Unified primal formulation-based
a priori and a posteriori error analysis
of mixed finite element methods

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Introduction

Model problem \((S\text{ inhomogeneous and anisotropic})\)

\[
\begin{align*}
\mathbf{u} &= -S \nabla p, \nabla \cdot \mathbf{u} = f \text{ in } \Omega \\
-\nabla \cdot (S \nabla p) &= f \text{ in } \Omega \\
p &= 0 \text{ on } \partial \Omega \\
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\end{align*}
\]

Mixed finite elements

\[
\begin{align*}
(S^{-1} \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
(\nabla \cdot \mathbf{u}_h, \phi_h) &= (f, \phi_h) \quad \forall \phi_h \in \Phi_h
\end{align*}
\]

Traditional analysis

- weak mixed formulation
  \[
  (S^{-1} \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = 0 \\
  \forall \mathbf{v} \in H(\text{div}, \Omega),
  \]

- inf–sup condition
  \[
  (\nabla \cdot \mathbf{u}, \phi) = (f, \phi) \quad \forall \phi \in L^2(\Omega)
  \]

Presented analysis

- classical weak formulation
  \[
  (S \nabla p, \nabla \phi) = (f, \phi) \\
  \forall \phi \in H^1_0(\Omega)
  \]

- postprocessing and discrete Friedrichs inequality

- \(\nabla \cdot \mathbf{V}_h = \Phi_h\)

**unified and optimal a priori and a posteriori error analysis**

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Unified a priori and a posteriori analysis of MFEs
Introduction

Model problem (S inhomogeneous and anisotropic)

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unified and optimal a priori and a posteriori error analysis

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Unified a priori and a posteriori analysis of MFEs
Outline

1. General framework
   - An abstract result for the flux variable
   - Postprocessing for the scalar variable

2. A priori error estimates
   - Lowest-order Raviart–Thomas case
   - General case

3. A posteriori error estimates
   - Estimates for the flux
   - Estimates for the potential
   - Local efficiency

4. Remarks
   - Comments on the estimates
   - \( L^2(\Omega) \) estimates
   - \( RT_0 \) and pure diffusion problems

5. Numerical experiments

6. Conclusions and future work
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Bilinear forms and weak solution

**Definition (Bilinear form $B$)**

$$B(p, \varphi) := \sum_{K \in \mathcal{T}_h} (\mathbf{S} \nabla p, \nabla \varphi)_K, \quad p, \varphi \in H^1(\mathcal{T}_h).$$

**Definition (Bilinear form $A$)**

$$A(u, v) := \sum_{K \in \mathcal{T}_h} (u, \mathbf{S}^{-1} v)_K, \quad u, v \in L^2(\Omega).$$

**Definition (Weak solution)**

$p \in H^1_0(\Omega)$ such that $B(p, \varphi) = (f, \varphi)$ \quad $\forall \varphi \in H^1_0(\Omega)$; or

$p \in H^1_0(\Omega)$ such that $A(\mathbf{S} \nabla p, \mathbf{S} \nabla \varphi) = (f, \varphi)$ \quad $\forall \varphi \in H^1_0(\Omega)$. 

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Bilinear forms and weak solution

**Definition (Bilinear form $\mathcal{B}$)**

$$\mathcal{B}(\rho, \varphi) := \sum_{K \in \mathcal{T}_h} (\mathbf{S} \nabla \rho, \nabla \varphi)_K, \quad \rho, \varphi \in H^1(\mathcal{T}_h).$$

**Definition (Bilinear form $\mathcal{A}$)**

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**Definition (Weak solution)**

$$\rho \in H^1_0(\Omega) \text{ such that } \mathcal{B}(\rho, \varphi) = (f, \varphi), \quad \forall \varphi \in H^1_0(\Omega);$$

or

$$\rho \in H^1_0(\Omega) \text{ such that } \mathcal{A}(\mathbf{S} \nabla \rho, \mathbf{S} \nabla \varphi) = (f, \varphi), \quad \forall \varphi \in H^1_0(\Omega).$$
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Energy norms

Definition (Energy semi-norm)
\[ \|\varphi\|^2 := \mathcal{B}(\varphi, \varphi), \quad \varphi \in H^1(T_h). \]

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Definition (Energy norm)
\[ \|v\|^*_2 := \mathcal{A}(v, v), \quad v \in L^2(\Omega). \]

Definition (Energy–div norm)
\[ \|v\|^2_{\ast, \text{div}} := \|v\|^*_2 + \|\nabla \cdot v\|^2, \quad v \in H(\text{div}, \Omega). \]
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Definition (Energy–div norm)

\[ ||| v \|||_*^{2,\text{div}} := ||| v \|||_*^2 + \| \nabla \cdot v \|^2, \quad v \in H(\text{div}, \Omega). \]
## Energy norms

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**Definition (Energy–div norm)**

\[ \|\| \mathbf{v} \|\|_{*, \text{div}}^2 := \|\| \mathbf{v} \|\|_2^* + \|\nabla \cdot \mathbf{v} \|_2^2, \quad \mathbf{v} \in H(\text{div}, \Omega). \]
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6. Conclusions and future work
An abstract result for the flux variable

Theorem (Abstract framework \textit{(scheme-independent)})

Let \( \mathbf{v}, \mathbf{w}, \mathbf{t} \in L^2(\Omega) \) be arbitrary. Then

\[
|||\mathbf{v} - \mathbf{w}|||^* \leq |||\mathbf{w} - \mathbf{t}|||^* + \left| \mathcal{A}\left(\mathbf{v} - \mathbf{w}, \frac{\mathbf{v} - \mathbf{t}}{|||\mathbf{v} - \mathbf{t}|||^*}\right) \right|
\]

A priori error estimate

\[
|||\mathbf{u} - \mathbf{u}_h|||^* \leq |||\mathbf{u} - \Pi_h \mathbf{u}|||^*
\]

