A priori and a posteriori error analysis in $\boldsymbol{H}(\mathrm{curl})$: localization, minimal regularity, and p-optimality

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École des Ponts ParisTech

Outline

The curl–curl problem (current density $\boldsymbol{j} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

The curl–curl problem

Find the magnetic vector potential $\boldsymbol{A} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ *such that*

$$
\begin{aligned}
\nabla \times (\nabla \times \boldsymbol{A}) &= \boldsymbol{j}, \quad \nabla \cdot \boldsymbol{A} = 0 \qquad && \text{in } \Omega, \\
\boldsymbol{A} \times \boldsymbol{n}_\Omega &= \boldsymbol{0}, && \text{on } \Gamma_\mathrm{D}, \\
(\nabla \times \boldsymbol{A}) \times \boldsymbol{n}_\Omega &= \boldsymbol{0}, \quad \boldsymbol{A} \cdot \boldsymbol{n}_\Omega = 0 && \text{on } \Gamma_\mathrm{N}.
\end{aligned}
$$

Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$
(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).
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Find the magnetic vector potential $\boldsymbol{A} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ *such that*

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{A}) &= \boldsymbol{j}, \quad \nabla\cdot\boldsymbol{A} = 0 && \text{in } \Omega, \\ \boldsymbol{A}\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, && \text{on } \Gamma_{\mathrm{D}}, \\ (\nabla\times\boldsymbol{A})\times\boldsymbol{n}_\Omega &= \boldsymbol{0}, \quad \boldsymbol{A}\cdot\boldsymbol{n}_\Omega = 0 && \text{on } \Gamma_{\mathrm{N}}. \end{aligned}$$

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Three key Sobolev spaces

$H^1(\Omega)$

scalar-valued $L^2(\Omega)$ functions with weak gradients in $\boldsymbol{L}^2(\Omega)$,
$H^1(\Omega) := \{v \in L^2(\Omega); \nabla v \in \boldsymbol{L}^2(\Omega)\}$

$\boldsymbol{H}(\mathrm{curl}, \Omega)$

vector-valued $\boldsymbol{L}^2(\Omega)$ functions with weak curls in $\boldsymbol{L}^2(\Omega)$,
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$\boldsymbol{H}(\mathrm{div}, \Omega)$

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Three key Sobolev spaces with BCs

$H^1_{0,\mathrm{N}}(\Omega)$

$H^1_{0,\mathrm{N}}(\Omega) := \{v \in H^1(\Omega);\ v = 0 \text{ on } \Gamma_\mathrm{N}\}$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$

$\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega);\ \boldsymbol{v} \times \boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N}$ in appropriate sense$\}$

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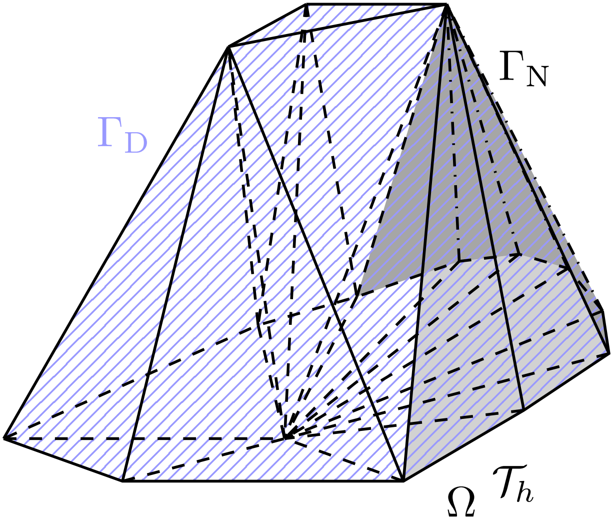
Meshes, elements, and patches



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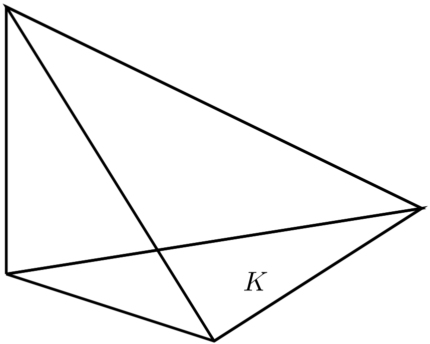

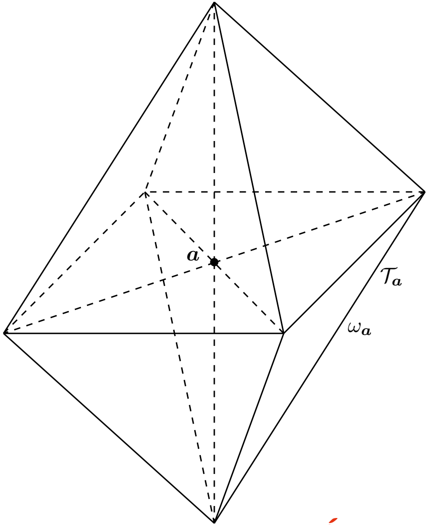

Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$

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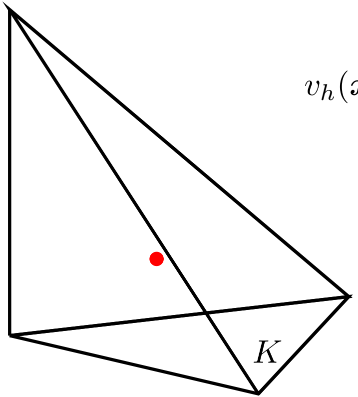

$$v_h(\boldsymbol{x}) = a$$

$$p = 0$$

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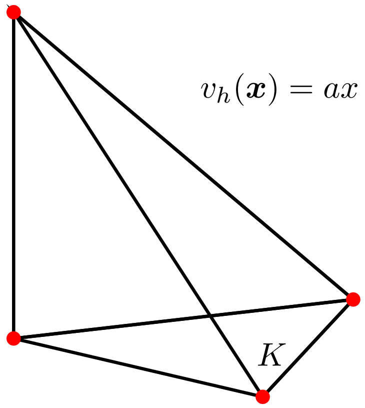
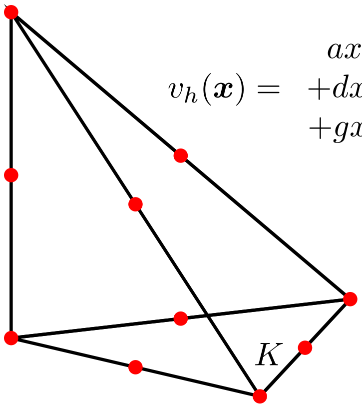


$$v_h(\boldsymbol{x}) = ax + by + cz + d$$

$p = 1$

Polynomial space $\mathcal{P}_p(K)$, $p \geq 0$



$$v_h(\boldsymbol{x}) = \begin{array}{l} ax^2 + by^2 + cz^2 \\ +dxy + eyz + fxz \\ +gx + hy + iz + j \end{array}$$

$p = 2$

Lagrange piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$, $p \geq 1$

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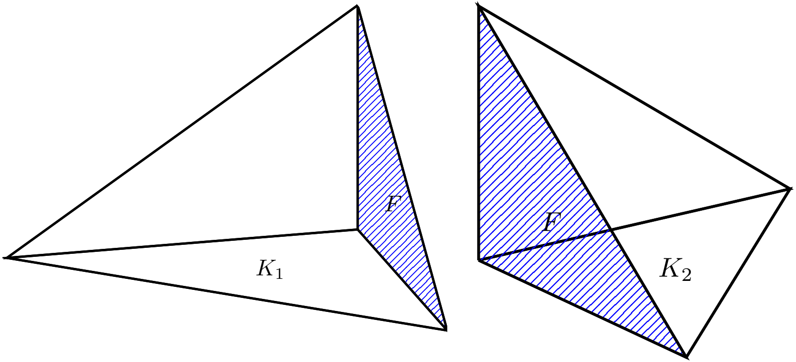

- $v \in H^1(K_1 \cup K_2)$ iff $v \in H^1(K_1)$, $v \in H^1(K_2)$, and $(v|_{K_1})|_F = (v|_{K_2})|_F$
- $\Rightarrow$ ensure this by putting sufficient DoFs at the face F

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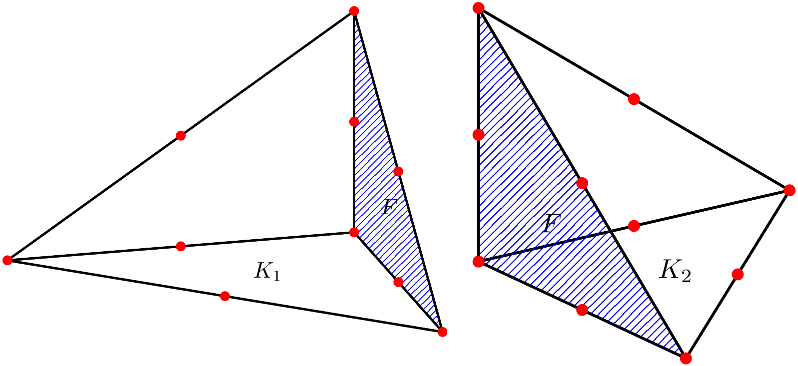

