_h^n\|_K,$$

$$\eta_{\mathrm{IC}} := \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}.$$

A posteriori estimate and its efficiency

Theorem (Estimate and its efficiency)

There holds

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}$$
$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n\right)^2 \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}}$$
$$\lesssim \|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_h^n\|_K,$$
$$\eta_{\mathrm{F},K}^n(t) := \|\nabla \beta(u_{h\tau}(t)) + \mathbf{t}_h^n\|_K,$$
$$\eta_{\mathrm{IC}} := \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}.$$

A posteriori estimate and its efficiency

Theorem (Estimate and its efficiency)

There holds

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}$$

$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n\right)^2 \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}}$$

$$\lesssim \|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_h^n\|_K,$$

$$\eta_{\mathrm{F},K}^n(t) := \|\nabla \beta(u_{h\tau}(t)) + \mathbf{t}_h^n\|_K,$$

$$\eta_{\mathrm{IC}} := \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}.$$

A posteriori estimate and its efficiency

Theorem (Estimate and its efficiency)

There holds

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}$$
$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n\right)^2 \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}}$$
$$\lesssim \|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_h^n\|_K,$$
$$\eta_{\mathrm{F},K}^n(t) := \|\nabla \beta(u_{h\tau}(t)) + \mathbf{t}_h^n\|_K,$$
$$\eta_{\mathrm{IC}} := \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}.$$

Distinguishing the different error components

Theorem (An estimate distinguishing the error components)

For time n, linearization k, and regularization ϵ, there holds

$$\|\mathcal{R}(u_{h\tau}^{n,\epsilon,k})\|_{X',I_n} \leq \eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{lin}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k}.$$

- $\boldsymbol{\sigma}^{n,\epsilon,k}$ a scheme linearized flux (not $\mathbf{H}(\mathrm{div},\Omega)$), $\mathbf{t}^{n,\epsilon,k}$ reconstructed $\mathbf{H}(\mathrm{div},\Omega)$ flux, Π^n interpolation

$$(\eta_{\mathrm{sp}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^{n,\epsilon,k} + \|\boldsymbol{\sigma}^{n,\epsilon,k} + \mathbf{t}^{n,\epsilon,k}\|_K\right)^2,$$

$$(\eta_{\mathrm{tm}}^{n,\epsilon,k})^2 := \int_{I_n} \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t) - \nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t^n)\|_K^2 \,\mathrm{d}t,$$

$$(\eta_{\mathrm{lin}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta_\epsilon(u_{h\tau}^{\epsilon,k})(t^n) - \boldsymbol{\sigma}^{n,\epsilon,k}\|_K^2,$$

$$(\eta_{\mathrm{reg}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t^n) - \nabla\Pi^n\beta_\epsilon(u_{h\tau}^{\epsilon,k})(t^n))\|_K^2$$

Distinguishing the different error components

Theorem (An estimate distinguishing the error components)

For time n, linearization k, and regularization ϵ, there holds

$$\|\mathcal{R}(u_{h\tau}^{n,\epsilon,k})\|_{X',I_n} \leq \eta_{\rm sp}^{n,\epsilon,k} + \eta_{\rm tm}^{n,\epsilon,k} + \eta_{\rm lin}^{n,\epsilon,k} + \eta_{\rm reg}^{n,\epsilon,k}.$$

- $\boldsymbol{\sigma}^{n,\epsilon,k}$ a scheme linearized flux (not $\mathbf{H}(\mathrm{div},\Omega)$), $\mathbf{t}^{n,\epsilon,k}$ reconstructed $\mathbf{H}(\mathrm{div},\Omega)$ flux, Π^n interpolation

$$(\eta_{\rm sp}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \left(\eta_{{\rm R},K}^{n,\epsilon,k} + \|\boldsymbol{\sigma}^{n,\epsilon,k} + \mathbf{t}^{n,\epsilon,k}\|_K\right)^2,$$

$$(\eta_{\rm tm}^{n,\epsilon,k})^2 := \int_{I_n} \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t) - \nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t^n)\|_K^2 \,\mathrm{d}t,$$

$$(\eta_{\rm lin}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta_\epsilon(u_{h\tau}^{\epsilon,k})(t^n) - \boldsymbol{\sigma}^{n,\epsilon,k}\|_K^2,$$

$$(\eta_{\rm reg}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}_h^n} \|\nabla\Pi^n\beta(u_{h\tau}^{\epsilon,k})(t^n) - \nabla\Pi^n\beta_\epsilon(u_{h\tau}^{\epsilon,k})(t^n))\|_K^2$$

Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

$$\frac{L_\beta}{2}\|u - u_{h\tau}\|^2_{X'} + \|\beta(u) - \beta(u_{h\tau})\|^2_{Q_T}$$
$$\leq \frac{L_\beta}{2}(2e^T - 1)\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'} + \|(u - u_{h\tau})(0)\|^2_{H^{-1}(\Omega)}\right)$$

Theorem (Temperature and enthalpy errors, tight Gronwall)

Let $u_{h\tau} \in Z$ be such that $\beta(u_{h\tau}) \in X$ be arbitrary. There holds

$$\frac{L_\beta}{2}\|u - u_{h\tau}\|^2_{X'} + \frac{L_\beta}{2}\|(u - u_{h\tau})(T)\|^2_{H^{-1}(\Omega)} + \|\beta(u) - \beta(u_{h\tau})\|^2_{Q_T}$$
$$+ 2\int_0^T\left(\|\beta(u) - \beta(u_{h\tau})\|^2_{Q_t} + \int_0^t \|\beta(u) - \beta(u_{h\tau})\|^2_{Q_s} e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t$$
$$\leq \frac{L_\beta}{2}\left\{(2e^T - 1)\|(u - u_{h\tau})(0)\|^2_{H^{-1}(\Omega)} + \|\mathcal{R}(u_{h\tau})\|^2_{X'}\right.$$
$$\left. + 2\int_0^T\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'_t} + \int_0^t \|\mathcal{R}(u_{h\tau})\|^2_{X'_s} e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t\right\}.$$

Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

$$\frac{L_\beta}{2}\|u-u_{h\tau}\|^2_{X'}+\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_T}$$
$$\leq\frac{L_\beta}{2}(2e^T-1)\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'}+\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)}\right)$$

Theorem (Temperature and enthalpy errors, tight Gronwall)

Let $u_{h\tau}\in Z$ be such that $\beta(u_{h\tau})\in X$ be arbitrary. There holds

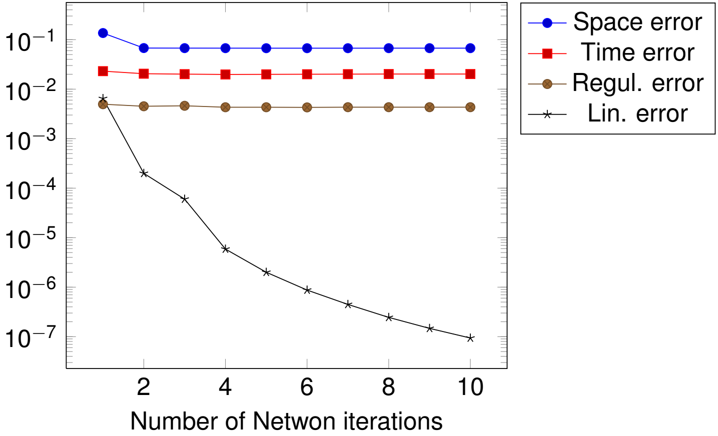
$$\frac{L_\beta}{2}\|u-u_{h\tau}\|^2_{X'}+\frac{L_\beta}{2}\|(u-u_{h\tau})(T)\|^2_{H^{-1}(\Omega)}+\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_T}$$
$$+2\int_0^T\left(\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_t}+\int_0^t\|\beta(u)-\beta(u_{h\tau})\|^2_{Q_s}e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t$$
$$\leq\frac{L_\beta}{2}\left\{(2e^T-1)\|(u-u_{h\tau})(0)\|^2_{H^{-1}(\Omega)}+\|\mathcal{R}(u_{h\tau})\|^2_{X'}\right.$$
$$\left.+2\int_0^T\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'_t}+\int_0^t\|\mathcal{R}(u_{h\tau})\|^2_{X'_s}e^{t-s}\,\mathrm{d}s\right)\,\mathrm{d}t\right\}.$$

Outline

Linearization stopping criterion

Linearization stopping criterion

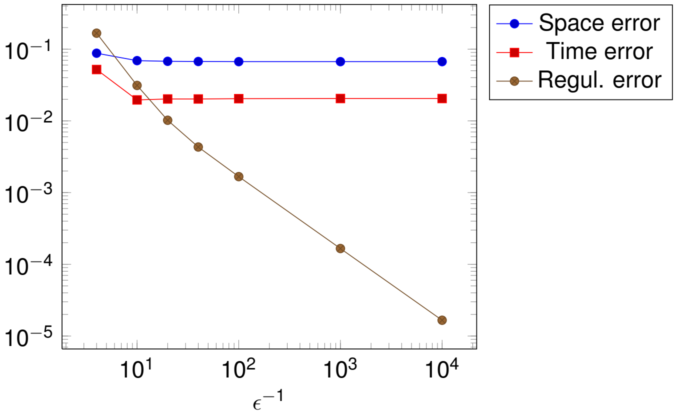
$$\eta_{\mathrm{lin}}^{n,\epsilon,k} \leq \gamma_{\mathrm{lin}}\left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k}\right)$$



Regularization stopping criterion

Regularization stopping criterion

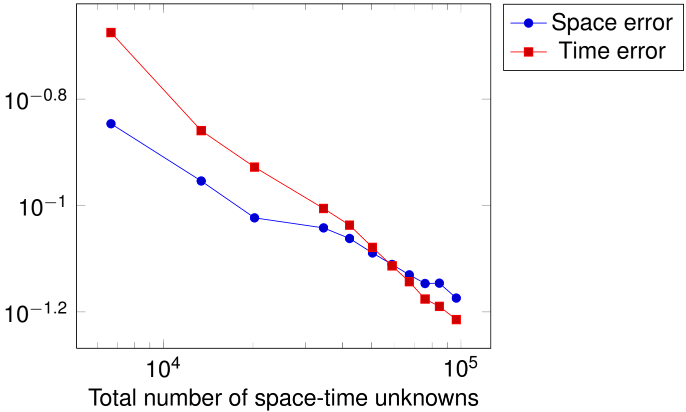
$$\eta_{\mathrm{reg}}^{n,\epsilon,k} \leq \gamma_{\mathrm{reg}} \left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} \right)$$