A posteriori error estimate

- put \( \mathbf{v} = \mathbf{u}, \mathbf{w} = \mathbf{u}_h, \mathbf{t} = -S\nabla s \) with \( s \in H^1_0(\Omega) \) arbitrary:

\[
|||\mathbf{u} - \mathbf{u}_h|||^* \leq |||\mathbf{u}_h + S\nabla s|||^* + \left| \mathcal{A}\left(\mathbf{u} - \mathbf{u}_h, \frac{\mathbf{u} + S\nabla s}{|||\mathbf{u} + S\nabla s|||^*}\right) \right|
\]

- notice that \( \mathcal{A}(\mathbf{u}, -S\nabla \varphi) = (f, \varphi) \) (here \( \varphi = p - s/|||p - s|||)\)

- notice that \( \mathcal{A}(\mathbf{u}_h, -S\nabla \varphi) = (\pi_I(f), \varphi) \)

- get

\[
|||\mathbf{u} - \mathbf{u}_h|||^* \leq \inf_{s \in H^1_0(\Omega)} |||\mathbf{u}_h + S\nabla s|||^* + \left\{ \sum_{K \in T_h} \frac{C_P h^2_K}{C_{S,K}} ||f - \pi_I(f)||^2_K \right\}^{1/2}
\]
An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $v, w, t \in L^2(\Omega)$ be arbitrary. Then

$$
|||v - w|||_* \leq |||w - t|||_* + \left| A \left( v - w, \frac{v - t}{||v - t||_*} \right) \right|.
$$

A priori error estimate

- put $v = u_h, w = u, t = \Pi_h u$:
  $$
  |||u_h - u|||_* \leq |||u - \Pi_h u|||_* + \left| A \left( u_h - u, \frac{u_h - \Pi_h u}{||u_h - \Pi_h u||_*} \right) \right|
  $$
- notice that $A(u_h - u, u_h - \Pi_h u) = 0$ in MFEs
- get $|||u - u_h|||_* \leq |||u - \Pi_h u|||_*$

A posteriori error estimate

- put $v = u, w = u_h, t = -S \nabla s$ with $s \in H^1_0(\Omega)$ arbitrary:
  $$
  |||u - u_h|||_* \leq |||u_h + S \nabla s|||_* + \left| A \left( u - u_h, \frac{u + S \nabla s}{||u + S \nabla s||_*} \right) \right|
  $$
- notice that $A(u - \nabla s, (f, p))$ (here $p = e(\|a\|)$).
An abstract result for the flux variable

Theorem (Abstract framework (scheme-independent))

Let $v, w, t \in L^2(\Omega)$ be arbitrary. Then

$$\|\|v - w\|\|_* \leq \|\|w - t\|\|_* + \left|\mathcal{A} \left( v - w, \frac{v - t}{\|v - t\|_*} \right) \right|. $$

A priori error estimate

$$\|\|u - u_h\|\|_* \leq \|\|u - \Pi_h u\|\|_*$$

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- notice that $\mathcal{A}(u, -S \nabla \varphi) = (f, \varphi)$ (here $\varphi = p - s/\|p - s\|)$

- notice that $\mathcal{A}(u_h, -S \nabla \varphi) = (\pi_I(f), \varphi)$

- get $\|\|u - u_h\|\|_* \leq \inf_{s \in H^1_0(\Omega)} \|\|u_h + S \nabla s\|\|_* + \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_p h_K^2}{c_{s,K}} \|f - \pi_I(f)\|_K^2 \right\}^{1/2}$

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Unified a priori and a posteriori analysis of MFEs
An abstract result for the flux variable

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- put $v = u$, $w = u_h$, $t = -S\nabla s$ with $s \in H^1_0(\Omega)$ arbitrary:

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$$

- notice that $A(u, -S\nabla \varphi) = (f, \varphi)$ (here $\varphi = p - s/||p - s||$)
- notice that $A(u_h, -S\nabla \varphi) = (\pi_I(f), \varphi)$
- get $|||u - u_h|||_* \leq \inf_{s \in H^1_0(\Omega)} |||u_h + S\nabla s|||_* + \left\{ \sum_{K \in T_h} \frac{c_P h_K^2}{c_{S,K}} ||f - \pi_I(f)||_K^2 \right\}^{1/2}$

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Postprocessing for the scalar variable

Postprocessing in mixed finite elements

- Arnold and Brezzi ’85: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates
- Bramble and Xu ’89: A local post-processing technique for improving the accuracy in mixed finite-element approximations
- Stenberg ’91: Postprocessing schemes for some mixed finite elements
- Arbogast and Chen ’95: On the implementation of mixed methods as nonconforming methods for second-order elliptic problems
- Chen ’96: Equivalence between and multigrid algorithms for nonconforming and mixed methods for second-order elliptic problems
Postprocessing in mixed finite elements

- Usually used in order to implement MFEMs and get superconvergence for the postprocessed variable.
- Usually not used in order to get a priori or a posteriori error estimates.
Postprocessing in mixed finite elements

- Usually used in order to implement MFEMs and get superconvergence for the postprocessed variable.
- Usually not used in order to get a priori or a posteriori error estimates.
Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K/|K| = p_K$ (mean of $\tilde{p}_h$ on $K$ is $p_K$).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H^1_0(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W^0_0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $\mathbf{u}_h$
- only valid in the lowest-order case on simplices or, when $\mathbf{S}$ is diagonal, on rectangular parallelepipeds
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- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W^0_0(\mathcal{T}_h)$

Proof:

\begin{align*}
0 &= - (\nabla \tilde{p}_h, v_{\sigma_{K,L}})_{K \cup L} - (\tilde{p}_h, \nabla \cdot v_{\sigma_{K,L}})_{K \cup L} \\
&= - \langle v_{\sigma_{K,L}} \cdot n, \tilde{p}_h \rangle_{\partial K} - \langle v_{\sigma_{K,L}} \cdot n, \tilde{p}_h \rangle_{\partial L} \\
&= \langle v_{\sigma_{K,L}} \cdot n_K, \tilde{p}_h |_L - \tilde{p}_h |_K \rangle_{\sigma_{K,L}}
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Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\not\in H^1_0(\Omega)$, only $\in H^1(T_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W^0_0(T_h)$
- the means are equal to the Lagrange multipliers from the hybridization

Remarks

- exact (not weak) connection of $\tilde{p}_h$ and $u_h$
- only valid in the lowest-order case on simplices or, when $S$ is diagonal, on rectangular parallelepipeds
General postprocessing

Definition (Postprocessed scalar variable $\tilde{p}_h$ (Arbogast & Chen))

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $(\tilde{p}_h, \phi_h)_K = (p_h, \phi_h)_K \quad \forall \phi_h \in \Phi_h(K)$.
- $(\tilde{p}_h, \mu_h)_\sigma = (\lambda_h, \mu_h)_\sigma \quad \forall \mu_h \in \Lambda_h(\sigma) \quad \forall \sigma \in \mathcal{E}_h^{\text{int}}$.