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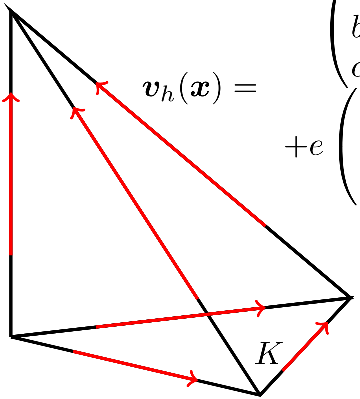
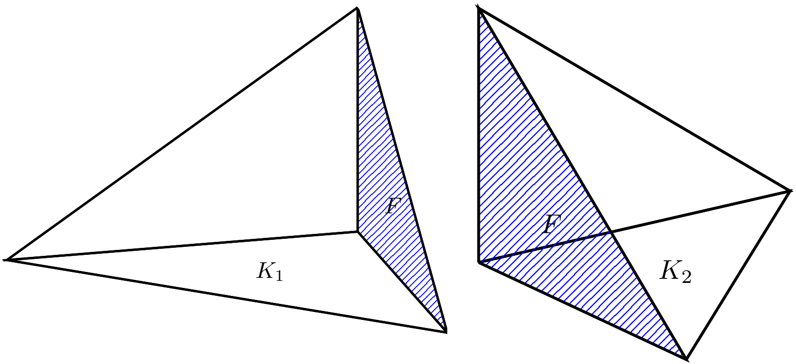


$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} 0 \\ z \\ -y \end{pmatrix} + e \begin{pmatrix} -z \\ 0 \\ x \end{pmatrix} + f \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

$p = 0$

Nédélec piecewise polynomial space $\mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$, $p \geq 0$



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_2)$, and $(\boldsymbol{v}|_{K_1} \times \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \times \boldsymbol{n}_F)|_F$ in appropriate sense
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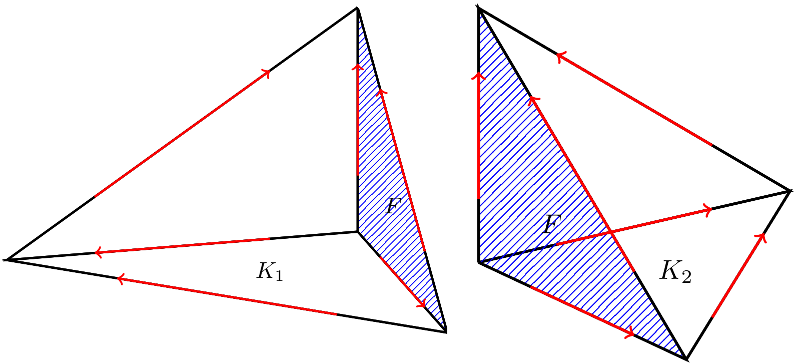

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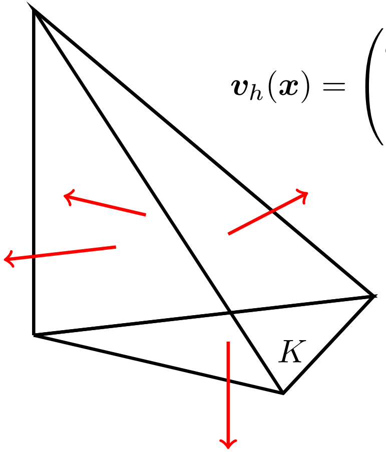
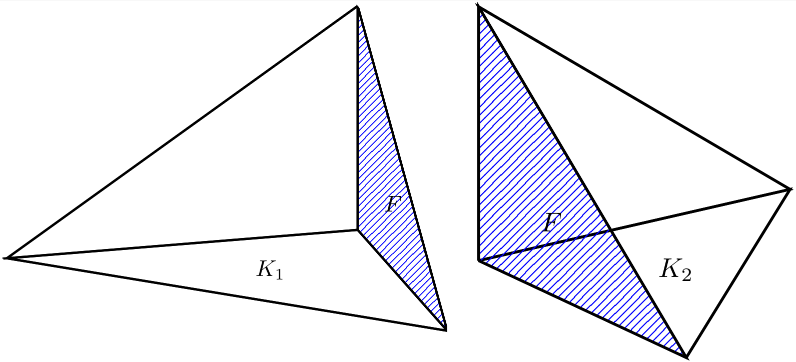


$$\boldsymbol{v}_h(\boldsymbol{x}) = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + d \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$p = 0$

Raviart–Thomas piecewise polynomial space $\mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega)$, $p \geq 0$



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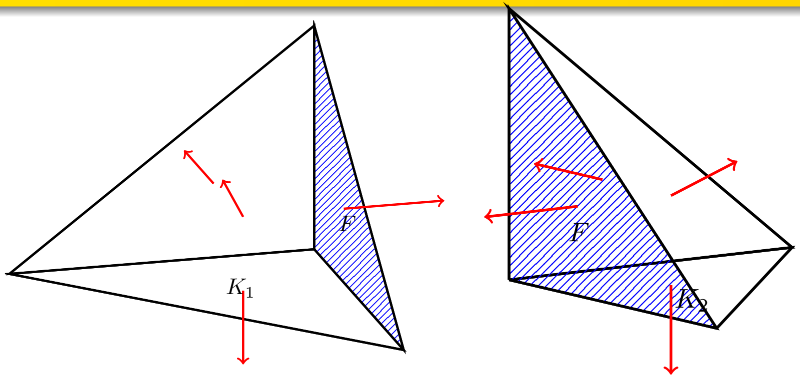

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Outline

1. Introduction
2. Approximation error estimates
3. A posteriori error estimates
4. Local-best–global-best equivalence
   - Context
   - Equivalence
5. A stable local commuting projector
   - Commuting de Rham diagram, wishlist, and context
   - A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$
6. Equilibration in $\boldsymbol{H}(\mathrm{curl})$
   - Patchwise equilibration
   - Main tool: stable (broken) $\boldsymbol{H}(\mathrm{curl})$ polynomial extensions
7. Numerical illustration
8. Conclusions

Outline

Approximation error estimates: context

h approximation estimate

Let $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \cap \boldsymbol{H}^s(\Omega)$*,* $s > 1/2$*. Then*

$$\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl},\Omega)} \|\boldsymbol{v} - \boldsymbol{v}_h\| \le C(\kappa_{\mathcal{T}_h}, s, p) \quad h^{\min\{p+1,s\}} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(\Omega)}.$$

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- Monk (1994, rectangular meshes)

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Approximation error estimates

Theorem (Local *hp*-optimal approximation under minimal Sobolev regularity)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ *with*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^s(K), \quad (\nabla\times\boldsymbol{v})|_K \in \boldsymbol{H}^t(K) \qquad \forall K \in \mathcal{T}_h$$

for $s \geq 0$ *and* $s \geq t \geq \max\{0, s-1\}$. *Then*

$$\min_{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)} \left[\|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla\times(\boldsymbol{v}-\boldsymbol{v}_h)\|_K \right)^2 \right]$$

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Comments

- *hp* case: $\Gamma_\mathrm{D} = \emptyset$ and convex patch subdomains $\omega_{\boldsymbol{a}}$ for all vertices

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Outline

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Reliability

$$\underbrace{\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\text{unknown error}} \leq C \underbrace{\eta}_{\text{computable estimator}}$$

Residual estimates (unknown constant C)

- Monk (1998)
- Beck, Hiptmair, Hoppe, & Wohlmuth (2000)
- Nicaise & Creusé (2003)

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Guaranteed upper bound via $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ s.t. $\nabla \times \boldsymbol{h}_h = \boldsymbol{j}$

$$\underbrace{\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}} \leq \underbrace{\|\nabla \times \boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\textit{computable estimator}}$$

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

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Guaranteed upper bound via $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ s.t. $\nabla \times \boldsymbol{h}_h = \boldsymbol{j}$

$$\underbrace{\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}} \leq \underbrace{\|\nabla \times \boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\textit{computable estimator}}$$

Functional estimates (**global** flux construction)

- Repin (2007)
- Hannukainen (2008)
- Neittaanmäki & Repin (2010)

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Guaranteed upper bound and efficiency via $\boldsymbol{h}_h \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ s.t. $\nabla \times \boldsymbol{h}_h = \boldsymbol{j}$

$$\underbrace{\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}} \leq \underbrace{\|\nabla \times \boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\textit{computable estimator}} \lesssim \underbrace{\|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}}$$

Equlibrated estimates (**local** flux construction)

- Braess & Schöberl (2008): lowest-order case $p = 0$
- Licht (2019): a conceptual discussion
- Gedicke, Geevers, & Perugia (2020): equilibrated-residual-style construction
- Gedicke, Geevers, Perugia, & Schöberl (2021): p-robust modification
- Ern, Chaumont-Frelet, Vohralík (2021): p-robust broken patchwise equil.