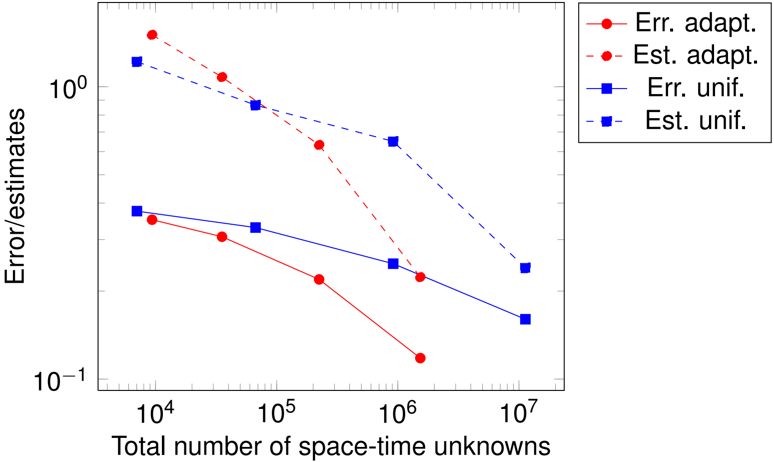
Equilibrating time and space errors

Equilibrating time and space errors

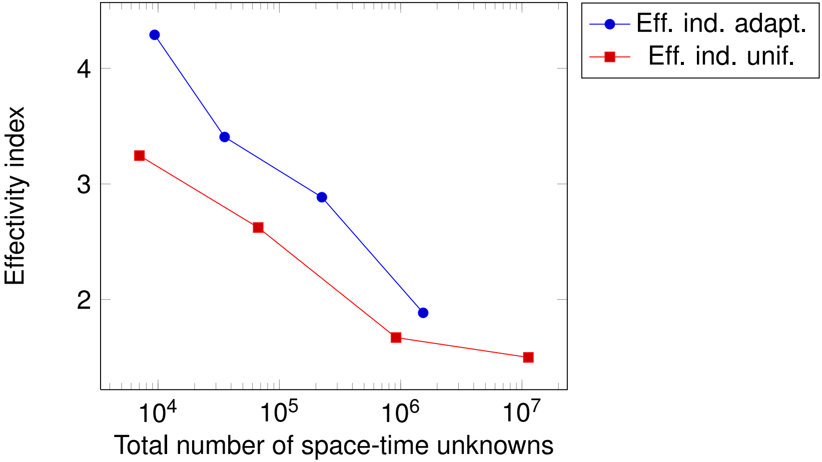
$$\gamma_{\min}\eta_{\rm sp}^{\epsilon,n,k} \leq \eta_{\rm tm}^{\epsilon,n,k} \leq \gamma_{\max}\eta_{\rm sp}^{\epsilon,n,k}$$



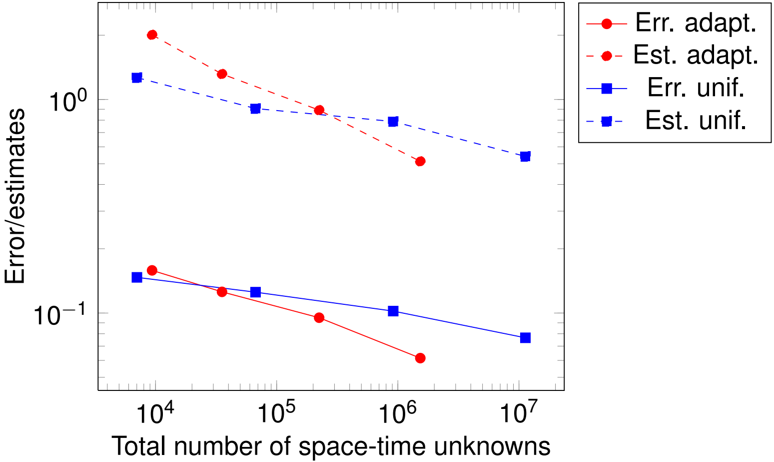
Error and estimate (dual norm)



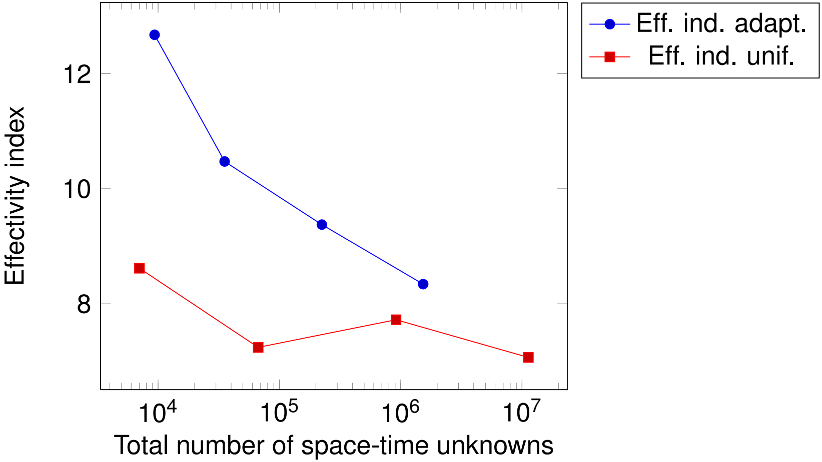
Effectivity indices (dual norm)



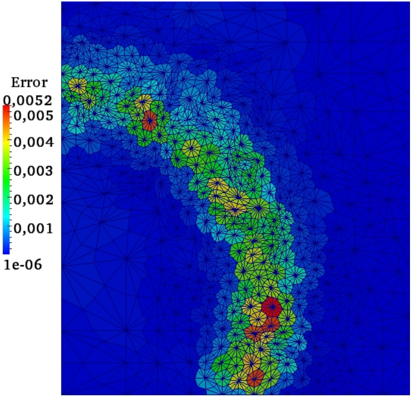
Error and estimate (energy norm)

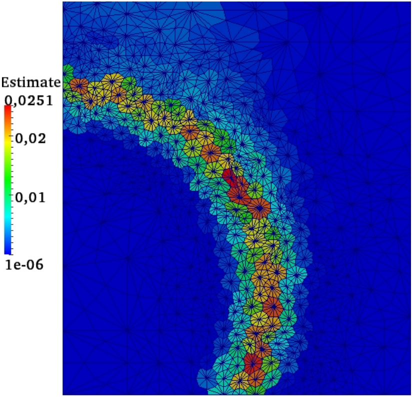


Effectivity indices (energy norm)



Actual and estimated error distribution





Computational efficiency

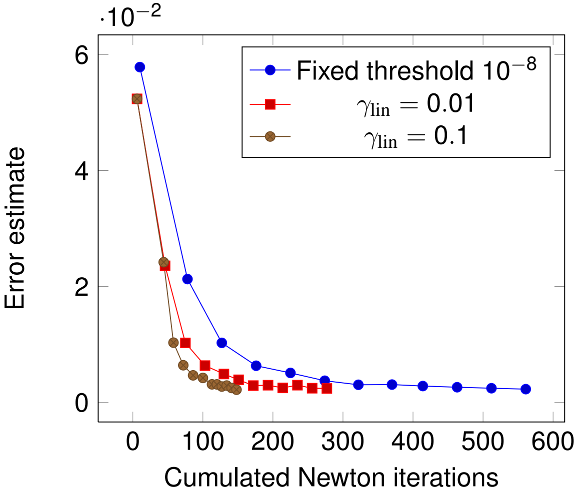


Figure: Number of cumulated Newton iterations vs. error estimate

Outline

Two-phase flow

The model

$$\begin{aligned}
\partial_t s_\alpha + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha(s_\alpha), & \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
\mathbf{u}_\alpha &= -\underline{\mathbf{K}}\eta_\alpha(s_\alpha)\nabla p_\alpha, & \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
s_\mathrm{n} + s_\mathrm{w} &= 1,\\
p_\mathrm{n} - p_\mathrm{w} &= \pi(s_\mathrm{n})
\end{aligned}$$