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, but $\tilde{p}_h \in W^k_0(\mathcal{T}_h)$
- $\tilde{p}_h$ is in general a nonconforming polynomial plus a bubble
- $\tilde{p}_h$ satisfies $-(S^{-1}u_h, v_h)_K = (\nabla \tilde{p}_h, v_h)_K \quad \forall v_h \in V_h(K)$

Remarks

- $u_h$ is a $P_{v_h,S^{-1}}$ projection of $-S\nabla \tilde{p}_h$ onto $V_h$, weak connection of $\tilde{p}_h$ and $u_h$
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Framework | A priori est. | A posteriori est. | Remarks | Exp. C. | Flux variable | Scalar variable
---|---|---|---|---|---|---

**General postprocessing**

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   - $L^2(\Omega)$ estimates
   - $RT_0$ and pure diffusion problems

5 Numerical experiments

6 Conclusions and future work

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Lowest-order Raviart–Thomas case

- \( \|p - \tilde{p}_h\| = \|u - u_h\| \leq \|u - \Pi_h u\| \leq Ch \)
- \( \tilde{p}_h \in W_0^0(\mathcal{T}_h) \): discrete Friedrichs inequality

\[
\|p - \tilde{p}_h\| \leq C_{DF}^{1/2} \left\{ \sum_{K \in \mathcal{T}_h} \| \nabla (p - \tilde{p}_h) \|^2_K \right\}^{1/2}
\]

- optimal value of \( C_{DF} \) (only depends on the shape regularity parameter and \( \inf_{b \in \mathbb{R}^d} \{ \text{thick}_b(\Omega) \} \)): Vohralík, NFAO 2005
- consequently: \( \left\{ \sum_{K \in \mathcal{T}_h} \|p - \tilde{p}_h\|^2_{1,K} \right\}^{1/2} \leq Ch \)
- superconvergence: \( \|p - \tilde{p}_h\| \leq Ch^2 \)
Lowest-order Raviart–Thomas case

- $|||p - \tilde{p}_h||| = |||u - u_h||| * \leq |||u - \Pi_h u||| * \leq Ch$
- $\tilde{p}_h \in W_0^0(I_h)$: discrete Friedrichs inequality

$$||p - \tilde{p}_h|| \leq C_{DF}^{1/2} \left\{ \sum_{K \in I_h} \|\nabla (p - \tilde{p}_h)\|_K^2 \right\}^{1/2}$$

- Optimal value of $C_{DF}$ (only depends on the shape regularity parameter and $\inf_{b \in \mathbb{R}^d} \{\text{thick}_b(\Omega)\}$): Vohralík, NFAO 2005

- Consequently: $\{ \sum_{K \in I_h} ||p - \tilde{p}_h||_{1,K}^2 \}^{1/2} \leq Ch$
- Superconvergence: $||p - \tilde{p}_h|| \leq Ch^2$
Lowest-order Raviart–Thomas case

\[ \| p - \tilde{p}_h \| = \| u - u_h \|_* \leq \| u - \Pi_h u \|_* \leq Ch \]

\[ \tilde{p}_h \in W^0_0(I_h): \text{discrete Friedrichs inequality} \]

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consequently: \[ \left\{ \sum_{K \in I_h} \| p - \tilde{p}_h \|_{1,K}^2 \right\}^{\frac{1}{2}} \leq Ch \]

superconvergence: \[ \| p - \tilde{p}_h \| \leq Ch^2 \]
Lowest-order Raviart–Thomas case

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superconvergence: \( \| p - \tilde{p}_h \| \leq Ch^2 \)
Lowest-order Raviart–Thomas case

\[ \|\rho - \tilde{\rho}_h\| = \|u - u_h\|_* \leq \|u - \Pi_h u\|_* \leq Ch \]

\[ \tilde{\rho}_h \in W^0_0(T_h): \text{discrete Friedrichs inequality} \]

\[ \|\rho - \tilde{\rho}_h\| \leq C_{DF}^{1/2} \left\{ \sum_{K \in T_h} \|\nabla (\rho - \tilde{\rho}_h)\|_K^2 \right\}^{1/2} \]

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General case

- a little bit more complicated since we only have
  \( u_h = -P_{V_h, S^{-1}}(S \nabla \tilde{p}_h) \) instead of \( u_h = -S \nabla \tilde{p}_h \)

- one still easily recovers all the known a priori error estimates for mixed finite elements
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Unified a priori and a posteriori analysis of MFEs
What is/should be an a posteriori error estimate

**Usual form**
- \[ \| p - p_h \|^2 \lesssim \sum_{K \in T_h} \eta_K(p_h)^2. \]
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

**Reliability**
- \[ \| p - p_h \|^2 \leq C \sum_{K \in T_h} \eta_K(p_h)^2 \]

**Guaranteed upper bound**
- \[ \| p - p_h \|^2 \leq \sum_{K \in T_h} \eta_K(p_h)^2 \]

**Local efficiency**
- \[ \eta_K(p_h)^2 \leq C_{\text{eff},K} \sum_{L \text{ close to } K} \| p - p_h \|_L^2 \]

**Asymptotic exactness**
- \[ \frac{\sum_{K \in T_h} \eta_K(p_h)^2}{\| p - p_h \|^2} \to 1 \]

**Robustness**
- independence of the data variation or mesh properties

**Negligible evaluation cost**
- estimators which can be evaluated locally
What is/should be an a posteriori error estimate

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- $\|p - p_h\|^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

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Reliability
- $\|p - p_h\|^2 \leq C \sum_{K \in T_h} \eta_K(p_h)^2$
- Problems:
  - What is $C$?
  - What does it depend on?
  - How does it depend on data?