A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

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A posteriori error estimates ($\boldsymbol{j} \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{j} = 0$)

Nédélec finite element discretization

$\boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$; $\boldsymbol{A}_h \in \boldsymbol{V}_h$ *satisfies*

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A posteriori error estimates

Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Nédélec finite element discretization

$\boldsymbol{A}_h \in \boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$, *satisfies*

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

$\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ s.t. $\nabla\times\boldsymbol{h}_h = \boldsymbol{j}$: local equilibrated flux reconstruction

Theorem (Guaranteed upper bound, efficiency, and p-robustness)

$$\underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\textit{unknown error}} \leq \underbrace{\|\nabla\times\boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\textit{computable estimator}} \leq C(\kappa_{\mathcal{T}_h}) \underbrace{\|\nabla\times(\boldsymbol{A}-\boldsymbol{A}_h)\|}_{\textit{unknown error}}$$

A posteriori error estimates

Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ *satisfies*

$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Nédélec finite element discretization

$\boldsymbol{A}_h \in \boldsymbol{V}_h := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$, $p \geq 0$, *satisfies*

$$(\nabla\times\boldsymbol{A}_h, \nabla\times\boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

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A posteriori error estimates

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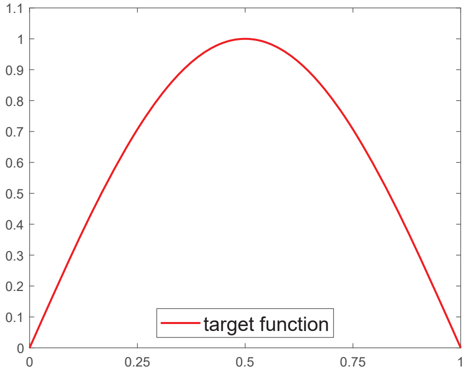
Theorem (Guaranteed upper bound, efficiency, and p-robustness)

$$\underbrace{\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}} \leq \underbrace{\|\nabla\times\boldsymbol{A}_h - \boldsymbol{h}_h\|}_{\textit{computable estimator}} \leq C(\kappa_{\mathcal{T}_h}) \underbrace{\|\nabla\times(\boldsymbol{A} - \boldsymbol{A}_h)\|}_{\textit{unknown error}}$$

Outline

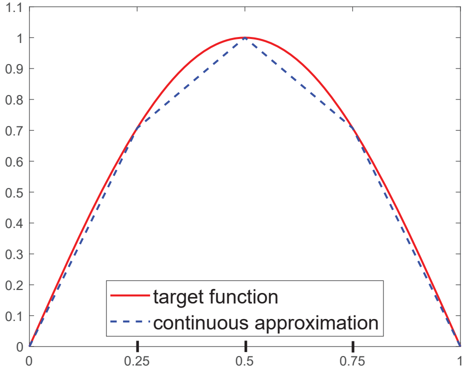
Outline

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**



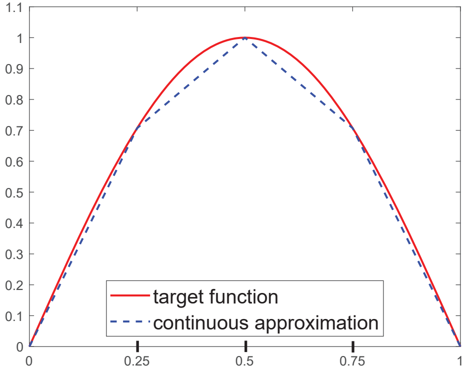
Target function in $H_0^1(\Omega)$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$: **1D**

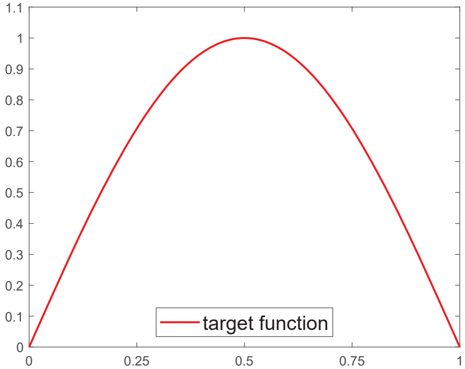


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

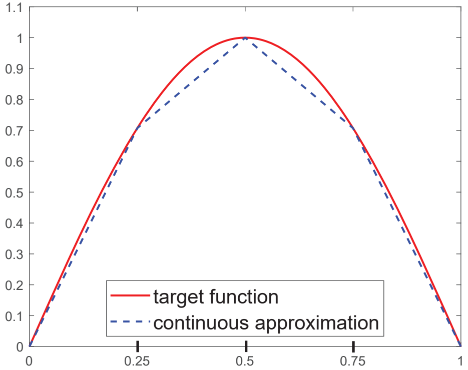


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

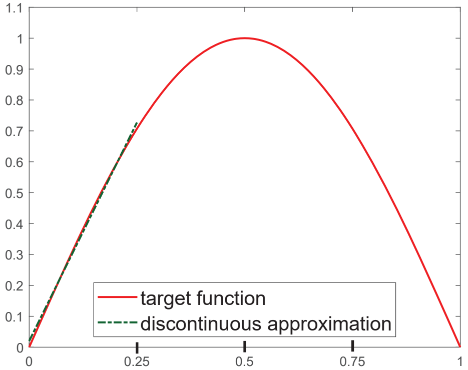


Target function in $H_0^1(\Omega)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

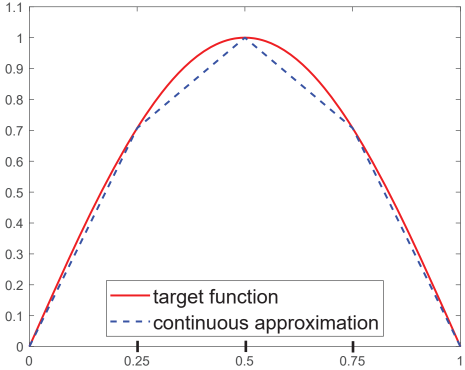


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

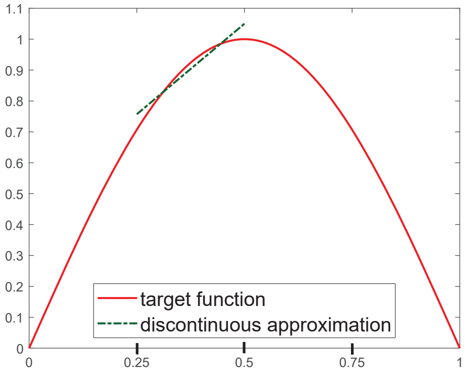


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

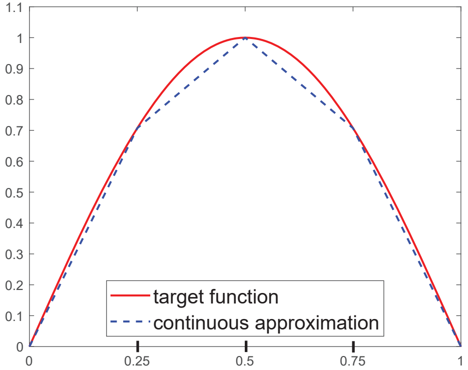


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

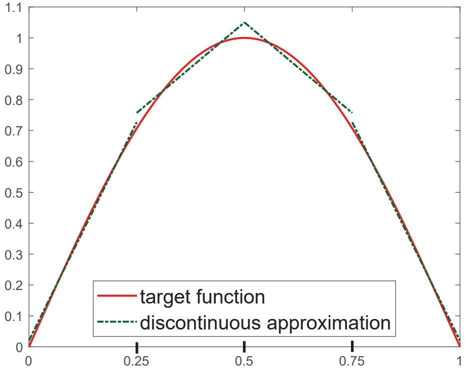


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$: **1D**

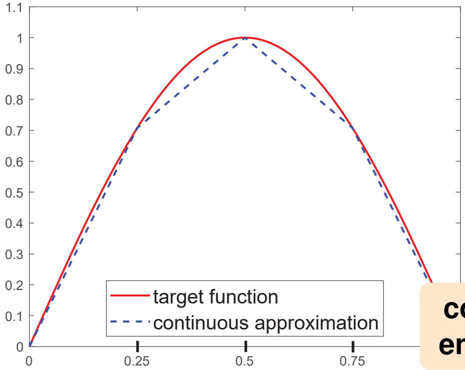


Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H_0^1(\Omega)$

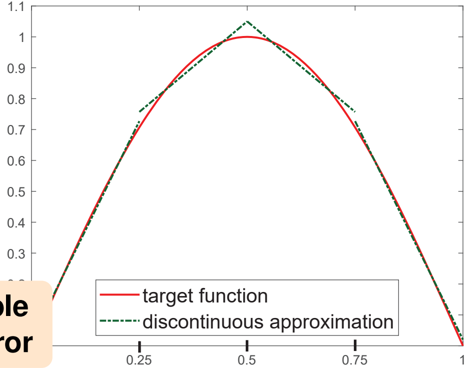


Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$: 1D



comparable energy error



Approximation by **continuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h) \cap H^1_0(\Omega)$

Approximation by **discontinuous** piecewise polynomials in $\mathcal{P}_1(\mathcal{T}_h)$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

$$\textit{bigger} \approx \textit{smaller}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

$$\min_{\textit{smaller space}} \approx \min_{\textit{bigger space}}$$

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

$$\min_{\textit{CG space}} \approx \min_{\textit{DG space}}$$

Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

Let $u \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint}}}.$$

- $\approx_p$: up to a generic constant that only depends on the shape-regularity of the mesh $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint}}}.$$

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Equivalence of local- and global-best approximations in $H^1_0(\Omega)$

Equivalence in H^1_0, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

Let $u \in H^1_0(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|^2_K}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint}}}.$$

