Guaranteed and robust L^2-norm a posteriori error estimates for 1D linear advection(-reaction) problems

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Outline

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Introduction

A posteriori error estimate

$$\underbrace{\|u - u_h\|}_{\text{unknown error}} \leq \overbrace{\eta}^{\text{estimator computable from } u_h}$$

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$$\underbrace{\|u - u_h\|}_{\text{unknown error}} \leq \underbrace{\eta}_{\text{estimator computable from } u_h} \leq C\|u - u_h\|$$

- guaranteed upper bound (reliability with constant one)
- efficiency

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- guaranteed upper bound (reliability with constant one)
- efficiency
- C independent of parameters: robustness

Some previous contributions

A posteriori error estimates

Süli (1999); Houston, Mackenzie, Süli, Warnecke (1999); Hauke, Fuster, Doweidar (2008); Burman (2009); John, Novo (2013); Zhang, Zhang (2015)

Adaptivity

Dahmen, Huang, Schwab, Welper (2012), Dahmen, Stevenson (2019)

Reconstructions

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The advection problem

The advection problem

Find $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{b}\cdot\nabla u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial_-\Omega. \end{aligned}$$

- $\boldsymbol{b} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R})$: divergence-free (constant since $d = 1$ for now) velocity field
- $f \in L^2(\Omega)$: source term
- $\partial_\pm\Omega := \{x \in \partial\Omega : \pm\boldsymbol{b}(x)\cdot\boldsymbol{n}(x) > 0\}$: inflow and outflow parts of the boundary
- $\partial_0\Omega := \{x \in \partial\Omega : \boldsymbol{b}(x)\cdot\boldsymbol{n}(x) = 0\}$: characteristic part of the boundary

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Functional setting

Sobolev spaces

$$H^1_-(\Omega) = \left\{ w \in H^1(\Omega), w = 0, \text{ on } \partial_-\Omega \right\},$$
$$H^1_+(\Omega) = \left\{ w \in H^1(\Omega), w = 0, \text{ on } \partial_+\Omega \right\}.$$

Integration by parts

$$(v, \boldsymbol{b}\cdot\nabla w)_\Omega + (\boldsymbol{b}\cdot\nabla v, w)_\Omega = (\boldsymbol{b}\cdot\boldsymbol{n} v, w)_{\partial\Omega} \qquad \forall v, w \in H^1(\Omega)$$

Poincaré–Friedrichs inequalities

$$\|v - \bar{v}\|_D \le h_D C_{\mathrm{P},D} \|\nabla v\|_D \qquad \forall v \in H^1(D), \quad C_{\mathrm{P},D} \le 1/\pi,$$
$$\|v\|_D \le h_D C_{\mathrm{F},D,\Gamma_D} \|\nabla v\|_D, \qquad \forall v \in \left\{ H^1(D), v|_{\Gamma_D} = 0, |\Gamma_D| \neq 0 \right\}, \quad C_{\mathrm{F},D,\Gamma_D} \le 1$$

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Numerical approximation

Example (Continuous trial Petrov–Galerkin (PG1) finite element)

Find $u_h \in X_h := H^1_-(\Omega) \cap \mathcal{P}^k(\mathcal{T}_h)$, $k \geq 2$, *such that*

$$(\boldsymbol{b}\cdot\nabla u_h, v_h) = (f, v_h) \qquad \forall v_h \in Y_h := \mathcal{P}^{k-1}(\mathcal{T}_h).$$

Example (Discontinuous trial Petrov–Galerkin (PG2) finite element)

Find $u_h \in X_h := \mathcal{P}^k(\mathcal{T}_h)$, $k \geq 0$, *such that*

$$-(u_h, \boldsymbol{b}\cdot\nabla v_h) = (f, v_h) \qquad \forall v_h \in Y_h := H^1_+(\Omega) \cap \mathcal{P}^{k+1}(\mathcal{T}_h).$$

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Numerical approximation

Example (dG finite element)