Local efficiency
- $\eta_K(p_h)^2 \leq C_{\text{eff},K}^2 \sum_{L \text{close to } K} \|p - p_h\|_L^2$

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Previous works on a posteriori analysis for MFEMs

Previous works . . .
- Alonso ’96
- Braess and Verfürth ’96
- Carstensen ’97
- Hoppe and Wohlmuth ’97, ’99
- Kirby ’03
- El Alaoui and Ern ’04
- Wheeler and Yotov ’05
- Lovadina and Stenberg ’06

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- evaluation of the constants (guaranteed upper bound)
- robustness
- asymptotic exactness
- an analysis of the convection–diffusion case
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Theorem (A first abstract estimate for the flux and its efficiency)

Let $u$ be the weak flux and let $u_h \in H(\text{div}, \Omega)$ be arbitrary. Then

\[
\|\|u - u_h\|\|_*^2 \leq \inf_{s \in H^1_0(\Omega)} \|\|u_h + S\nabla s\|\|_*^2 + \frac{C_{F,\Omega} h^2}{c_{S,\Omega}} \|f - \nabla \cdot u_h\|_2
\]

\[
\leq \|\|u - u_h\|\|_*^2 + \frac{C_{F,\Omega} h^2}{c_{S,\Omega}} \|f - \nabla \cdot u_h\|_2.
\]

Properties

- Guaranteed upper bound (no undetermined constant).
- $\|\|u_h + S\nabla s\|\|_*$ penalizes $u_h \neq -S\nabla s$ for some $s \in H^1_0(\Omega)$.
- Advantage: scheme-independent (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).
- Disadvantage: $C_{F,\Omega}^{1/2} h^2/c_{S,\Omega}^{1/2} \|f - \nabla \cdot u_h\|$ too big.
A first abstract estimate for the flux

Theorem (A first abstract estimate for the flux and its efficiency)

Let $\mathbf{u}$ be the weak flux and let $\mathbf{u}_h \in H(\text{div}, \Omega)$ be arbitrary. Then

$$\|\|\mathbf{u} - \mathbf{u}_h\|\|_*^2 \leq \inf_{s \in H^1_0(\Omega)} \|\|\mathbf{u}_h + S\nabla s\|\|_*^2 + \frac{C_{F,\Omega}h^2}{c_{S,\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2$$

$$\leq \|\|\mathbf{u} - \mathbf{u}_h\|\|_*^2 + \frac{C_{F,\Omega}h^2}{c_{S,\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|^2.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- $\|\|\mathbf{u}_h + S\nabla s\|\|_*$ penalizes $\mathbf{u}_h \neq -S\nabla s$ for some $s \in H^1_0(\Omega)$.
- Advantage: scheme-independent (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used).
- Disadvantage: $C^{1/2}_{F,\Omega}h/\sqrt{c_{S,\Omega}} \|f - \nabla \cdot \mathbf{u}_h\|$ too big.
An improved abstract estimate for the flux

Theorem (An improved abstract estimate for the flux and its efficiency)

Let \( \mathbf{u} \) be the weak flux and let \( \mathbf{u}_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot \mathbf{u}_h = \pi_l(f) \) be arbitrary. Then

\[
\|\| \mathbf{u} - \mathbf{u}_h \|\|_*^2 \leq \inf_{s \in H^1_0(\Omega)} \|\| \mathbf{u}_h + S\nabla s \|\|_*^2 + \eta_R^2 \leq \|\| \mathbf{u} - \mathbf{u}_h \|\|_*^2 + \eta_R^2,
\]

where

\[
\eta_R := \left\{ \sum_{K \in T_h} \frac{C_P h_K^2}{C_s, K} \| f - \pi_l(f) \|_K^2 \right\}^{\frac{1}{2}}.
\]

Properties

- No global Galerkin orthogonality needed, just local conservativity.
- \( \eta_R \) is in general a higher-order term for RT methods.
- \( \eta_R \) is not in general a higher-order term for BDM methods.
An improved abstract estimate for the flux

**Theorem (An improved abstract estimate for the flux and its efficiency)**

Let \( u \) be the weak flux and let \( u_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot u_h = \pi_l(f) \) be arbitrary. Then

\[
\| |u - u_h| |^2 \leq \inf_{s \in H_0^1(\Omega)} \| |u_h + S\nabla s| |^2 + \eta_R^2 \leq \| |u - u_h| |^2 + \eta_R^2,
\]

where

\[
\eta_R := \left\{ \sum_{K \in \mathcal{T}_h} \frac{C_P h_K^2}{c_{S,K}} \| f - \pi_l(f) \|_K^2 \right\}^{\frac{1}{2}}.
\]

**Properties**

- No global Galerkin orthogonality needed, just local conservativity.
- \( \eta_R \) is in general a higher-order term for RT methods.
- \( \eta_R \) is not in general a higher-order term for BDM methods.
An energy–div norm abstract estimate for the flux

Theorem (An energy–div norm abstract estimate for the flux and its efficiency)

Let \( \mathbf{u} \) be the weak flux and let \( \mathbf{u}_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot \mathbf{u}_h = \pi_I(f) \) be arbitrary. Then

\[
\left\| \mathbf{u} - \mathbf{u}_h \right\|_{*, \text{div}}^2 \leq \inf_{s \in H_0^1(\Omega)} \left\| \mathbf{u}_h + \mathbf{S} \nabla s \right\|_{*}^2 + \| f - \pi_I(f) \|_2^2 + \eta_R^2 \\
\leq \left\| \mathbf{u} - \mathbf{u}_h \right\|_{*, \text{div}}^2 + \eta_R^2.
\]

Properties

- \( \eta_R \) gets always a higher-order term.