- $\approx_p$: up to a generic constant that only depends on the shape-regularity of the mesh $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Equivalence of local- and global-best approximations in $H_0^1(\Omega)$

Equivalence in H_0^1, Carstensen, Peterseim, Schedensack (2012), Aurada, Feischl, Kemetmüller, Page, Praetorius (2013), Veeser (2016)

Let $u \in H_0^1(\Omega)$ and $p \geq 1$ be arbitrary. Then,

$$\underbrace{\min_{v_h \in \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)} \|\nabla(u - v_h)\|^2}_{\substack{\text{global-best on } \Omega \\ \text{trace-continuity constraint}}} \approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(u - v_h)\|_K^2}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no trace-continuity constraint}}}.$$

- $\approx_p$: up to a generic constant that only depends on the shape-regularity of the mesh $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Outline

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

$$\textit{bigger} \approx \textit{smaller}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\text{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\text{curl})$)

$$\min_{\textit{smaller space } \textbf{with curl constraints}} \approx \min_{\textit{bigger space } \textbf{without curl constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

$$\min_{\text{conforming Nédélec space with curl constraints}} \approx \min_{\text{broken Nédélec space without curl constraints}}$$

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times \boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p (\nabla \times \boldsymbol{v})\|_K \right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \text{curl constraint}}}$$

$$\approx_p \sum_{K \in \mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{N}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla \times \boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p (\nabla \times \boldsymbol{v})\|_K \right)^2 \right]}_{\substack{\text{local-best on each } K \in \mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}}.$$

- $\approx_p$: only depends on shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ *and* $p \geq 0$ *be arbitrary. Then,*

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \Pi^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K \right)^2}_{\text{global-best on } \Omega,\ \text{tangential-trace-continuity constraint},\ \text{curl constraint}}$$

global-best on Ω
tangential-trace-continuity constraint
curl constraint

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \Pi^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K \right)^2 \right]}_{\text{local-best on each } K\in\mathcal{T}_h}.$$

local-best on each $K \in \mathcal{T}_h$
no tangential-trace-continuity constraint
no curl constraint

- $\approx_p$: only depends on shape-regularity $\kappa_{\mathcal{T}_h}$ and the polynomial degree p

Global-best approximation ≈ local-best approximation in $\boldsymbol{H}(\mathrm{curl})$

Theorem (Constrained equivalence in $\boldsymbol{H}(\mathrm{curl})$)

Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ and $p \geq 0$ be arbitrary. Then,

$$\underbrace{\min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \Pi^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_K \right)^2}_{\substack{\text{global-best on } \Omega \\ \text{tangential-trace-continuity constraint} \\ \text{curl constraint}}}$$

$$\approx_p \sum_{K\in\mathcal{T}_h} \underbrace{\left[\min_{\boldsymbol{v}_h \in \mathcal{N}_p(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \Pi^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_K \right)^2 \right]}_{\substack{\text{local-best on each } K\in\mathcal{T}_h \\ \text{no tangential-trace-continuity constraint} \\ \text{no curl constraint}}}.$$

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Outline

Outline

Commuting de Rham diagram and wishlist for $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Commuting de Rham diagram

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\downarrow P_h^{p+1,\mathrm{grad}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \mathcal{N}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \mathcal{RT}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

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\big\downarrow P_h^{p+1,\mathrm{grad}} & & \big\downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \big\downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \big\downarrow \Pi_h^p \\
\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
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\end{array}$$

- $\boldsymbol{P}_h^{p,\mathrm{div}}$: Ern, Gudi, Smears, Vohralík (2022)

Commuting de Rham diagram and wishlist for $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Commuting de Rham diagram

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\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
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\mathcal{P}_{p+1}(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega)
\end{array}
$$

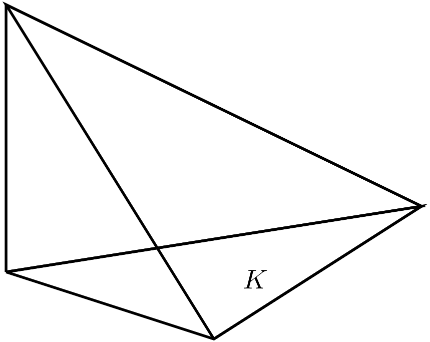
Requirements on $\boldsymbol{P}_h^{p,\mathrm{curl}}$

1. be defined over the **entire** infinite-dimensional space $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$
2. be defined **locally** (in neighborhood of mesh elements)
3. be defined simply (starting from elementwise polynomial projections)
4. have optimal approximation properties, that of **elementwise curl-unconstrained L^2-orthogonal projector** (local–global equivalence)
5. be **stable in $L^2(\Omega)$** (up to data oscillation)
6. satisfy the **commuting properties** expressed by the arrows
7. be **projector**, i.e., leave intact piecewise polynomials

Stable local commuting projectors defined on $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$

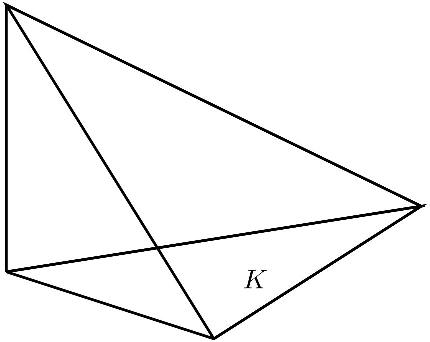
- Schöberl (2001, 2005): not local
- Christiansen and Winther (2008): not local
- Bespalov and Heuer (2011): low regularity but still not $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$
- Falk and Winther (2014): local and $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$-stable but not $\boldsymbol{L}^2$-stable
- Ern and Guermond (2016): not local
- Ern and Guermond (2017): $\boldsymbol{H}(\mathrm{div})/\boldsymbol{H}(\mathrm{curl})$ regularity but not commuting
- Licht (2019): essential boundary conditions on part of $\partial\Omega$
- Arnold and Guzmán (2021): $\boldsymbol{L}^2$-stable

Classical elementwise interpolation



- $\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2$
- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega) \Rightarrow \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{curl}, K) \Rightarrow$ so interpolate $\boldsymbol{v}|_K$

Classical elementwise interpolation



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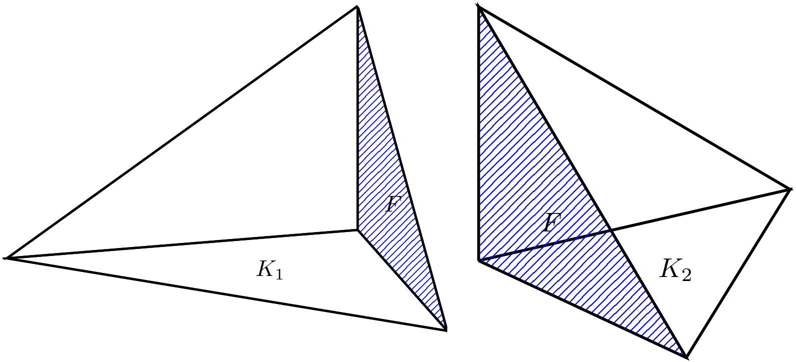
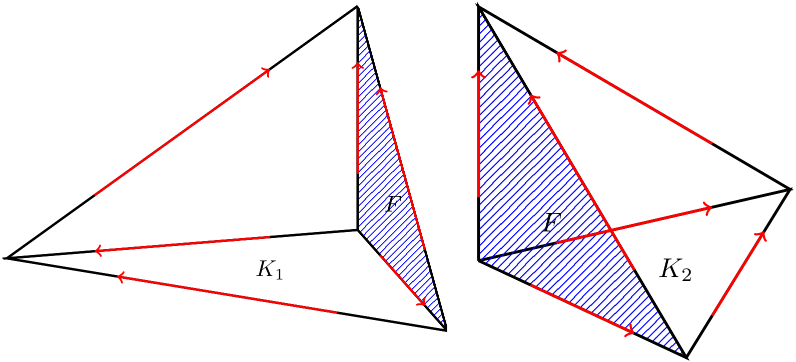
Classical elementwise interpolation: conformity enforcement



- $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1 \cup K_2)$ iff $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_1)$, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K_2)$, and $(\boldsymbol{v}|_{K_1} \times \boldsymbol{n}_F)|_F = (\boldsymbol{v}|_{K_2} \times \boldsymbol{n}_F)|_F$ in appropriate sense ($p = 0$)

Classical elementwise interpolation: conformity enforcement



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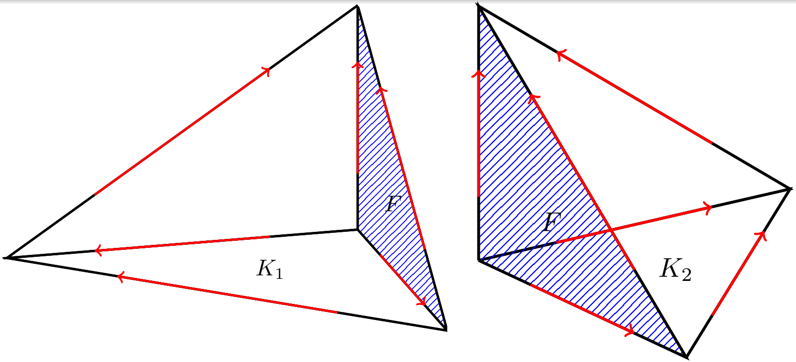

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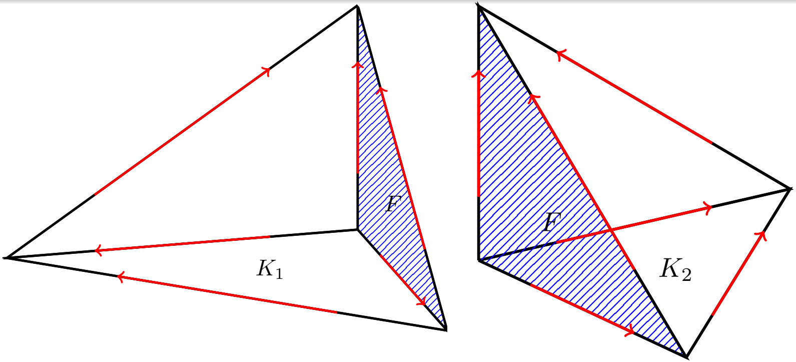
Clash

Edge integrals not available in $\boldsymbol{H}(\mathrm{curl})$.