Find $u_h \in X_h := \mathcal{P}^k(\mathcal{T}_h)$, $k \geq 1$, *such that*

$$\mathcal{B}_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in Y_h := \mathcal{P}^k(\mathcal{T}_h),$$

where

$$\mathcal{B}_h(u_h, v_h) := -\sum_{K \in \mathcal{T}_h} (u_h, \boldsymbol{b}\cdot\nabla v_h)_K - \sum_{e \in \mathcal{E}_h^{\mathrm{int}}} \boldsymbol{b}\cdot\boldsymbol{n}\{\!\{u_h\}\!\}[\![v_h]\!] + \sum_{e \in \mathcal{E}_h^{\mathrm{int}}} \frac{1}{2}|\boldsymbol{b}\cdot\boldsymbol{n}|[\![u_h]\!][\![v_h]\!] + \sum_{e \in \mathcal{E}_h^{\mathrm{bnd}}} (\boldsymbol{b}\cdot\boldsymbol{n})^+ u_h v_h.$$

- u_h^-, u_h^+: trace value from left and from right
- $\{\!\{u_h\}\!\} := (u_h^- + u_h^+)/2$: average
- $[\![u_h]\!] := u_h^+ - u_h^-$: jump
- upwind dG (Lax–Friedrichs) flux applied on the cell interfaces

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Weak solution and residual

Ultra-weak solution

Find $u \in L^2(\Omega)$ such that

$$-(u, \boldsymbol{b}\cdot\nabla v) = (f, v) \qquad \forall v \in H^1_+(\Omega).$$

Residual

- $u_h \in L^2(\Omega)$ arbitrary
- $\mathcal{R}(u_h) \in H^1_+(\Omega)'$,

$$\langle \mathcal{R}(u_h), v\rangle := (f, v) + (u_h, \boldsymbol{b}\cdot\nabla v), \qquad v \in H^1_+(\Omega)$$

- dual norm (velocity-scaled)

$$\|\mathcal{R}(u_h)\|_{\boldsymbol{b};\, H^1_+(\Omega)'} := \sup_{v \in H^1_+(\Omega)\setminus\{0\}} \frac{\langle \mathcal{R}(u_h), v\rangle}{\|\boldsymbol{b}\cdot\nabla v\|}$$

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Error–residual equivalence

Theorem (Error–residual equivalence)

Let u be the ultra-weak solution. Then

$$\|u - u_h\| = \|\mathcal{R}(u_h)\|_{\boldsymbol{b};\, H^1_+(\Omega)'} \qquad \forall u_h \in L^2(\Omega).$$

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Hat functions orthogonality of the residual

Assumption ($\psi_{\mathbf{a}}$-orthogonality of the residual)

The residual $\mathcal{R}(u_h) \in H^1_+(\Omega)'$ *satisfies*

$$\langle \mathcal{R}(u_h), \psi_{\mathbf{a}} \rangle = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} + (u_h, \mathbf{b}\cdot\nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}} \cup \mathcal{V}_h^{\partial_-\Omega}.$$

- holds for the PG1, PG2, and dG schemes

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Patchwise potential reconstruction

Definition (Patchwise potential reconstruction)

Let $u_h \in L^2(\Omega)$ satisfy the $\psi_{\mathbf{a}}$-orthogonality assumption. For all vertices $\mathbf{a} \in \mathcal{V}_h$, let $s_h^{\mathbf{a}} \in X_h^{\mathbf{a}}$ be the solution of the advection–reaction problem on the patch $\omega_{\mathbf{a}}$

$$(\mathbf{b}\cdot\nabla(\psi_{\mathbf{a}} s_h^{\mathbf{a}}), v_h)_{\omega_{\mathbf{a}}} = (f\psi_{\mathbf{a}} + (\mathbf{b}\cdot\nabla\psi_{\mathbf{a}})\, u_h, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in Y_h^{\mathbf{a}},$$

with $X_h^{\mathbf{a}} := \mathcal{P}^{k'}(\mathcal{T}_{\mathbf{a}}) \cap H^1(\omega_{\mathbf{a}})$, $Y_h^{\mathbf{a}} := \mathcal{P}^{k'}(\mathcal{T}_{\mathbf{a}})$, and $k' \geq 0$. Then define

$$s_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \psi_{\mathbf{a}} s_h^{\mathbf{a}} \in \mathcal{P}^{k'+1}(\mathcal{T}_h) \cap H^1_-(\Omega).$$