- two immiscible, incompressible fluids
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $\pi(\cdot)$: the nonlinear capillary pressure function
- $\eta_\mathrm{n}(\cdot)$, $\eta_\mathrm{w}(\cdot)$: the nonlinear phase mobilities functions
- $\underline{\mathbf{K}}$ permeability tensor, $q_\mathrm{n}(\cdot)$, $q_\mathrm{w}(\cdot)$ sources

Two-phase flow

The model

$$\begin{aligned}
\partial_t s_\alpha + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha(s_\alpha), && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
\mathbf{u}_\alpha &= -\underline{\mathbf{K}}\eta_\alpha(s_\alpha)\nabla p_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
s_\mathrm{n} + s_\mathrm{w} &= 1,\\
p_\mathrm{n} - p_\mathrm{w} &= \pi(s_\mathrm{n})
\end{aligned}$$

- two immiscible, incompressible fluids
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $\pi(\cdot)$: the nonlinear capillary pressure function
- $\eta_\mathrm{n}(\cdot)$, $\eta_\mathrm{w}(\cdot)$: the nonlinear phase mobilities functions
- $\underline{\mathbf{K}}$ permeability tensor, $q_\mathrm{n}(\cdot)$, $q_\mathrm{w}(\cdot)$ sources

Two-phase flow

The model

$$
\begin{aligned}
\partial_t s_\alpha + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha(s_\alpha), && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
\mathbf{u}_\alpha &= -\underline{\mathbf{K}}\eta_\alpha(s_\alpha)\nabla p_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
s_\mathrm{n} + s_\mathrm{w} &= 1,\\
p_\mathrm{n} - p_\mathrm{w} &= \pi(s_\mathrm{n})
\end{aligned}
$$

- two immiscible, incompressible fluids
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $\pi(\cdot)$: the nonlinear capillary pressure function
- $\eta_\mathrm{n}(\cdot)$, $\eta_\mathrm{w}(\cdot)$: the nonlinear phase mobilities functions
- $\underline{\mathbf{K}}$ permeability tensor, $q_\mathrm{n}(\cdot)$, $q_\mathrm{w}(\cdot)$ sources

Outline

Notation and transformations

Notation and transformations

- $s := s_{\mathrm{n}}$
- $$f(s) := \frac{\eta_{\mathrm{n}}(s)}{\eta_{\mathrm{n}}(s) + \eta_{\mathrm{w}}(1-s)}, \quad \lambda(s) := \eta_{\mathrm{w}}(1-s) f(s)$$
- Kirchhoff transform
$$\varphi(s) := \int_0^s \lambda(a)\pi'(a)\,\mathrm{d}a$$
- global pressure
$$P := P(s, p_{\mathrm{n}}) := p_{\mathrm{n}} - \int_0^{\pi(s)} \frac{\eta_{\mathrm{w}}(1-\pi^{-1}(a))}{\eta_{\mathrm{n}}(\pi^{-1}(a)) + \eta_{\mathrm{w}}(1-\pi^{-1}(a))}\,\mathrm{d}a$$
- $M(s) := \eta_{\mathrm{w}}(1-s) + \eta_{\mathrm{n}}(s)$
- $q_{\mathrm{t}}(s) := q_{\mathrm{n}}(s) + q_{\mathrm{w}}(1-s)$
- $f, \lambda, \varphi, P, M, q_{\mathrm{t}}$ only needed for the theoretical analysis, not in the scheme
- s^0: initial condition
- $\overline{s}, \overline{P}$: Dirichlet boundary conditions

Notation and transformations

Notation and transformations

- $s := s_{\mathrm{n}}$
- $$f(s) := \frac{\eta_{\mathrm{n}}(s)}{\eta_{\mathrm{n}}(s) + \eta_{\mathrm{w}}(1-s)}, \quad \lambda(s) := \eta_{\mathrm{w}}(1-s)f(s)$$
- Kirchhoff transform
$$\varphi(s) := \int_0^s \lambda(a)\pi'(a)\,\mathrm{d}a$$
- global pressure
$$P := P(s, p_{\mathrm{n}}) := p_{\mathrm{n}} - \int_0^{\pi(s)} \frac{\eta_{\mathrm{w}}(1-\pi^{-1}(a))}{\eta_{\mathrm{n}}(\pi^{-1}(a)) + \eta_{\mathrm{w}}(1-\pi^{-1}(a))}\,\mathrm{d}a$$
- $M(s) := \eta_{\mathrm{w}}(1-s) + \eta_{\mathrm{n}}(s)$
- $q_{\mathrm{t}}(s) := q_{\mathrm{n}}(s) + q_{\mathrm{w}}(1-s)$
- f, λ, φ, P, M, q_{t} only needed for the theoretical analysis, not in the scheme
- s^0: initial condition
- $\overline{s}$, $\overline{P}$: Dirichlet boundary conditions