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An energy–div norm abstract estimate for the flux

Theorem (An energy–div norm abstract estimate for the flux and its efficiency)

Let \( u \) be the weak flux and let \( u_h \in H(\text{div}, \Omega) \) such that
\[
\nabla \cdot u_h = \pi_I(f)
\]
be arbitrary. Then
\[
\|\| u - u_h \|\|_{*, \text{div}}^2 \leq \inf_{s \in H^1_0(\Omega)} \|\| u_h + S \nabla s \|\|_{*}^2 + \| f - \pi_I(f) \|_2^2 + \eta_R^2
\]
\[
\leq \|\| u - u_h \|\|_{*, \text{div}}^2 + \eta_R^2.
\]

Properties

- \( \eta_R \) gets always a higher-order term.
A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $u$ be the weak flux and let $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_I(f)$ be arbitrary. Then

$$
\ |||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 \right),
$$

$$
\ |||u - u_h|||^2_{*, \text{div}} \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 + \eta_{D,K}^2 \right).
$$

- potential estimator
  $$
  \eta_{P,K} := |||u_h + S \nabla (I_{O}(\tilde{\phi}_h))|||_{*, K}.
  $$
  $I_{O}(\tilde{\phi}_h)$: Oswald interpolate $P_n(I_h) \rightarrow P_n(I_h) \cap H^1_0(\Omega)$

- residual estimator
  $$
  \eta_{R,K} := \frac{C_{p,K}^{1/2}}{C_{S,K}^{1/2}} \|f - \pi_I(f)\|_K.
  $$

- divergence estimator
  $$
  \eta_{D,K} := \|f - \pi_I(f)\|_K.
  $$
A fully computable estimate for the flux

**Theorem (A fully computable estimate for the flux)**

Let $\mathbf{u}$ be the weak flux and let $\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega)$ such that $\nabla \cdot \mathbf{u}_h = \pi_l(f)$ be arbitrary. Then

\[ ||| \mathbf{u} - \mathbf{u}_h |||_2^2 \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 \right), \]

\[ ||| \mathbf{u} - \mathbf{u}_h |||_{*,\text{div}}^2 \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 + \eta_{D,K}^2 \right). \]

- **potential estimator**
  \[ \eta_{P,K} := ||| \mathbf{u}_h + S \nabla (\mathcal{I}_{Os}(\tilde{p}_h)) |||_{*,K} \]
  \( \mathcal{I}_{Os}(\tilde{p}_h) \): Oswald interpolate \( \mathbb{P}_n(\mathcal{T}_h) \rightarrow \mathbb{P}_n(\mathcal{T}_h) \cap H^1_0(\Omega) \)

- **residual estimator**
  \[ \eta_{R,K} := \frac{C_P^{1/2} h_K}{C_S^{1/2}} || f - \pi_l(f) ||_K \]

- **divergence estimator**
  \[ \eta_{D,K} := || f - \pi_l(f) ||_K \]
A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let \( u \) be the weak flux and let \( u_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot u_h = \pi_I(f) \) be arbitrary. Then

\[
\|\| u - u_h \|\|^2 \leq \sum_{K \in T_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 \right),
\]

\[
\|\| u - u_h \|\|^2_{\text{div}} \leq \sum_{K \in T_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 + \eta_{D,K}^2 \right).
\]

- potential estimator
  \[
  \eta_{P,K} := \| u_h + S \nabla (I_{Os}(\tilde{p}_h)) \|_{\ast,K},
  \]
  \( I_{Os}(\tilde{p}_h) \): Oswald interpolate \( \mathbb{P}_n(T_h) \rightarrow \mathbb{P}_n(T_h) \cap H^1_0(\Omega) \)

- residual estimator
  \[
  \eta_{R,K} := \frac{C_p^{1/2} h_K}{C_s^{1/2}} \| f - \pi_I(f) \|_K,
  \]

- divergence estimator
  \[
  \eta_{D,K} := \| f - \pi_I(f) \|_K.
  \]
A fully computable estimate for the flux

Theorem (A fully computable estimate for the flux)

Let $u$ be the weak flux and let $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_I(f)$ be arbitrary. Then

$$\|u - u_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 \right),$$

$$\|u - u_h\|^2_{\text{div}} \leq \sum_{K \in \mathcal{T}_h} \left( \eta_{P,K}^2 + \eta_{R,K}^2 + \eta_{D,K}^2 \right).$$

- **potential estimator**
  
  $\eta_{P,K} := \|u_h + S \nabla (I_{Os} (\tilde{\phi}_h))\|_{*,K}$
  
  $I_{Os}(\tilde{\phi}_h)$: Oswald interpolate $P_n(T_h) \rightarrow P_n(T_h) \cap H^1_0(\Omega)$

- **residual estimator**
  
  $\eta_{R,K} := C_{p}^{1/2} h_K^{1/2} \|f - \pi_I(f)\|_K$

- **divergence estimator**
  
  $\eta_{D,K} := \|f - \pi_I(f)\|_K$
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Theorem (Abstract a posteriori estimate for the potential and its efficiency)

Let $p$ be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\|\| p - \tilde{p}_h \|\|^2 \leq \inf_{s \in H^1_0(\Omega)} \|\| \tilde{p}_h - s \|\|^2
$$

$$
+ \inf_{t \in H(\text{div}, \Omega)} \sup_{\varphi \in H^1_0(\Omega), \|\| \varphi \|\|=1} \left( (f - \nabla \cdot t, \varphi) - (S \nabla \tilde{p}_h + t, \nabla \varphi) \right)^2
$$

$$
\leq 2\|\| p - \tilde{p}_h \|\|^2.
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of $p_h$. 
An abstract estimate for the potential

Theorem (Abstract a posteriori estimate for the potential and its efficiency)

Let $p$ be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
|||p - \tilde{p}_h|||^2 \leq \inf_{s \in H^1_0(\Omega)} |||\tilde{p}_h - s|||^2 \\
+ \inf_{t \in H^{(\text{div},\Omega)}} \sup_{\varphi \in H^1(\Omega), ||\varphi||=1} ((f - \nabla \cdot t, \varphi) - (S \nabla \tilde{p}_h + t, \nabla \varphi))^2 \\
\leq 2|||p - \tilde{p}_h|||^2.
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of $p_h$. 