- s_h matches with the usual weak formulation:

$$(f - \mathbf{b}\cdot\nabla s_h, v_h)_K = 0 \qquad \forall v_h \in \mathcal{P}^{k'}(K), \quad \forall K \in \mathcal{T}_h$$

- the hat-function-weighted difference $\psi_{\mathbf{a}}(s_h^{\mathbf{a}} - u_h)$ is a lifting of the local hat-function-weighted residual by a local advection problem:

$$(\psi_{\mathbf{a}}(u_h - s_h^{\mathbf{a}}), \mathbf{b}\cdot\nabla v_h)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}(u_h), \psi_{\mathbf{a}} v_h\rangle = (f, \psi_{\mathbf{a}} v_h)_{\omega_{\mathbf{a}}} + (u_h, \mathbf{b}\cdot\nabla(\psi_{\mathbf{a}} v_h))_{\omega_{\mathbf{a}}}$$

$$\forall v_h \in Y_h^{\mathbf{a}} \cap H^1(\omega_{\mathbf{a}}),\ v_h(\mathbf{a}) = 0 \text{ when } \mathbf{a} \in \mathcal{V}_h^{\partial_+\Omega}$$

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A posteriori error estimate: reliability

Theorem (Guaranteed a posteriori error estimate)

Let $u \in L^2(\Omega)$ be the ultra-weak solution and let $u_h \in L^2(\Omega)$ be arbitrary subject to the $\psi_{\mathbf{a}}$-orthogonality assumption. Furthermore, let s_h be the patchwise potential reconstruction with $k' \geq 0$. Then

$$\|u - u_h\| \leq \eta := \left\{ \sum_{K \in \mathcal{T}_h} \left(\eta_{\mathrm{NC},K} + \eta_{\mathrm{osc},K}\right)^2 \right\}^{1/2}.$$

- $\eta_{\mathrm{NC},K} := \|u_h - s_h\|_K$: comparison of approximation u_h and reconstruction s_h
- $\eta_{\mathrm{osc},K} := \frac{h_K}{\pi|\mathbf{b}|}\|(I - \Pi_{\mathcal{P}^{k'}(\mathcal{T}_h)})f\|_K$: data oscillation; $\Pi_{\mathcal{P}^{k'}(\mathcal{T}_h)}$ is the $L^2(\Omega)$-orthogonal projection onto $\mathcal{P}^{k'}(\mathcal{T}_h)$

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A posteriori error estimate: efficiency and robustness

Theorem (Global and local efficiency and robustness)

Let the reliability assumptions hold. Let, additionally, $u_h \in \mathcal{P}^k(\mathcal{T}_h)$, $k \geq 0$, and $k' \geq k$. Then

$$\|u_h - s_h\| \leq 2 C_{\rm cont,PF} \|u - u_h\| + \textit{data oscillation},$$

where $C_{\rm cont,PF}$ only depends on mesh shape-regularity,

$$C_{\rm cont,PF} := \max_{\mathbf{a} \in \mathcal{V}_h} \left(1 + C_{{\rm PF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_\infty\right) \leq 3 \textit{ for uniform meshes.}$$

More precisely, for all mesh elements $K \in \mathcal{T}_h$,

$$\eta_{{\rm NC},K} \leq C_{\rm cont,PF} \sum_{\mathbf{a} \in \mathcal{V}_K} \|u - u_h\|_{\omega_{\mathbf{a}}} + \sum_{\mathbf{a} \in \mathcal{V}_K} \frac{h_{\omega_{\mathbf{a}}}}{\pi |\mathbf{b}|} \|(I - \Pi_{\mathcal{P}^{k'}(\mathcal{T}_{\mathbf{a}})})(f \psi_{\mathbf{a}})\|_{\omega_{\mathbf{a}}}.$$

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Outline

Effectivity index

$$I_{\mathrm{eff}} := \frac{\eta}{\|u - u_h\|}$$

$f(x) = x^2 + x + \sin(2\pi x_{i-1})$ on K_i, $1 \le i \le n$: robustness wrt b

$k = k' = 1$, PG2		b | I_{eff}				
# Elements	DOF(u_h)	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
4	8	1.234	1.234	1.234	1.234	1.234
16	32	1.058	1.058	1.058	1.058	1.058
64	128	1.014	1.014	1.014	1.014	1.014
256	512	1.004	1.004	1.004	1.004	1.004