Notation and transformations

Notation and transformations

- $s := s_{\mathrm{n}}$
- $$f(s) := \frac{\eta_{\mathrm{n}}(s)}{\eta_{\mathrm{n}}(s) + \eta_{\mathrm{w}}(1-s)}, \quad \lambda(s) := \eta_{\mathrm{w}}(1-s) f(s)$$
- Kirchhoff transform
$$\varphi(s) := \int_0^s \lambda(a)\pi'(a)\,\mathrm{d}a$$
- global pressure
$$P := P(s, p_{\mathrm{n}}) := p_{\mathrm{n}} - \int_0^{\pi(s)} \frac{\eta_{\mathrm{w}}(1-\pi^{-1}(a))}{\eta_{\mathrm{n}}(\pi^{-1}(a)) + \eta_{\mathrm{w}}(1-\pi^{-1}(a))}\,\mathrm{d}a$$
- $M(s) := \eta_{\mathrm{w}}(1-s) + \eta_{\mathrm{n}}(s)$
- $q_{\mathrm{t}}(s) := q_{\mathrm{n}}(s) + q_{\mathrm{w}}(1-s)$
- f, λ, φ, P, M, q_{t} only needed for the theoretical analysis, not in the scheme
- s^0: initial condition
- $\overline{s}$, $\overline{P}$: Dirichlet boundary conditions

Weak formulation

Functional space for the weak solution

$$\mathcal{E} := \big\{ (s, P) \mid s \in \mathcal{C}([0,T]; L^2(\Omega)),\ \partial_t s \in L^2((0,T); H^{-1}(\Omega)),$$
$$\varphi(s) - \varphi(\overline{s}) \in L^2((0,T); H_0^1(\Omega)),\ P - \overline{P} \in L^2((0,T); H_0^1(\Omega)) \big\}$$

Definition (Weak solution)

A weak solution is a pair $(s, P) \in \mathcal{E}$ such that $s(\cdot, 0) = s^0$ and for all $\psi \in L^2((0,T); H_0^1(\Omega))$,

$$\int_0^T \langle \partial_t s(\cdot,\theta); \psi(\cdot,\theta) \rangle_{H^{-1},H_0^1} \,\mathrm{d}\theta + \iint_{Q_T} \underline{\mathbf{K}}(\eta_{\mathrm{n}}(s)\nabla P + \nabla\varphi(s))\cdot\nabla\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta$$
$$= \iint_{Q_T} q_{\mathrm{n}}(s)\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta,$$

$$\iint_{Q_T} \underline{\mathbf{K}} M(s)\nabla P\cdot\nabla\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta = \iint_{Q_T} q_{\mathrm{t}}(s)\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta.$$

Weak formulation

Functional space for the weak solution

$$\mathcal{E} := \big\{ (s, P) \mid s \in \mathcal{C}([0,T]; L^2(\Omega)),\ \partial_t s \in L^2((0,T); H^{-1}(\Omega)),$$
$$\varphi(s) - \varphi(\overline{s}) \in L^2((0,T); H_0^1(\Omega)),\ P - \overline{P} \in L^2((0,T); H_0^1(\Omega)) \big\}$$

Definition (Weak solution)

A weak solution is a pair $(s, P) \in \mathcal{E}$ such that $s(\cdot, 0) = s^0$ and for all $\psi \in L^2((0,T); H_0^1(\Omega))$,

$$\int_0^T \langle \partial_t s(\cdot,\theta); \psi(\cdot,\theta) \rangle_{H^{-1}, H_0^1} \,\mathrm{d}\theta + \iint_{Q_T} \underline{\mathbf{K}}(\eta_{\mathrm{n}}(s)\nabla P + \nabla\varphi(s))\cdot\nabla\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta$$
$$= \iint_{Q_T} q_{\mathrm{n}}(s)\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta,$$
$$\iint_{Q_T} \underline{\mathbf{K}} M(s)\nabla P\cdot\nabla\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta = \iint_{Q_T} q_{\mathrm{t}}(s)\psi \,\mathrm{d}\mathbf{x}\mathrm{d}\theta.$$

Outline

A posteriori error estimate

Functional space for the approximate solution

$$\mathcal{E}_\tau := \{(s, P) \mid s \in V_\tau, \text{ pw affine-in-time subspace of } \mathcal{C}([0,T]; L^2(\Omega)),$$
$$\varphi(s) - \varphi(\overline{s}) \in L^2((0,T); H_0^1(\Omega)),\ P - \overline{P} \in L^2((0,T); H_0^1(\Omega))\}$$

Theorem (A posteriori error estimate)

Let (s, P) be the weak solution. Let $(s_{h\tau}, P_{h\tau}) \in \mathcal{E}_\tau$ be arbitrary. Then there exists $C > 0$ such that

$$\|s_{h\tau} - s\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|P_{h\tau} - P\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\varphi(s_{h\tau}) - \varphi(s)\|^2_{L^2(Q_T)}$$
$$\leq C\|s_{h\tau}(\cdot, 0) - s^0\|^2_{H^{-1}(\Omega)}$$
$$+ C(|||\mathcal{R}^n_{\mathrm{n}}(s_{h\tau}, P_{h\tau})|||^2 + |||\mathcal{R}^n_{\mathrm{t}}(s_{h\tau}, P_{h\tau})|||^2).$$

A posteriori error estimate

Functional space for the approximate solution

$$\mathcal{E}_\tau := \big\{(s,P) \mid s \in V_\tau,\ \text{pw affine-in-time subspace of } \mathcal{C}([0,T];L^2(\Omega)), \\ \varphi(s)-\varphi(\overline{s}) \in L^2((0,T);H_0^1(\Omega)),\ P-\overline{P} \in L^2((0,T);H_0^1(\Omega))\big\}$$

Theorem (A posteriori error estimate)

Let (s,P) be the weak solution. Let $(s_{h\tau},P_{h\tau}) \in \mathcal{E}_\tau$ be arbitrary. Then there exists $C>0$ such that

$$\|s_{h\tau}-s\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|P_{h\tau}-P\|^2_{L^2(0,T;H_0^1(\Omega))}+\|\varphi(s_{h\tau})-\varphi(s)\|^2_{L^2(Q_T)} \\ \leq C\|s_{h\tau}(\cdot,0)-s^0\|^2_{H^{-1}(\Omega)} \\ +C(|||\mathcal{R}^n_{\mathrm{n}}(s_{h\tau},P_{h\tau})|||^2+|||\mathcal{R}^n_{\mathrm{t}}(s_{h\tau},P_{h\tau})|||^2).$$