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Theorem (A first computable estimate for the potential)

Let \( p \) be the weak potential and let \( \tilde{p}_h \in H^1(\mathcal{T}_h) \) be arbitrary. Take any \( t_h \in H(\text{div}, \Omega) \) and any \( s_h \in H^1_0(\Omega) \). Then

\[
|||p - \tilde{p}_h|||^2 \leq |||\tilde{p}_h - s_h|||^2 + \left( \frac{C_{F,\Omega}^{1/2} h_{\Omega}}{C_{S,\Omega}^{1/2}} |||f - \nabla \cdot t_h||| + |||S \nabla \tilde{p}_h + t_h|||* \right)^2.
\]

Properties

- \( |||S \nabla \tilde{p}_h + t_h|||* \) penalizes \(-S \nabla \tilde{p}_h \not\in H(\text{div}, \Omega)\).
- \( |||\tilde{p}_h - s_h||| \) penalizes \( \tilde{p}_h \not\in H^1_0(\Omega) \).
- Advantage: scheme-independent.
- Disadvantage: \( C_{F,\Omega}^{1/2} h_{\Omega}/C_{S,\Omega}^{1/2} |||f - \nabla \cdot t_h||| \) too big.
A first computable estimate for the potential

Theorem (A first computable estimate for the potential)

Let \( p \) be the weak potential and let \( \tilde{p}_h \in H^1(\mathcal{T}_h) \) be arbitrary. Take any \( t_h \in H(\text{div}, \Omega) \) and any \( s_h \in H^1_0(\Omega) \). Then

\[
|||p - \tilde{p}_h|||^2 \leq |||\tilde{p}_h - s_h|||^2 + \left( \frac{C_{1/2}^{1/2} h_\Omega}{c_{s, \Omega}^{1/2}} \| f - \nabla \cdot t_h \| + |||S \nabla \tilde{p}_h + t_h|||_* \right)^2.
\]

Properties

- \( |||S \nabla \tilde{p}_h + t_h|||_* \) penalizes \( -S \nabla \tilde{p}_h \not\in H(\text{div}, \Omega) \).
- \( |||\tilde{p}_h - s_h||| \) penalizes \( \tilde{p}_h \not\in H^1_0(\Omega) \).
- Advantage: scheme-independent.
- Disadvantage: \( C_{F, \Omega}^{1/2} h_\Omega / c_{s, \Omega}^{1/2} \| f - \nabla \cdot t_h \| \) too big.
A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let $p$ be the weak potential and let $\tilde{p}_h \in H^1(T_h)$ and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_1(f)$ be arbitrary. Then

$$\|\|p - \tilde{p}_h\|\|^2 \leq \sum_{K \in T_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.$$

- nonconformity estimator
  $$\eta_{NC,K} := \|\|\tilde{p}_h - I_{Os}(\tilde{p}_h)\|\|_K$$
- diffusive flux estimator
  $$\eta_{DF,K} := \|\|u_h + S\nabla \tilde{p}_h\|\|_{*,K}$$
- residual estimator
  $$\eta_{R,K} := \frac{C_p^{1/2} h_K^{1/2}}{C_s^{1/2}} \|f - \pi_1(f)\|_K$$
A fully computable estimate for the potential

**Theorem (A fully computable estimate for the potential)**

Let \( p \) be the weak potential and let \( \tilde{p}_h \in H^1(T_h) \) and \( u_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot u_h = \pi_1(f) \) be arbitrary. Then

\[
|||p - \tilde{p}_h|||^2 \leq \sum_{K \in T_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.
\]

- **nonconformity estimator**
  \[ \eta_{NC,K} := |||\tilde{p}_h - I_{Os}(\tilde{p}_h)|||_K \]

- **diffusive flux estimator**
  \[ \eta_{DF,K} := |||u_h + S\nabla \tilde{p}_h|||_{*,K} \]

- **residual estimator**
  \[ \eta_{R,K} := \frac{C_p^{1/2} h_K}{C_s^{1/2} S_{K}} \|f - \pi_1(f)\|_K \]
A fully computable estimate for the potential

Theorem (A fully computable estimate for the potential)

Let $p$ be the weak potential and let $\tilde{p}_h \in H^1(\mathcal{T}_h)$ and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_l(f)$ be arbitrary. Then

$$|||p - \tilde{p}_h|||^2 \leq \sum_{K \in \mathcal{T}_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.$$

- nonconformity estimator
  - $\eta_{NC,K} := |||\tilde{p}_h - I_{Os}(\tilde{p}_h)|||_K$

- diffusive flux estimator
  - $\eta_{DF,K} := |||u_h + S\nabla \tilde{p}_h|||_{*,K}$

- residual estimator
  - $\eta_{R,K} := \frac{C^1_{p} \|h_K\|}{C_{s,K}^{1/2}} ||f - \pi_l(f)||_K$
A fully computable estimate for the potential

**Theorem (A fully computable estimate for the potential)**

Let \( p \) be the weak potential and let \( \tilde{p}_h \in H^1(I_h) \) and \( u_h \in H(\text{div}, \Omega) \) such that \( \nabla \cdot u_h = \pi_I(f) \) be arbitrary. Then

\[
|||p - \tilde{p}_h|||^2 \leq \sum_{K \in I_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.
\]

- **nonconformity estimator**
  \[
  \eta_{NC,K} := |||\tilde{p}_h - I_{Os}(\tilde{p}_h)|||_K
  \]

- **diffusive flux estimator**
  \[
  \eta_{DF,K} := |||u_h + S\nabla \tilde{p}_h|||_{*,K}
  \]

- **residual estimator**
  \[
  \eta_{R,K} := \frac{C_p^{1/2} h_K}{C_s^{1/2}} ||f - \pi_I(f)||_K
  \]
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Theorem (Local efficiency of the estimates)

Let $p, u$ be the weak potential and flux, respectively, and let $u_h$ be the MFE flux and $\tilde{p}_h$ the postprocessed potential. Then

$$
\eta_{DF,K} \leq \|u - u_h\|_{*,K} + \|p - \tilde{p}_h\|_{K},
$$

$$
\eta_{P,K} \leq \eta_{DF,K} + \eta_{NC,K},
$$

$$
\eta_{NC,K} \leq C\sqrt{\frac{C_{S,K}}{C_{S,K,T_k}}} \|p - \tilde{p}_h\|_{T_k},
$$

$$
\eta_{R,K} \leq C\sqrt{\frac{C_{S,K}}{C_{S,K}}} \|u - u_h\|_{*,K},
$$

where $C$ depends only on the space dimension $d$, the maximal polynomial degree $n$ of $\tilde{p}_h$, the shape regularity parameter $\kappa_T$, and the polynomial degree $m$ of $f$. 