$k = k' = 1$, dG		b | I_{eff}				
# Elements	DOF(u_h)	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
4	8	1.126	1.126	1.126	1.126	1.126
16	32	1.032	1.032	1.032	1.032	1.032
64	128	1.008	1.008	1.008	1.008	1.008
256	512	1.002	1.002	1.002	1.002	1.002

$f(x) = x^2 + x + \sin(2\pi x_{i-1})$ on K_i, $1 \le i \le n$: robustness wrt b

$k = k' = 1$, PG2		b | $I_{\rm eff}$				
# Elements	DOF(u_h)	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
4	8	1.234	1.234	1.234	1.234	1.234
16	32	1.058	1.058	1.058	1.058	1.058
64	128	1.014	1.014	1.014	1.014	1.014
256	512	1.004	1.004	1.004	1.004	1.004

$k = k' = 1$, dG		b | $I_{\rm eff}$				
# Elements	DOF(u_h)	10^{-4}	10^{-2}	10^{0}	10^{2}	10^{4}
4	8	1.126	1.126	1.126	1.126	1.126
16	32	1.032	1.032	1.032	1.032	1.032
64	128	1.008	1.008	1.008	1.008	1.008
256	512	1.002	1.002	1.002	1.002	1.002

$f(x) = \tan^{-1}(x)$, $\boldsymbol{b} = 1$, PG2: robustness wrt k

$k = k' = 0$						
# Elements	# DOF(u_h)	$\|u - u_h\|$	η	η_{NC}	η_{osc}	I_{eff}
4	4	3.562e-02	3.951e-02	3.574e-02	4.601e-03	1.11
16	16	8.934e-03	9.161e-03	8.936e-03	2.877e-04	1.03
64	64	2.234e-03	2.248e-03	2.234e-03	1.798e-05	1.01
256	256	5.585e-04	5.593e-05	5.585e-04	1.124e-06	1.00
1024	1024	1.396e-04	1.397e-05	1.396e-04	7.025e-08	1.00

$k = k' = 1$						
4	8	1.868e-03	1.955e-03	1.867e-03	9.783e-05	1.05
16	32	1.167e-04	1.181e-04	1.167e-04	1.531e-06	1.02
64	128	7.294e-06	7.315e-06	7.294e-06	2.393e-08	1.00
256	512	4.559e-07	4.562e-07	4.559e-07	3.739e-10	1.00
1024	2048	2.849e-08	2.849e-08	2.849e-08	5.843e-12	1.00

$f(x) = \tan^{-1}(x)$, $\boldsymbol{b} = 1$, PG2: robustness wrt k

# Elements	# DOF(u_h)	$\|u - u_h\|$	η	η_{NC}	η_{osc}	I_{eff}
$k = k' = 2$						
4	12	2.600e-05	2.844e-05	2.598e-05	3.967e-06	1.09
16	48	4.066e-07	4.154e-07	4.066e-07	1.558e-08	1.02
64	192	6.354e-09	6.387e-09	6.354e-09	6.091e-11	1.01
256	768	9.928e-11	9.941e-11	9.928e-11	2.379e-13	1.00
1024	3072	1.552e-12	1.551e-12	1.551e-12	9.294e-16	1.00
$k = k' = 3$						
4	16	7.859e-07	9.299e-07	7.852e-07	1.803e-07	1.18
16	64	3.085e-09	3.213e-09	3.085e-09	1.775e-10	1.04
64	256	1.205e-11	1.217e-11	1.205e-11	1.735e-13	1.01
256	1024	4.730e-14	4.730e-14	4.718e-14	1.694e-16	1.00
$k = k' = 4$						
4	20	2.851e-08	3.517e-08	2.847e-08	8.486e-09	1.23
16	80	2.804e-11	2.948e-11	2.804e-11	2.095e-12	1.05
64	320	2.753e-14	2.776e-14	2.742e-14	5.118e-16	1.01