Classical elementwise interpolation: conformity enforcement



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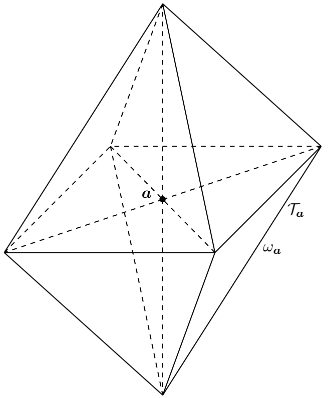
Conclusion

Not a single tetrahedron $K \in \mathcal{T}_h$ if the minimal regularity $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$ requested.

Classical patchwise interpolation (Clément)

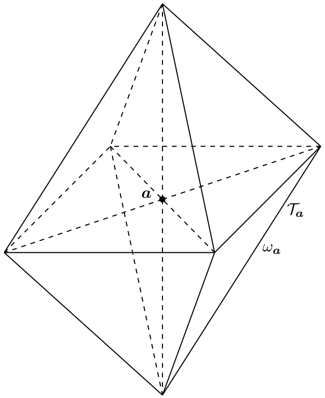


- some local-best polynomial approximation on $\omega_{\boldsymbol{a}}$
- values on $\omega_{\boldsymbol{a}}$ as coefficients for basis functions supported on $\omega_{\boldsymbol{a}}$

Conclusion

Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

Classical patchwise interpolation (Clément)



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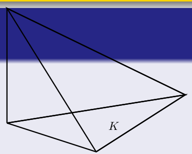
Allows the minimal regularity but breaks the projection property, the elementwise structure, and the commuting diagram.

Outline

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Definition (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)



Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ be given (minimal regularity).

1. For each $K \in \mathcal{T}_h$, prepare the datum $\boldsymbol{\tau}_h|_K$

$$\boldsymbol{\tau}_h|_K := \arg \min_{\substack{\boldsymbol{w}_h \in \mathcal{RT}_p(K) \\ \nabla\cdot\boldsymbol{w}_h = 0}} \|\nabla\times\boldsymbol{v} - \boldsymbol{w}_h\|_K$$

   and define $\boldsymbol{\iota}_h|_K$ by the **elementwise (constrained) projection**

$$\boldsymbol{\iota}_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(K) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K$$

   (discrete, tangential-trace discontinuous).

2. Obtain $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ by applying the **flux equilibration procedure** to $\boldsymbol{\iota}_h$; in particular, $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) := \boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}$, where $\boldsymbol{h}_h^{\boldsymbol{a}}$ are obtained by local energy minimizations on the patch subdomains $\omega_{\boldsymbol{a}}$.

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

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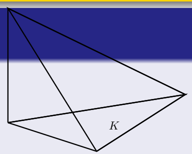

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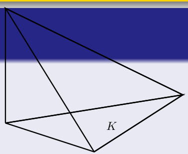

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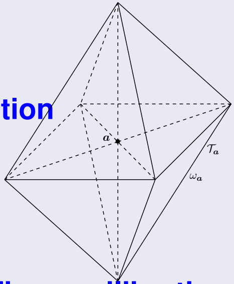

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A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Theorem (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)

$\boldsymbol{P}_h^{p,\mathrm{curl}}$ *is a* **commuting projector** *since*

$$\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) \qquad \forall \boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega),$$

$$\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v}\in\mathcal{N}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega).$$

Moreover, it has **local-best approximation properties** *and is* **L^2 stable** *up to data oscillation, since, for all* $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ *and* $K\in\mathcal{T}_h$*,*

$$\|\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times(\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}))\|_K\right)^2$$

$$\lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\min_{\boldsymbol{v}_h\in\mathcal{N}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\Pi_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\},$$

$$\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 \lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\|\boldsymbol{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\Pi_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}.$$

A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$

Theorem (A stable local commuting projector $\boldsymbol{P}_h^{p,\mathrm{curl}}$)

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$$\begin{aligned} \nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) &= \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) && \forall \boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega),\\ \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) &= \boldsymbol{v} && \forall \boldsymbol{v}\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega).\end{aligned}$$

Moreover, it has **local-best approximation properties** *and is* **L^2 stable** *up to data oscillation, since, for all $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ and $K\in\mathcal{T}_h$,*

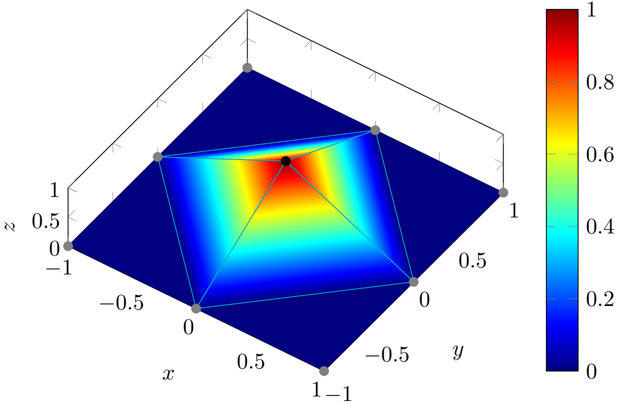
$$\begin{aligned} &\|\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times(\boldsymbol{v}-\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}))\|_K\right)^2\\ &\lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\},\end{aligned}$$

$$\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|_K^2 \lesssim_p \sum_{K'\in\mathcal{T}_K}\left\{\|\boldsymbol{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}.$$

Outline

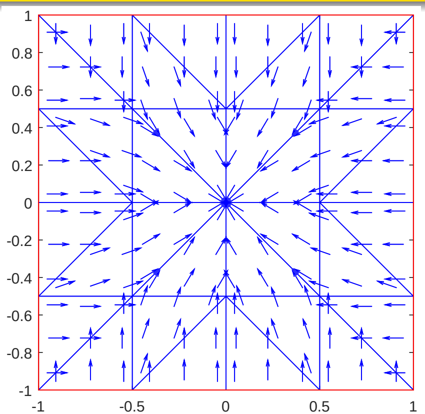
Partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1$$



Hat basis function $\psi^{\boldsymbol{a}}$

Equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{div})$ Destuynder and Métivet (1998), Braess & Schöberl (2008)

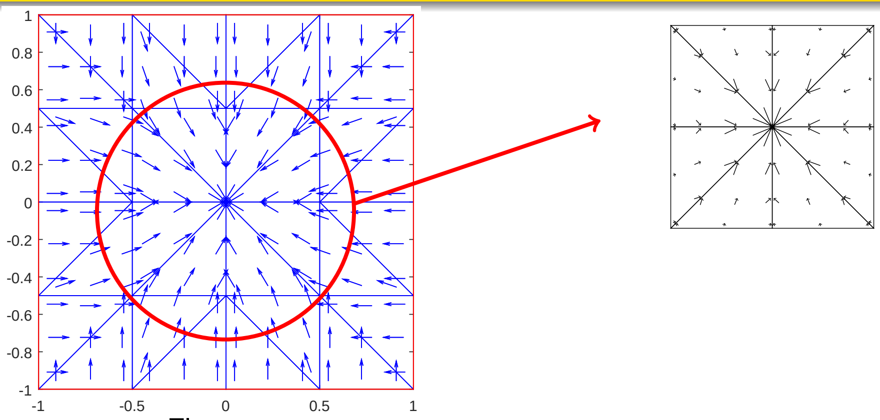


Flux $\boldsymbol{\iota}_h$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0\ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

Equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{div})$

Destuynder and Métivet (1998), Braess & Schöberl (2008)



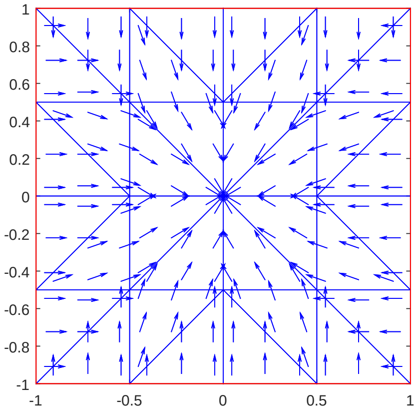
Flux $\boldsymbol{\iota}_h$

$\psi^{\boldsymbol{a}} \boldsymbol{\iota}_h$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f, \psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\iota}_h, \nabla \psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}}$$

Equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{div})$

Destuynder and Métivet (1998), Braess & Schöberl (2008)



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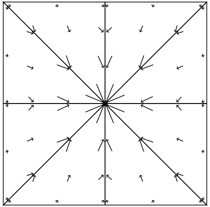

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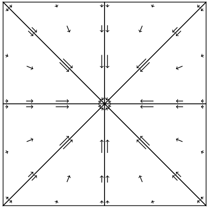

$\boldsymbol{h}_h^{\boldsymbol{a}}$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0 \ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}$$

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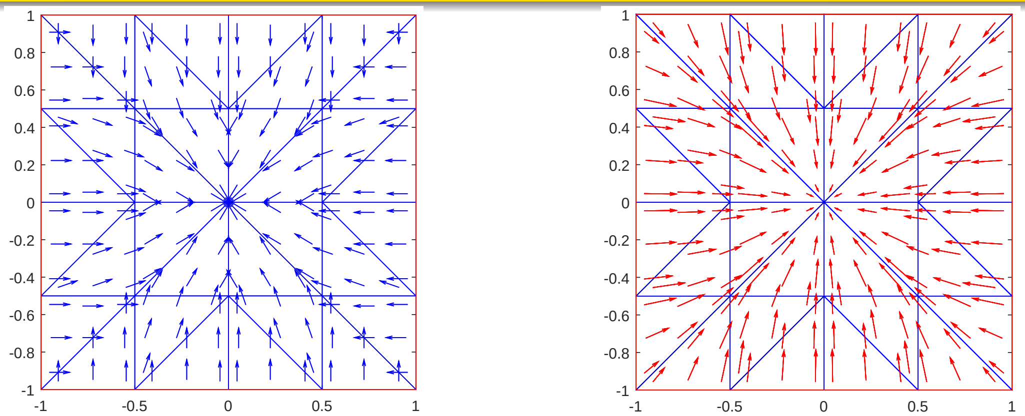

Flux $\boldsymbol{\iota}_h$ Equilibrated flux reconstruction $\boldsymbol{h}_h$

$$\underbrace{\boldsymbol{\iota}_h \in \mathcal{RT}_p(\mathcal{T}_h), f \in \mathcal{P}_{p+1}(\mathcal{T}_h)}_{(f,\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}+(\boldsymbol{\iota}_h,\nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=0 \ \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \to \boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}), \nabla\cdot\boldsymbol{h}_h = f$$

Equilibration – the bottom line

$\boldsymbol{H}$(div)-case

- When there exists $\boldsymbol{v}_h \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ such that $\nabla \cdot \boldsymbol{v}_h = j_h^{\boldsymbol{a}}$?
- When $j_h^{\boldsymbol{a}} \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}})$ and $(j_h^{\boldsymbol{a}}, 1)_{\omega_{\boldsymbol{a}}} = 0$ if $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$.

$\boldsymbol{H}$(curl)-case

- When there exists $\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ such that $\nabla \times \boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}$?
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Outline

Patchwise equilibrated fluxes

Continuous level

- $\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ satisfies
$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v})\ \forall \boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$
- Thus $\nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ with $\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}$.
- Take $\boldsymbol{h}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}) \in \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ and note that $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}^{\boldsymbol{a}} = \nabla\times\boldsymbol{A}$.
- Rewritten implicitly,
$$\boldsymbol{h}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})-\boldsymbol{v}\|^2_{\omega_{\boldsymbol{a}}}$$
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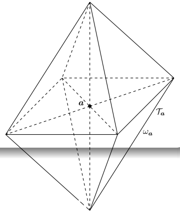

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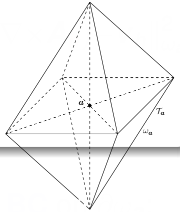

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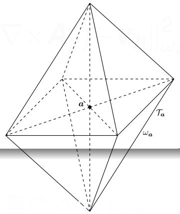
For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the local constrained minimization pb



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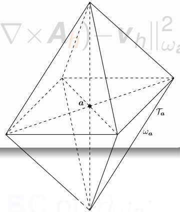

Key points

- **homogeneous tangential BC on** $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
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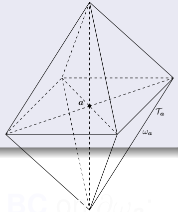

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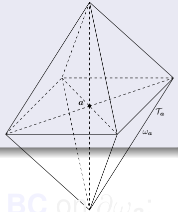

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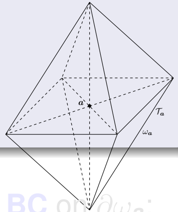

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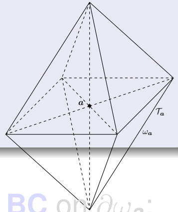

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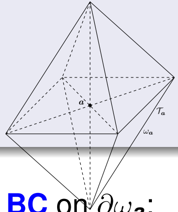


Key points

- **homogeneous tangential BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
- **global equilibrium** $\nabla\times\boldsymbol{h}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\times\boldsymbol{h}_h^{\boldsymbol{a}}$
  $$= \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\psi^{\boldsymbol{a}}\boldsymbol{j} + \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)) = \boldsymbol{j}$$

Patchwise equilibrated fluxes

Continuous level

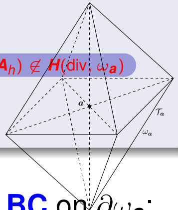
- $\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ satisfies
$$(\nabla\times\boldsymbol{A}, \nabla\times\boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \, \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$
- Thus $\nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ with $\nabla\times(\nabla\times\boldsymbol{A}) = \boldsymbol{j}$.
- Take $\boldsymbol{h}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}) \in \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ and note that $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}^{\boldsymbol{a}} = \nabla\times\boldsymbol{A}$.
- Rewritten implicitly,
$$\boldsymbol{h}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}=\boldsymbol{j}^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A})-\boldsymbol{v}\|^2_{\omega_{\boldsymbol{a}}}$$
with
$$\boldsymbol{j}^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j} + \nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}).$$

Definition (Chaumont-Frelet, Vohralík (2022))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization pb**
$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}_h=\psi^{\boldsymbol{a}}\boldsymbol{j}+\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)}} \|\psi^{\boldsymbol{a}}(\nabla\times\boldsymbol{A}_h)-\boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}$$

- $\psi^{\boldsymbol{a}}\boldsymbol{j} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ but $\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h) \notin \boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



Key points

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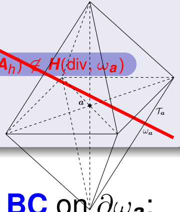
$$\boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



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Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)$ to a Raviart–Thomas space

For all vertices $\boldsymbol{a}\in\mathcal{V}_h$*, consider* $p' := \min\{p,1\}$*-degree patchwise minimizations:*

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})\\ \nabla\cdot\boldsymbol{v}_h=-\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{j}\\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K=(\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h),\boldsymbol{r}_h)_K \quad \forall\boldsymbol{r}_h\in[\mathcal{P}_0(K)]^3,\,\forall K\in\mathcal{T}_{\boldsymbol{a}}}} \|\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)-\boldsymbol{v}_h\|^2_{\omega_{\boldsymbol{a}}}.$$

Comments

- $\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A}_h)\notin\mathcal{RT}_{p'}(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$
- remainder $\boldsymbol{\delta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{\theta}_h^{\boldsymbol{a}}$
  - should be zero ($\sim$ partition of unity $\sum_{\boldsymbol{a}\in\mathcal{V}_h}\{\nabla\psi^{\boldsymbol{a}}\times(\nabla\times\boldsymbol{A})\}=0$), but is not
  - $\boldsymbol{\delta}_h\in\mathcal{RT}_{p'}(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and $\nabla\cdot\boldsymbol{\delta}_h=0$
- additional orthogonality constraint
  - crucial for stage 2 below
  - only possible thanks to the lowest-order Galerkin orthogonality of $\boldsymbol{A}_h$
  - requests $\min\{p,1\}$ (and not simply p)

Stage 1: overconstrained Raviart–Thomas projection

Projection of $\nabla\psi^{\mathbf{a}} \times (\nabla \times \boldsymbol{A}_h)$ to a Raviart–Thomas space

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- $\nabla\psi^{\mathbf{a}} \times (\nabla \times \boldsymbol{A}_h) \notin \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_{\mathbf{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\mathbf{a}})$
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Stage 2: **divergence-free decomposition** of the given divergence-free Raviart-Thomas piecewise polynomial $\boldsymbol{\delta}_h$

Divergence-free decomposition of $\boldsymbol{\delta}_h$

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$-degree elementwise minimizations:

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_1(K) \\ \nabla \cdot \boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = \boldsymbol{I}^1_{\mathcal{RT}}(\psi^{\boldsymbol{a}} \boldsymbol{\delta}_h) \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \boldsymbol{I}^1_{\mathcal{RT}}(\psi^{\boldsymbol{a}} \boldsymbol{\delta}_h)\|_K^2 \quad \forall \boldsymbol{a} \in \mathcal{V}_K \text{ when } p = 0,$$

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_{p+1}(K) \\ \nabla \cdot \boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = \psi^{\boldsymbol{a}} \boldsymbol{\delta}_h \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \psi^{\boldsymbol{a}} \boldsymbol{\delta}_h\|_K^2 \quad \forall \boldsymbol{a} \in \mathcal{V}_K \text{ when } p \geq 1.$$