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Local efficiency of the estimates

Proof for $\eta_{NC,K}$.

Oswald interpolate (Karakashian and Pascal ’03, Burman and Ern ’07):

$$
\| \nabla (\phi_h - I_{Os}(\phi_h)) \|_K^2 \leq C \sum_{\sigma \in \tilde{E}_K} h_\sigma^{-1} \| \phi_h \|_{\sigma}^2
$$

Achdou, Bernardi, Coquel ’03:

$$
h_\sigma^{-\frac{1}{2}} \| [\tilde{p}_h] \|_{\sigma} \leq C \sum_{L; \sigma \in E_L} \| \nabla (\tilde{p}_h - \phi) \|_L
$$

$$
\eta_{NC,K}^2 = \| \tilde{p}_h - I_{Os}(\tilde{p}_h) \|_K^2 \leq C C_{S,K} \sum_{\sigma \in \tilde{E}_K} h_\sigma^{-1} \| [\tilde{p}_h] \|_{\sigma}^2
$$

$$
\leq C C_{S,K} \sum_{L \in T_K} \| \nabla (p - \tilde{p}_h) \|_L^2 \leq C \frac{C_{S,K}}{c_{S,T_K}} \sum_{L \in T_K} \| p - \tilde{p}_h \|_L^2
$$
Local efficiency of the estimates

Proof for $\eta_{R,K}$.

- $\| f - \pi_I(f) \|_K = \| f - \nabla \cdot u_h \|_K \leq C C_{s,K}^{1/2} h_K^{-1} \| |u - u_h| |^*,K$
- element bubble functions
- equivalence of norms on finite-dimensional spaces
- weak solution definition
- Green theorem
- Cauchy–Schwarz inequality
- energy norm definition
- inverse inequality

- residual estimator is always efficient (also for BDM)
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Comments on the estimates

General comments

- \( p \in H^1(\Omega) \), no additional regularity
- no convexity of \( \Omega \) needed
- no saturation assumption
- no Helmholtz decomposition
- no shape-regularity needed for the upper bounds (only for the efficiency proofs)
- polynomial degree-independent upper bound
- no “monotonicity” hypothesis on inhomogeneities distribution
- the only important tool: optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
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Theorem (Estimate for $\tilde{p}_h$ in the $L^2(\Omega)$-norm)

Let $p$ be the weak potential and let $\tilde{p}_h \in W^0_0(\mathcal{T}_h)$ and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_l(f)$ be arbitrary. Then

$$\|p - \tilde{p}_h\|^2 \leq \frac{C_{DF}}{c_{S,\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.$$ 

Theorem (Estimate for $p_h$ in the $L^2(\Omega)$-norm)

Let $p$ be the weak potential and let $p_h \in \Phi_h$, $\tilde{p}_h \in W^0_0(\mathcal{T}_h)$, and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_l(f)$ be arbitrary. Then

$$\|p - p_h\| \leq \left\{ \frac{C_{DF}}{c_{S,\Omega}} \sum_{K \in \mathcal{T}_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\} \right\}^{\frac{1}{2}} + \|\tilde{p}_h - p_h\|.$$
### Theorem (Estimate for $\tilde{p}_h$ in the $L^2(\Omega)$-norm)

Let $p$ be the weak potential and let $\tilde{p}_h \in W^0_0(I_h)$ and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_1(f)$ be arbitrary. Then

$$
\|p - \tilde{p}_h\|^2 \leq \frac{C_{\text{DF}}}{c_{S,\Omega}} \sum_{K \in T_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\}.
$$

### Theorem (Estimate for $p_h$ in the $L^2(\Omega)$-norm)

Let $p$ be the weak potential and let $p_h \in \Phi_h$, $\tilde{p}_h \in W^0_0(I_h)$, and $u_h \in H(\text{div}, \Omega)$ such that $\nabla \cdot u_h = \pi_1(f)$ be arbitrary. Then

$$
\|p - p_h\| \leq \left\{ \frac{C_{\text{DF}}}{c_{S,\Omega}} \sum_{K \in T_h} \left\{ \eta_{NC,K}^2 + (\eta_{R,K} + \eta_{DF,K})^2 \right\} \right\}^{\frac{1}{2}} + \|\tilde{p}_h - p_h\|.
$$
Some additional comments

- We believe that $L^2(\Omega)$ norm is not optimal for a posteriori error estimates in mixed finite elements.
- We believe that trying to directly and only derive estimates for $\rho_h$ in the $L^2(\Omega)$-norm was the bottleneck of a lot of previous works.
- $\|\| u_h + S \nabla \tilde{\rho}_h \|_{*,K}$ or $\|\| u_h + S \nabla (I_{Os}(\tilde{\rho}_h)) \|_{*,K}$ (our estimates): clear physical meaning
- $h_K \|\| u_h + S \nabla \rho_h \|_{*,K} = h_K \|\| u_h \|_{*,K}$ in $RT_0$ (some previous works): no good sense
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Pure diffusion problem: \(-\nabla \cdot (S \nabla p) = f, \quad p = 0 \text{ on } \partial \Omega\)

**Theorem (Mixed FEM for the diffusion problem)**

There holds
\[
|||p - \tilde{p}_h||| \leq \inf_{s \in H^1_0(\Omega)} |||\tilde{p}_h - s||| + \left\{ \sum_{K \in \mathcal{T}_h} C_P \frac{h^2_K}{c_{\text{S},K}} |||f - f_K|||_K^2 \right\}^{1/2}.
\]

**Theorem (Galerkin FEM for the diffusion problem)**

There holds
\[
|||p - p_h||| \leq \inf_{s_h \in V_h} |||p - s_h|||.
\]