A posteriori error estimate

Functional space for the approximate solution

$$\mathcal{E}_\tau := \{(s,P) \mid s \in V_\tau, \text{ pw affine-in-time subspace of } \mathcal{C}([0,T];L^2(\Omega)), \varphi(s)-\varphi(\overline{s}) \in L^2((0,T);H_0^1(\Omega)),\ P-\overline{P} \in L^2((0,T);H_0^1(\Omega))\}$$

Theorem (A posteriori error estimate)

Let (s,P) be the weak solution. Let $(s_{h\tau}, P_{h\tau}) \in \mathcal{E}_\tau$ be arbitrary. Then there exists $C > 0$ such that

$$\|s_{h\tau}-s\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|P_{h\tau}-P\|^2_{L^2(0,T;H_0^1(\Omega))} + \|\varphi(s_{h\tau})-\varphi(s)\|^2_{L^2(Q_T)}$$
$$\leq C\|s_{h\tau}(\cdot,0)-s^0\|^2_{H^{-1}(\Omega)}$$
$$+ C(|||\mathcal{R}_{\mathrm{n}}^n(s_{h\tau},P_{h\tau})|||^2 + |||\mathcal{R}_{\mathrm{t}}^n(s_{h\tau},P_{h\tau})|||^2).$$

A posteriori error estimate

Functional space for the approximate solution

$$\mathcal{E}_\tau := \big\{(s,P) \mid s \in V_\tau, \text{ pw affine-in-time subspace of } \mathcal{C}([0,T];L^2(\Omega)),$$
$$\varphi(s)-\varphi(\overline{s}) \in L^2((0,T);H_0^1(\Omega)),\ P-\overline{P} \in L^2((0,T);H_0^1(\Omega))\big\}$$

Theorem (A posteriori error estimate)

Let (s,P) be the weak solution. Let $(s_{h\tau},P_{h\tau}) \in \mathcal{E}_\tau$ be arbitrary. Then there exists $C>0$ such that

$$\|s_{h\tau}-s\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|P_{h\tau}-P\|^2_{L^2(0,T;H_0^1(\Omega))}+\|\varphi(s_{h\tau})-\varphi(s)\|^2_{L^2(Q_T)}$$
$$\le C\|s_{h\tau}(\cdot,0)-s^0\|^2_{H^{-1}(\Omega)}$$
$$+\ C(|||\mathcal{R}^n_{\mathrm{n}}(s_{h\tau},P_{h\tau})|||^2+|||\mathcal{R}^n_{\mathrm{t}}(s_{h\tau},P_{h\tau})|||^2).$$

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s_{h\tau}^{k,i}, P_{h\tau}^{k,i})$. Let there exist equilibrated fluxes reconstructions $\mathbf{u}_{\alpha,h}^{n,k,i}$ for each phase $\alpha \in \mathrm{n}, \mathrm{w}$. Split them as

$$\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$\begin{aligned}(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}} \\ \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.\end{aligned}$$

Moreover, if $\eta_{\mathrm{tm}}^{n,k,i}$, $\eta_{\mathrm{lin}}^{n,k,i}$, and $\eta_{\mathrm{alg}}^{n,k,i}$ do not dominate, then

$$\begin{aligned}\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i} \\ \leq C(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}.\end{aligned}$$

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s_{h\tau}^{k,i}, P_{h\tau}^{k,i})$. Let there exist equilibrated fluxes reconstructions $\mathbf{u}_{\alpha,h}^{n,k,i}$ for each phase $\alpha \in \mathrm{n}, \mathrm{w}$. Split them as

$$\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$\begin{aligned} &(|||\mathcal{R}_{\mathrm{n}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}} \\ &\leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}. \end{aligned}$$

Moreover, if $\eta_{\mathrm{tm}}^{n,k,i}$, $\eta_{\mathrm{lin}}^{n,k,i}$, and $\eta_{\mathrm{alg}}^{n,k,i}$ do not dominate, then

$$\begin{aligned} &\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i} \\ &\leq C(|||\mathcal{R}_{\mathrm{n}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}. \end{aligned}$$

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s_{h\tau}^{k,i}, P_{h\tau}^{k,i})$. Let there exist equilibrated fluxes reconstructions $\mathbf{u}_{\alpha,h}^{n,k,i}$ for each phase $\alpha \in \mathrm{n}, \mathrm{w}$. Split them as

$$\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$\left(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 \right)^{\frac{1}{2}} \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Moreover, if $\eta_{\mathrm{tm}}^{n,k,i}$, $\eta_{\mathrm{lin}}^{n,k,i}$, and $\eta_{\mathrm{alg}}^{n,k,i}$ do not dominate, then

$$\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i} \leq C\left(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 \right)^{\frac{1}{2}}.$$

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s_{h\tau}^{k,i}, P_{h\tau}^{k,i})$. *Let there exist equilibrated fluxes reconstructions* $\mathbf{u}_{\alpha,h}^{n,k,i}$ *for each phase* $\alpha \in \mathrm{n}, \mathrm{w}$. *Split them as*

$$\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$\begin{aligned}&(|||\mathcal{R}_{\mathrm{n}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}\\ &\leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.\end{aligned}$$

Moreover, if $\eta_{\mathrm{tm}}^{n,k,i}$, $\eta_{\mathrm{lin}}^{n,k,i}$, *and* $\eta_{\mathrm{alg}}^{n,k,i}$ *do not dominate, then*

$$\begin{aligned}&\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}\\ &\leq C(|||\mathcal{R}_{\mathrm{n}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^n(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}.\end{aligned}$$

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s_{h\tau}^{k,i}, P_{h\tau}^{k,i})$. Let there exist equilibrated fluxes reconstructions $\mathbf{u}_{\alpha,h}^{n,k,i}$ for each phase $\alpha \in \mathrm{n}, \mathrm{w}$. Split them as

$$\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}} \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Moreover, if $\eta_{\mathrm{tm}}^{n,k,i}$, $\eta_{\mathrm{lin}}^{n,k,i}$, and $\eta_{\mathrm{alg}}^{n,k,i}$ do not dominate, then

$$\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i} \leq C(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}.$$