Comments

- patchwise contributions

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla \cdot \boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h$$

Stage 2: **divergence-free decomposition** of the given divergence-free Raviart-Thomas piecewise polynomial δ_h

Divergence-free decomposition of δ_h

For all tetrahedra $K \in \mathcal{T}_h$, consider $(p+1)$-degree elementwise minimizations:

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Comments

- patchwise contributions

$$\delta_h^{\boldsymbol{a}} \in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla\cdot\delta_h^{\boldsymbol{a}} = 0 \quad \forall \boldsymbol{a} \in \mathcal{V}_h$$

- $\delta_h^{\boldsymbol{a}}$ form a **divergence-free decomposition** of δ_h: $\delta_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \delta_h^{\boldsymbol{a}}$

Stage 2: **divergence-free decomposition** of the given divergence-free Raviart-Thomas piecewise polynomial δ_h

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Comments

- patchwise contributions

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Comments

- patchwise contributions

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Stage 2: divergence-free decomposition of the given divergence-free current density $\boldsymbol{j}$

Divergence-free decomposition of the current density $\boldsymbol{j}$

Set

$$\boldsymbol{j}_h^{\boldsymbol{a}} := \psi^{\boldsymbol{a}}\boldsymbol{j} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}.$$

Then

$$\begin{aligned} \boldsymbol{j}_h^{\boldsymbol{a}} &\in \mathcal{RT}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}), \\ \nabla\cdot\boldsymbol{j}_h^{\boldsymbol{a}} &= 0, \\ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} &= \boldsymbol{j}. \end{aligned}$$

Stage 3: discrete patchwise equilibrated fluxes

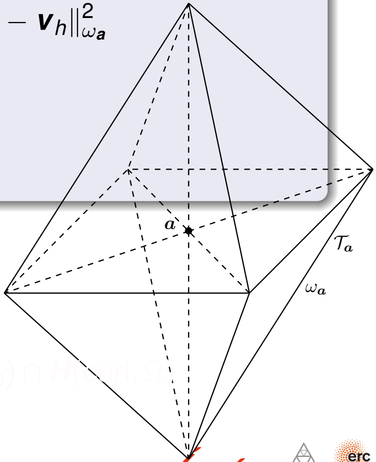
Definition (Chaumont-Frelet, Vohralík (2021))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

and combine

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



Key points

- **homogeneous tangential BC** on $\partial\omega_{\boldsymbol{a}}$: $\boldsymbol{h}_h \in \mathcal{N}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}, \Omega)$
- **global equilibrium** $\nabla \times \boldsymbol{h}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \times \boldsymbol{h}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}$

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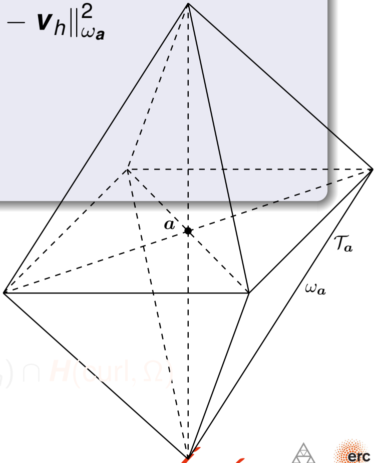

Key points

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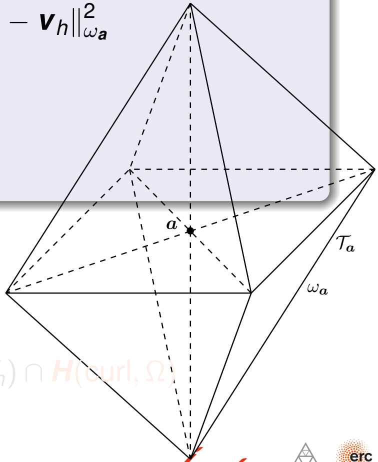

Key points

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- **global equilibrium** $\nabla \times \boldsymbol{h}_h = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \nabla \times \boldsymbol{h}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{j}_h^{\boldsymbol{a}} = \boldsymbol{j}$

Stage 3: discrete patchwise equilibrated fluxes

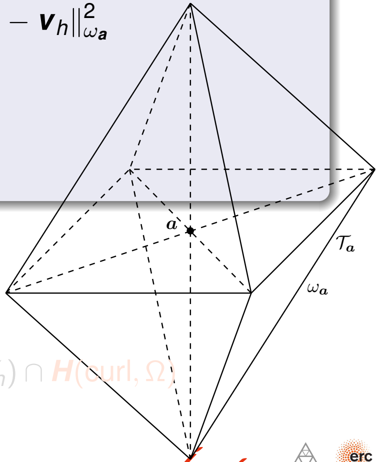
Definition (Chaumont-Frelet, Vohralík (2021))

For each vertex $\boldsymbol{a} \in \mathcal{V}_h$, solve the **local constrained minimization problem**

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}}) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{j}_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}(\nabla \times \boldsymbol{A}_h) - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}}^2$$

and combine

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}.$$



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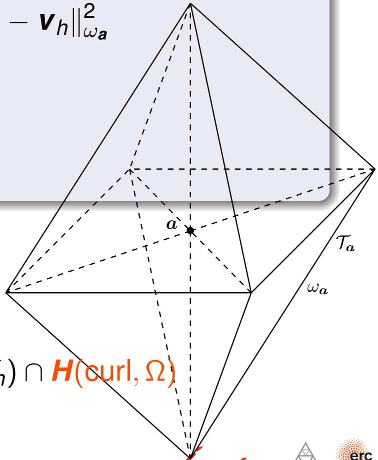

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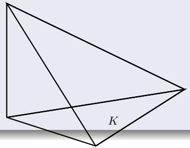
Outline

$\boldsymbol{H}(\mathrm{curl})$ polynomial extensions on a tetrahedron

Theorem ($\boldsymbol{H}(\mathrm{curl})$ polynomial extension on a single tetrahedron Costabel & Mc-Intosh (2010); Demkowicz, Gopalakrishnan, & Schöberl (2009); Braess, Pillwein, & Schöberl (2009); Chaumont-Frelet, Ern, & Vohralík (2020)**)**

Let $\emptyset \subseteq \mathcal{F} \subseteq \mathcal{F}_K$ be a (sub)set of faces of a tetrahedron K. Then, for every polynomial degree $p \geq 0$, for all $\boldsymbol{r}_K \in \mathcal{RT}_p(K)$ such that $\nabla \cdot \boldsymbol{r}_K = 0$, and for all $\boldsymbol{r}_{\mathcal{F}} \in \mathcal{N}_p^{\tau}(\Gamma_{\mathcal{F}})$ such that $\boldsymbol{r}_K \cdot \boldsymbol{n}_F = \mathrm{curl}_F(\boldsymbol{r}_F)$ for all $F \in \mathcal{F}$, there holds



$$\min_{\substack{\boldsymbol{v}_p \in \mathcal{N}_p(K) \\ \nabla \times \boldsymbol{v}_p = \boldsymbol{r}_K \\ \boldsymbol{v}_p|^{\tau}_{\mathcal{F}} = \boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}_p\|_K \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K) \\ \nabla \times \boldsymbol{v} = \boldsymbol{r}_K \\ \boldsymbol{v}|^{\tau}_{\mathcal{F}} = \boldsymbol{r}_{\mathcal{F}}}} \|\boldsymbol{v}\|_K.$$

Comments

- C_{st} only depends on the shape-regularity of K
- p-robustness: for (pw) p-polynomial data $\boldsymbol{r}_K$, $\boldsymbol{r}_{\mathcal{F}}$, minimization over $\mathcal{N}_p(K)$ is up to C_{st} as good as minimization over the entire $\boldsymbol{H}(\mathrm{curl}, K)$
- extension to an edge patch: Chaumont-Frelet, Ern, & Vohralík (2021)
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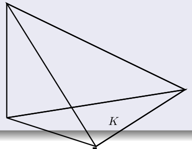

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$\boldsymbol{H}$(curl) polynomial extensions on a tetrahedron and on patches

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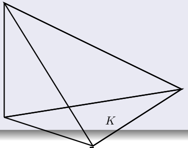

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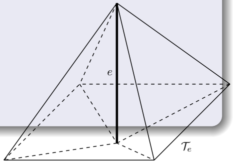

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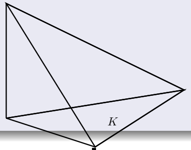

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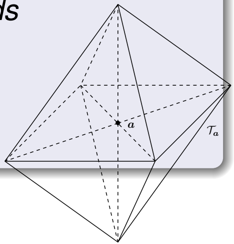