**Mixed FEM 1D:**
- no nonconformity, \(\tilde{p}_h \in H^1_0(\Omega)\)
- \(|||p - \tilde{p}_h||| \leq C h^2\) when \(f \in H^1(\mathcal{T}_h)\)
- \(\tilde{p}_h = p\), the exact solution, for pw constant \(S\) (arbitrary inhomogeneities) and pw constant \(f\)

**Galerkin FEM 1D:**
- \(|||p - \tilde{p}_h||| \leq C h\)
Pure diffusion problem $-\nabla \cdot (S \nabla p) = f$, $p = 0$ on $\partial \Omega$

### Theorem (Mixed FEM for the diffusion problem)

There holds
\[
\|p - \tilde{p}_h\| \leq \inf_{s \in H^1_0(\Omega)} \|\tilde{p}_h - s\| + \left\{ \sum_{K \in T_h} \frac{h_K^2}{C_{s,K}} \|f - f_K\|_K^2 \right\}^{1/2}.
\]

### Theorem (Galerkin FEM for the diffusion problem)

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**Galerkin FEM 1D:**

- \(\|\|p - \tilde{p}_h\|\| \leq Ch\)
Outline

1. General framework
   - An abstract result for the flux variable
   - Postprocessing for the scalar variable

2. A priori error estimates
   - Lowest-order Raviart–Thomas case
   - General case

3. A posteriori error estimates
   - Estimates for the flux
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   - Local efficiency

4. Remarks
   - Comments on the estimates
   - $L^2(\Omega)$ estimates
   - $RT_0$ and pure diffusion problems

5. Numerical experiments

6. Conclusions and future work
Problem with discontinuous and inhomogeneous diffusion tensor

- consider the pure diffusion equation
  \[-\nabla \cdot (S \nabla \rho) = 0 \quad \text{in} \quad \Omega = (-1, 1) \times (-1, 1)\]
- discontinuous and inhomogeneous $S$, two cases:

  \[
  \begin{bmatrix}
  s_1 = 5 & s_2 = 1 \\
  s_3 = 5 & s_4 = 1
  \end{bmatrix}
  \quad \begin{bmatrix}
  s_1 = 100 & s_2 = 1 \\
  s_3 = 100 & s_4 = 1
  \end{bmatrix}
  \]

- analytical solution: singularity at the origin
  \[\rho(r, \theta) |_{\Omega_i} = r^\alpha (a_i \sin(\alpha \theta) + b_i \cos(\alpha \theta))\]

  - $(r, \theta)$ polar coordinates in $\Omega$
  - $a_i, b_i$ constants depending on $\Omega_i$
  - $\alpha$ regularity of the solution
Analytical solutions

M. Vohralík
Unified a priori and a posteriori analysis of MFEs
Estimated and actual error distribution on an adaptively refined mesh, case 1
Approximate solution and the corresponding adaptively refined mesh, case 2
Estimated and actual error against the number of elements in uniformly/adaptively refined meshes

- **Inhomogeneous diffusion**
- **Dominating convection**

### Energy error
- **error uniform**
- **estimate uniform**
- **error adapt.**
- **estimate adapt.**

### Number of triangles

<table>
<thead>
<tr>
<th>Number of triangles</th>
<th>Energy error</th>
<th>error uniform</th>
<th>estimate uniform</th>
<th>error adapt.</th>
<th>estimate adapt.</th>
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<td>$10^0$</td>
<td>$10^1$</td>
<td>$10^0$</td>
<td>$10^0$</td>
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</tr>
<tr>
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<td>$10^{-1}$</td>
<td>$10^0$</td>
<td>$10^{-1}$</td>
<td>$10^{-1}$</td>
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<tr>
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<td>$10^{-3}$</td>
<td>$10^{-3}$</td>
<td>$10^{-2}$</td>
</tr>
</tbody>
</table>
Global efficiency of the estimates

Inhomogeneous diffusion
Dominating convection

Global efficiency of the estimates

Number of triangles
Efficiency

- efficiency uniform
- efficiency adapt.

M. Vohralík
Unified a priori and a posteriori analysis of MFEs
Convection-dominated problem

- consider the convection–diffusion–reaction equation
  \[-\varepsilon \Delta p + \nabla \cdot (p(0, 1)) + p = f \quad \text{in} \quad \Omega = (0, 1) \times (0, 1)\]

- analytical solution: layer of width $a$
  \[p(x, y) = 0.5 \left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)\]

- consider
  - $\varepsilon = 1, \ a = 0.5$
  - $\varepsilon = 10^{-2}, \ a = 0.05$
  - $\varepsilon = 10^{-4}, \ a = 0.02$

- unstructured grid of 46 elements given, uniformly/adaptively refined
Analytical solutions, $\varepsilon = 1$, $a = 0.5$ and $\varepsilon = 10^{-4}$, $a = 0.02$
Estimated and actual error distribution, $\varepsilon = 1$, $a = 0.5$
Modified Oswald interpolate: estimated and actual error against the number of elements and global efficiency of the estimates, $\varepsilon = 1$, $a = 0.5$
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Estimated and actual error distribution, $\varepsilon = 10^{-2}$, $a = 0.05$
Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$
Estimated and actual error against the number of elements in uniformly/adaptively refined meshes, $\varepsilon = 10^{-2}$, $a = 0.05$ and $\varepsilon = 10^{-4}$, $a = 0.02$
Global efficiency of the estimates, $\varepsilon = 10^{-2}, a = 0.05$ and $\varepsilon = 10^{-4}, a = 0.02$
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- unified framework for a priori and a posteriori error control in mixed finite elements
- optimality of the framework for a posteriori error estimation: guaranteed upper bound, local efficiency, asymptotic exactness, robustness, negligible evaluation cost
- directly implementable—all constants evaluated
- parallel work for finite volumes, discontinuous Galerkin finite elements, and continuous finite elements

Future work

- full asymptotic exactness and robustness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Navier–Lamé, Maxwell)
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Papers

- VOHRALÍK M., Unified primal formulation-based a priori and a posteriori error analysis of mixed finite element methods, to be submitted.

Thank you for your attention!