Estimators

- *spatial estimators*

$$\eta_{\mathrm{sp,n},D}^{n,k,i} := \|\mathbf{d}_{\mathrm{n},h}^{n,k,i} - \underline{\mathbf{K}}(\eta(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i} + \nabla\varphi(s_{h\tau}^{n,k,i}))(t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{sp,t},D}^{n,k,i} := \|\mathbf{d}_{\mathrm{t},h}^{n,k,i} - \underline{\mathbf{K}}M(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i}(t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *temporal estimators*

$$\eta_{\mathrm{tm,n},D}^{n,k,i}(t) := \|\underline{\mathbf{K}}(\eta(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i} + \nabla\varphi(s_{h\tau}^{n,k,i}))(t - t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{tm,t},D}^{n,k,i}(t) := \|\underline{\mathbf{K}}M(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i}(t - t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *linearization estimators*

$$\eta_{\mathrm{lin,n},D}^{n,k,i} := \|\mathbf{l}_{\mathrm{n},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{lin,t},D}^{n,k,i} := \|\mathbf{l}_{\mathrm{t},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg,n},D}^{n,k,i} := \|\mathbf{a}_{\mathrm{n},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{alg,t},D}^{n,k,i} := \|\mathbf{a}_{\mathrm{t},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

Outline

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}|\frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|,$$

$$F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},K'} + \pi(s^n_{\mathrm{w},K'}) - (\bar{p}^n_{\mathrm{n},K} + \pi(s^n_{\mathrm{n},K}))}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|.$$

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|,$$

$$F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},K'} + \pi(s^n_{\mathrm{w},K'}) - (\bar{p}^n_{\mathrm{n},K} + \pi(s^n_{\mathrm{n},K}))}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|.$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) := F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) + \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) + \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}).$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{e_{KK'}} := F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$

$$((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{e_{KK'}} := F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$

$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$-\eta_{\mathrm{r,w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,$$

$$p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) = \bar{p}_{\mathrm{w},K}^{n,k,i},$$

$$-\eta_{\mathrm{r,n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,$$

$$p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) = \pi(s_{\mathrm{w},K}^{n,k,i})+\bar{p}_{\mathrm{w},K}^{n,k,i}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

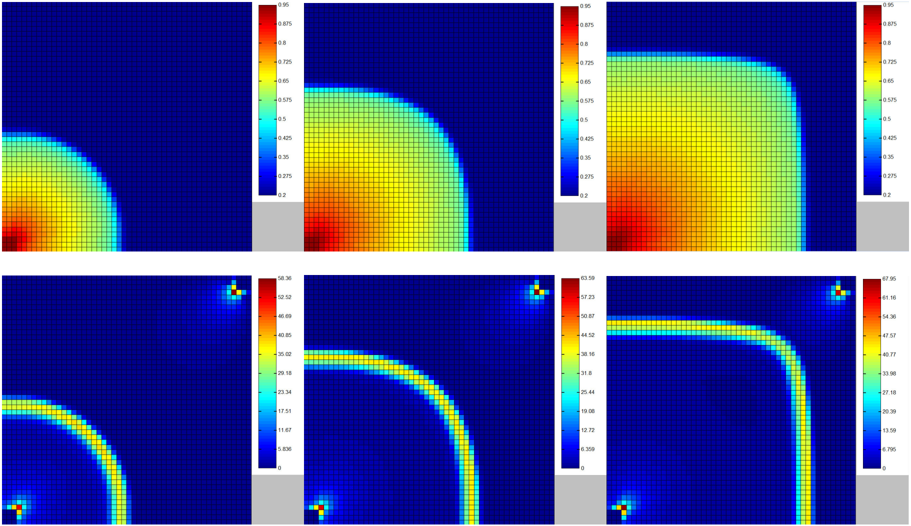
Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

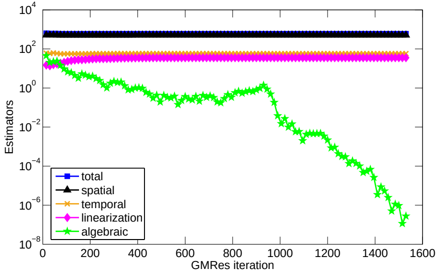
$$
\begin{aligned}
-\eta_{\mathrm{r,w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,\\
p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) &= \bar{p}_{\mathrm{w},K}^{n,k,i},
\end{aligned}
$$

$$
\begin{aligned}
-\eta_{\mathrm{r,n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,\\
p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) &= \pi(s_{\mathrm{w},K}^{n,k,i}) + \bar{p}_{\mathrm{w},K}^{n,k,i}
\end{aligned}
$$

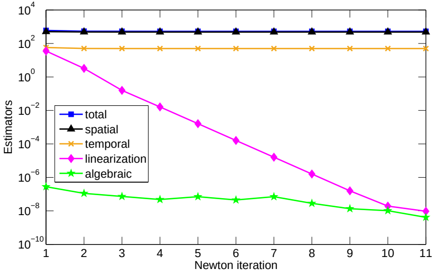
Water saturation/estimators evolution



Estimators and stopping criteria

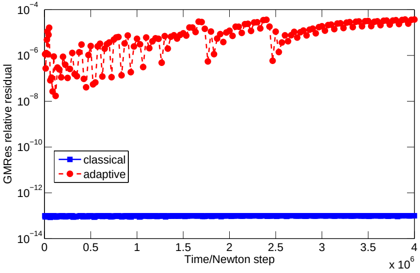


Estimators in function of GMRes iterations

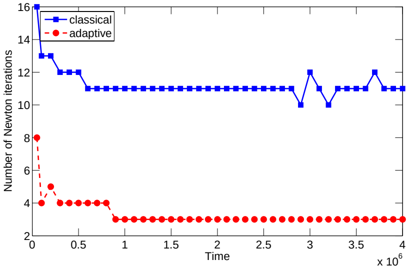


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

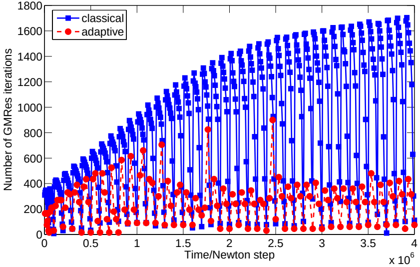


GMRes relative residual

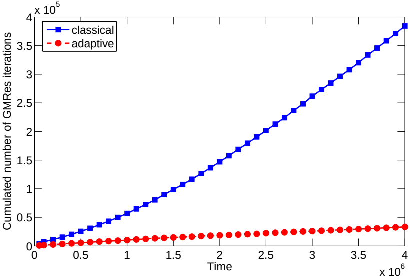


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D$$
$$+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\Big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Fluxes reconstructions