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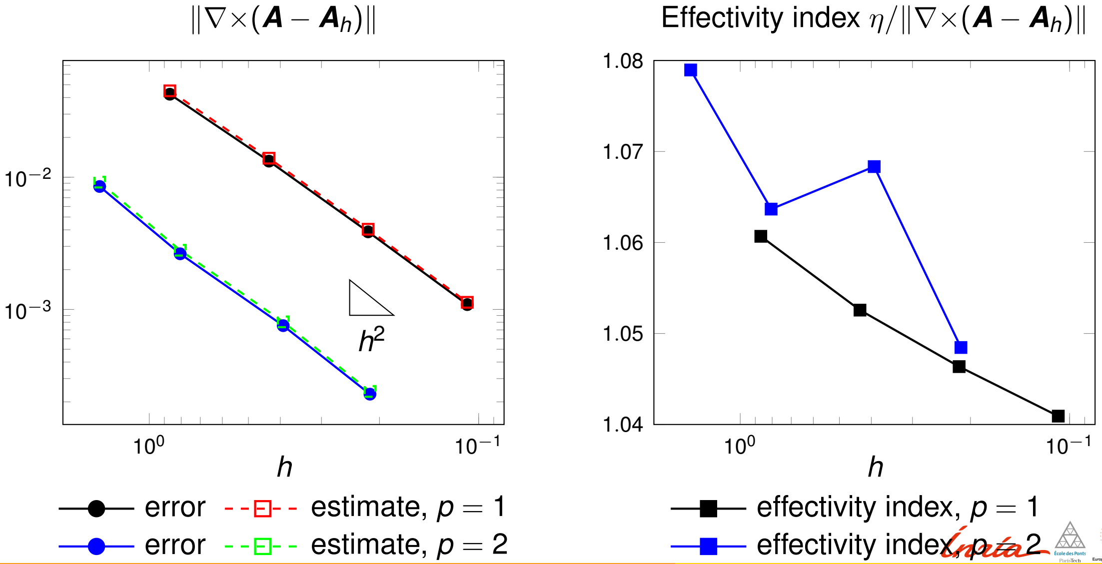
Outline

Patchwise equilibration, $\boldsymbol{H}^3$ solution, h-refinement



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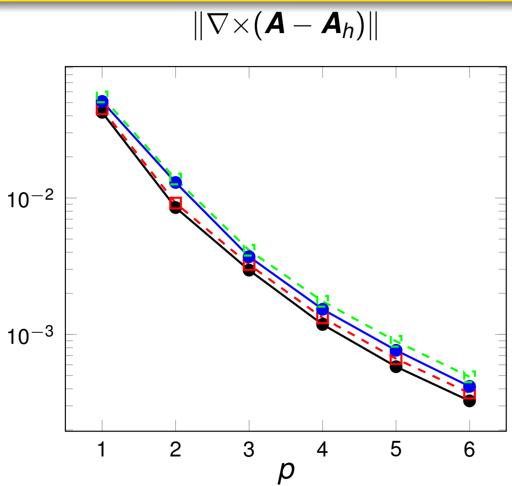
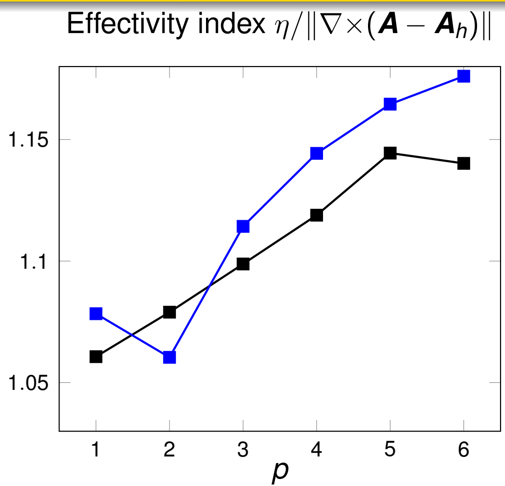
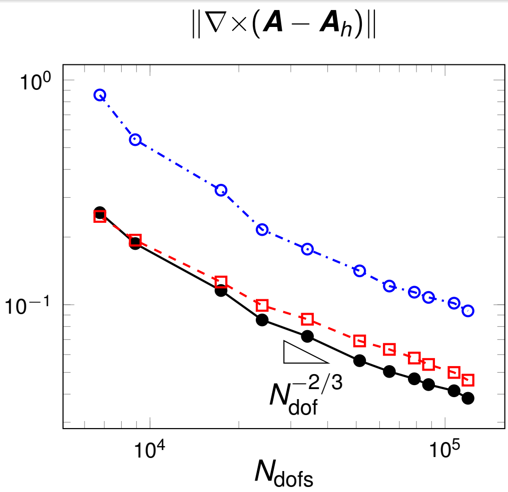
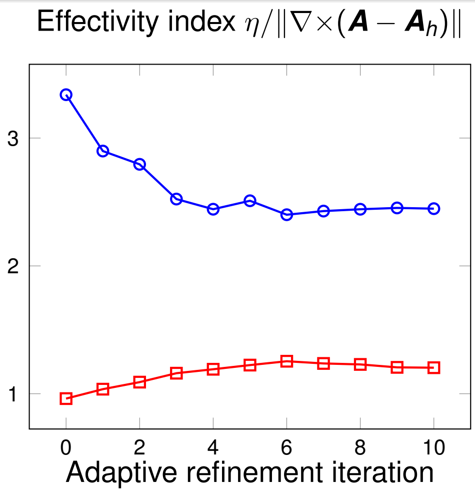




Patchwise equilibration, **singular solution**, **adap.** refinement ($p = 2$)





error estimate est.+d. osc. no d. osc. d. osc.

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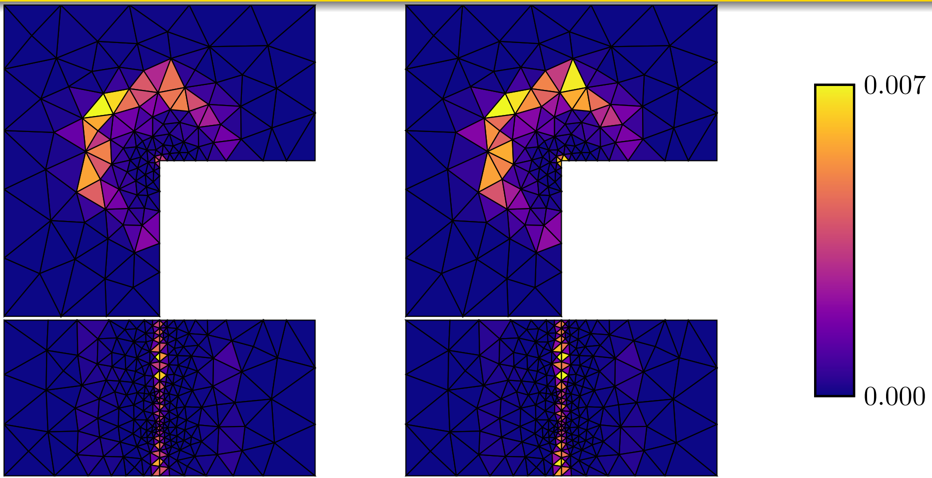

Estimators (left) and actual error (right), adaptive mesh refinement iteration #10.
Top view (top) and side view (bottom)

Outline

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- local *hp*-optimal approximation under minimal Sobolev regularity
- guaranteed, locally efficient, and *p*-robust a posteriori estimates

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Outline

The grad–grad problem: primal and dual approximations

Problem (source $f \in L^2(\Omega)$)

Find $u : \Omega \to \mathbb{R}$ such that

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Weak formulation

Find $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega)$$

Properties of the weak solution

$u \in H^1_0(\Omega)$ (primal variable), $\boldsymbol{\sigma} := -\nabla u \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f$ (dual variable)

Primal approximation

Find $u_h \in V_h := \mathcal{P}_p(\mathcal{T}_h) \cap H^1_0(\Omega)$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h$$

Dual approximation

$$\boldsymbol{\sigma}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{v}_h\|^2$$

gives

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{RT}_p(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_p f}} \|\boldsymbol{\sigma} - \boldsymbol{v}_h\|$$

The grad–grad problem: primal and dual approximations

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Weak formulation
Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution
$u \in H_0^1(\Omega)$ (primal variable), $\boldsymbol{\sigma} := -\nabla u \in \boldsymbol{H}(\mathrm{div}, \Omega)$ with $\nabla \cdot \boldsymbol{\sigma} = f$ (dual variable)

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The curl–curl problem: primal and dual approximations

Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ such that

$$(\nabla\times\boldsymbol{A},\nabla\times\boldsymbol{v}) = (\boldsymbol{j},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega).$$

Properties of the weak solution

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ (primal variable), $\boldsymbol{h} := \nabla\times\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ with $\nabla\times\boldsymbol{h} = \boldsymbol{j}$ (dual variable)

Primal approximation

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Weak formulation

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ such that

$$(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v}) = (\boldsymbol{j}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega).$$

Properties of the weak solution

$\boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ (primal variable), $\boldsymbol{h} := \nabla \times \boldsymbol{A} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ with $\nabla \times \boldsymbol{h} = \boldsymbol{j}$ (dual variable)

Primal approximation

$\boldsymbol{A}_h \in \boldsymbol{V}_h := \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl}, \Omega)$ satisfies

$$(\nabla \times \boldsymbol{A}_h, \nabla \times \boldsymbol{v}_h) = (\boldsymbol{j}, \boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h$$

Dual approximation

$$\boldsymbol{h}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{\Pi}_p \boldsymbol{j}}} \|\boldsymbol{v}_h\|^2$$

gives

$$\|\boldsymbol{h} - \boldsymbol{h}_h\| = \min_{\substack{\boldsymbol{v}_h \in \mathcal{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{\Pi}_p \boldsymbol{j}}} \|\boldsymbol{h} - \boldsymbol{v}_h\|$$