$f(x) = \tan^{-1}(x)$, $\boldsymbol{b} = 1$, dG: robustness wrt k

$k = k' = 1$						
# Elements	# DOF(u_h)	$\|u - u_h\|$	η	η_{NC}	η_{osc}	I_{eff}
4	8	3.021e-03	3.136e-03	3.048e-03	9.783e-05	1.04
16	32	1.901e-04	1.919e-03	1.906e-04	1.531e-06	1.01
64	128	1.190e-05	1.193e-05	1.191e-05	2.393e-08	1.00
256	512	7.444e-07	7.447e-07	7.445e-07	3.739e-10	1.00
1024	2048	4.653e-08	4.653e-08	4.653e-08	5.843e-12	1.00
$k = k' = 2$						
4	12	4.045e-05	4.260e-05	4.210e-05	3.967e-06	1.05
16	48	6.307e-07	6.386e-07	6.299e-07	1.558e-08	1.01
64	192	9.847e-09	9.877e-09	9.844e-09	6.091e-11	1.00
256	768	1.538e-10	1.539e-10	1.538e-10	2.379e-13	1.00
1024	3072	2.403e-12	2.403e-12	2.403e-12	9.294e-16	1.00

$f(x) = \tan^{-1}(x)$, $\boldsymbol{b} = 1$, dG: robustness wrt k

$k = k' = 3$						
# Elements	# DOF(u_h)	$\|u - u_h\|$	η	η_{NC}	η_{osc}	I_{eff}
4	16	1.169e-06	1.328e-06	1.186e-06	1.803e-07	1.14
16	64	4.647e-09	4.791e-09	4.664e-09	1.775e-10	1.03
64	256	1.821e-11	1.834e-11	1.822e-11	1.735e-13	1.01
256	1024	7.181e-14	7.184e-14	7.172e-14	1.694e-16	1.00
$k = k' = 4$						
4	20	4.252e-08	4.895e-08	4.240e-08	8.486e-09	1.15
16	80	4.180e-11	4.323e-11	4.179e-11	2.095e-12	1.03
64	320	4.094e-14	4.117e-14	4.083e-14	5.118e-16	1.01

Outline

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Functional setting

Advection operator and its formal adjoint

$$\mathcal{L}: v \mapsto \boldsymbol{b}\cdot\nabla v,$$
$$\mathcal{L}^*: v \mapsto -\nabla\cdot(\boldsymbol{b}v) = -\boldsymbol{b}\cdot\nabla v$$

Graph spaces

$$H(\mathcal{L},\Omega) := \left\{v \in L^2(\Omega),\, \mathcal{L}v \in L^2(\Omega)\right\},$$
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Graph spaces with boundary conditions

$$H_0(\mathcal{L},\Omega) := \{v \in H(\mathcal{L},\Omega),\, v = 0 \text{ on } \partial_-\Omega\},$$
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Integration by parts

$$(v,\boldsymbol{b}\cdot\nabla w) + (\boldsymbol{b}\cdot\nabla v, w) = (\boldsymbol{b}\cdot\boldsymbol{n}v, w) \qquad \forall v \in H(\mathcal{L},\Omega),\, w \in H(\mathcal{L}^*,\Omega)$$

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Functional setting

Advective field $\boldsymbol{b}$

- $\boldsymbol{b} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R})$ is divergence-free
- $\boldsymbol{b}$ is Ω-filling and there exists a unit vector $\boldsymbol{k} \in \mathbb{R}^d$ such that, for $\alpha > 0$,

$$\forall x \in \overline{\Omega}, \qquad \boldsymbol{b}(x)\cdot\boldsymbol{k} \geq \alpha$$

Streamline Poincaré inequality

$$\|v\| \leq C_{\mathrm{P},\boldsymbol{b},\Omega}\|\boldsymbol{b}\cdot\nabla v\| \qquad \forall v \in H_0(\mathcal{L},\Omega), \qquad C_{\mathrm{P},\boldsymbol{b},\Omega} \leq 2h_\Omega/\alpha$$

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Weak solution and residual

Ultra-weak solution

Find $u \in L^2(\Omega)$ such that

$$-(u, \boldsymbol{b}{\cdot}\nabla v) = (f, v) \qquad \forall v \in H_0(\mathcal{L}^*, \Omega).$$