Total fluxes

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1\Big)_e,$$

$$((\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D, 1)_e := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_e,$$

$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu} + \mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu} - (\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

Wetting fluxes

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_e,$$

$$((\mathbf{d}_{\mathrm{w},h}^{n,k,i} + \mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1)_e := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_e,$$

$$\mathbf{a}_{\mathrm{w},h}^{n,k,i} := 0$$

Fluxes reconstructions

Total fluxes

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_e := -\big((\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i}))\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\big)_e,$$

$$((\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D,1)_e := -\big((\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1}))\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\big)_e,$$

$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu}+\mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu}-(\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

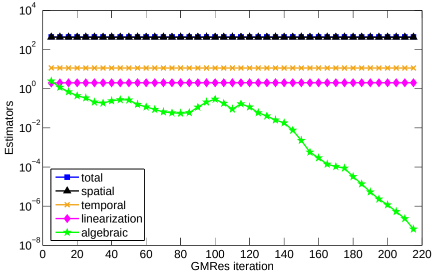
Wetting fluxes

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_e := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_e,$$

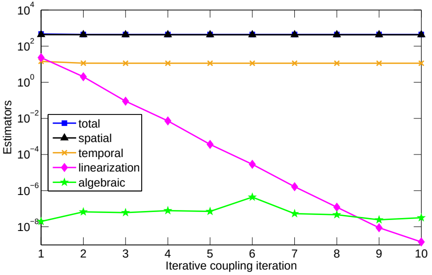
$$((\mathbf{d}_{\mathrm{w},h}^{n,k,i}+\mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1)_e := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_e,$$

$$\mathbf{a}_{\mathrm{w},h}^{n,k,i} := 0$$

Estimators and stopping criteria

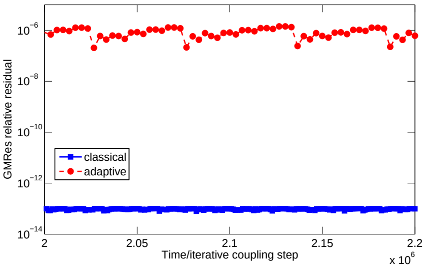


Estimators in function of GMRes iterations

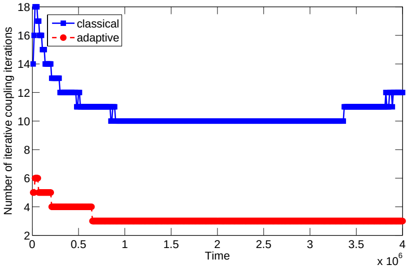


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

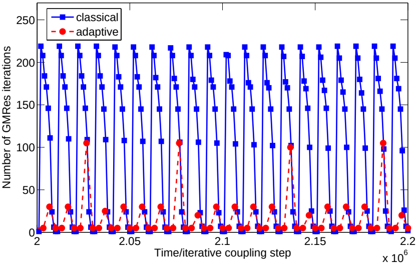


GMRes relative residual

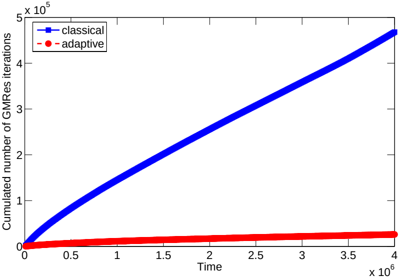


Iterative coupling iterations

GMRes iterations



Per time and iterative coupling step



Cumulated

Outline

Conclusions

Complete adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **optimal** choice of the **regularization parameter**
- **space-time** mesh **adaptivity**
- **“smart online decisions”**: algebraic step / linearization step / regularization / time step refinement / space mesh refinement
- important **computational savings**
- guaranteed and robust upper bound via **a posteriori error estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

Conclusions

Complete adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **optimal** choice of the **regularization parameter**
- **space-time** mesh **adaptivity**
- **“smart online decisions”**: algebraic step / linearization step / regularization / time step refinement / space mesh refinement
- important **computational savings**
- guaranteed and robust upper bound via **a posteriori error estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

Bibliography I

- ERN A., VOHRALÍK M., Adaptive inexact Newton methods for discretizations of nonlinear diffusion PDEs. I. General theory and a posteriori stopping criteria, HAL Preprint 00681422 & Adaptive inexact Newton methods for discretizations of nonlinear diffusion PDEs. II. Applications, HAL Preprint 00681426.
- DI PIETRO D. A., VOHRALÍK M., YOUSEF S., Adaptive regularization, linearization, and discretization and a posteriori error control for the two-phase Stefan problem, HAL Preprint 00690862.
- CANCÈS C., POP I. S., VOHRALÍK M., An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow, accepted for *Math. Comp.*
- VOHRALÍK M., WHEELER M. F., A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows, HAL Preprint 00633594.

Bibliography II

- ERN A., VOHRALÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.* **48** (2010), 198–223.
- JIRÁNEK P., STRAKOŠ Z., VOHRALÍK M., A posteriori error estimates including algebraic error and stopping criteria for iterative solvers. *SIAM J. Sci. Comput.* **32** (2010), 1567–1590.
- EL ALAOUI L., ERN A., VOHRALÍK M., Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems. *Comput. Methods Appl. Mech. Engrg.* **200** (2011), 2782–2795.

Thank you for your attention!