Residual

- $u_h \in L^2(\Omega)$ arbitrary
- $\mathcal{R}(u_h) \in H_0(\mathcal{L}^*, \Omega)'$,

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) + (u_h, \boldsymbol{b}{\cdot}\nabla v), \qquad v \in H_0(\mathcal{L}^*, \Omega)$$

- dual norm (velocity-scaled)

$$\|\mathcal{R}(u_h)\|_{\boldsymbol{b};\, H_0(\mathcal{L}^*,\Omega)'} := \sup_{v \in H_0(\mathcal{L}^*,\Omega)\setminus\{0\}} \frac{\langle \mathcal{R}(u_h), v \rangle}{\|\boldsymbol{b}{\cdot}\nabla v\|}$$

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Error–residual equivalence

Theorem (Error–residual equivalence)

Let u be the ultra-weak solution. Then

$$\|u - u_h\| = \|\mathcal{R}(u_h)\|_{\boldsymbol{b};\, H_0(\mathcal{L}^*,\Omega)'} \qquad \forall u_h \in L^2(\Omega).$$

Outline

Patchwise potential reconstruction

Definition (Patchwise potential reconstruction)

Let $u_h \in L^2(\Omega)$. For all vertices $\boldsymbol{a} \in \mathcal{V}_h$, let $s_h^{\boldsymbol{a}} \in X_h^{\boldsymbol{a}}$ be the solution of the following least-squares problem on the patch subdomain $\omega_{\boldsymbol{a}}$:

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in X_h^{\boldsymbol{a}}} \left\{ \|\psi_{\boldsymbol{a}}(u_h - v_h)\|_{\omega_{\boldsymbol{a}}}^2 + C_{\mathrm{opt}}^2 \|f\psi_{\boldsymbol{a}} + (\boldsymbol{b}\cdot\nabla\psi_{\boldsymbol{a}})\, u_h - \boldsymbol{b}\cdot\nabla(\psi_{\boldsymbol{a}} v_h)\|_{\omega_{\boldsymbol{a}}}^2 \right\}$$

with $X_h^{\boldsymbol{a}} := \mathcal{P}^{k'}(\mathcal{T}_{\boldsymbol{a}}) \cap H_0(\mathcal{L}, \omega_{\boldsymbol{a}})$ when $\boldsymbol{a}$ lies in the inflow boundary $\partial_-\Omega$ and $X_h^{\boldsymbol{a}} := \mathcal{P}^{k'}(\mathcal{T}_{\boldsymbol{a}}) \cap H(\mathcal{L}, \omega_{\boldsymbol{a}})$ otherwise, $k' \geq 0$. Then define

$$s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \psi_{\boldsymbol{a}} s_h^{\boldsymbol{a}} \in \mathcal{P}^{k'+1}(\mathcal{T}_h) \cap H_0(\mathcal{L}, \Omega).$$

- we choose $C_{\mathrm{opt}} = 2h_\Omega/\alpha$

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A posteriori error estimate: reliability

Theorem (Guaranteed a posteriori error estimate)

Let $u \in L^2(\Omega)$ be the ultra-weak solution and let $u_h \in L^2(\Omega)$ be arbitrary. Furthermore, let s_h be the patchwise potential reconstruction with $k' \geq 0$. Then

$$\|u - u_h\| \leq \eta := \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{1/2}.$$

- $\eta_{\mathrm{NC},K} := \|u_h - s_h\|_K$: comparison of approximation u_h and reconstruction s_h
- $\eta_{\mathrm{R},K} := C_{\mathrm{P},\boldsymbol{b},\Omega} \|f - \boldsymbol{b} \cdot \nabla s_h\|_K$: not data oscillation, may be large; recall $C_{\mathrm{P},\boldsymbol{b},\Omega} \leq 2 h_\Omega / \alpha$
- heuristic modification: $\eta_{\mathrm{R},K}^{\mathrm{mod}} := (C' h_K / \alpha) \|f - \boldsymbol{b} \cdot \nabla s_h\|_K$ with $C' = 2$

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Outline

Smooth solution $u(x,y) = \sin(\pi x)\sin(\pi y)$, $\boldsymbol{b} = (1,1)^{t}$, dG

# Elements	# DOF	$\|u-u_h\|$		η_{NC}			I_{eff}
$k=1, k'=2$							
8	24	1.097e-01		9.365e-02			2.67
32	96	2.963e-02		2.584e-02			4.03
128	384	7.553e-03		6.786e-03			6.54
512	1536	1.897e-03		1.727e-03			11.8
2048	6144	4.749e-04		4.347e-04			22.7
8192	24576	1.187e-04		1.088e-04			44.7
$k=2, k'=3$							
8	48	1.882e-02		2.271e-02			3.81
32	192	2.476e-03		3.106e-03			4.50
128	768	3.135e-04		3.972e-04			7.58
512	3072	3.929e-05		4.995e-05			14.4
2048	12288	4.934e-06		6.253e-06			28.5
8192	49152	6.270e-07		7.822e-07			56.6

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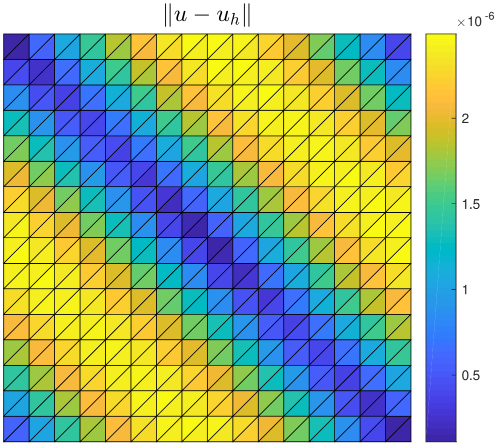
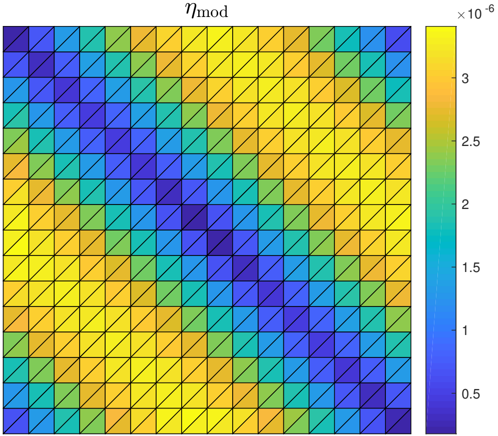
$k = 1, k' = 2$							
# Elements	# DOF	$\|u - u_h\|$	η_{mod}	η_{NC}	$\eta_{\mathrm{R}}^{\mathrm{mod}}$	$I_{\mathrm{eff}}^{\mathrm{mod}}$	I_{eff}
8	24	1.097e-01	2.284e-01	9.365e-02	2.083e-01	2.08	2.67
32	96	2.963e-02	4.894e-02	2.584e-02	4.156e-02	1.65	4.03
128	384	7.553e-03	1.101e-02	6.786e-03	8.666e-03	1.45	6.54
512	1536	1.897e-03	2.630e-03	1.727e-03	1.983e-03	1.38	11.8
2048	6144	4.749e-04	6.456e-04	4.347e-04	4.773e-04	1.35	22.7
8192	24576	1.187e-04	1.601e-04	1.088e-04	1.173e-04	1.34	44.7

$k = 2, k' = 3$							
8	48	1.882e-02	5.317e-02	2.271e-02	4.807e-02	2.82	3.81
32	192	2.476e-03	4.896e-03	3.106e-03	3.785e-03	1.97	4.50
128	768	3.135e-04	5.742e-04	3.972e-04	4.147e-04	1.83	7.58
512	3072	3.929e-05	7.076e-05	4.995e-05	5.012e-05	1.80	14.4
2048	12288	4.934e-06	8.817e-06	6.253e-06	6.216e-06	1.78	28.5
8192	49152	6.270e-07	1.107e-06	7.822e-07	7.843e-07	1.76	56.6

Smooth sol. $u(x, y) = \sin(\pi x)\sin(\pi y)$, $\boldsymbol{b} = (1, 1)^{\mathrm{t}}$, dG, $k = 2$, $k' = 3$



L^2 error



estimate

Smooth solution $u(x, y) = \sin(\pi x)\sin(\pi y)$, dG

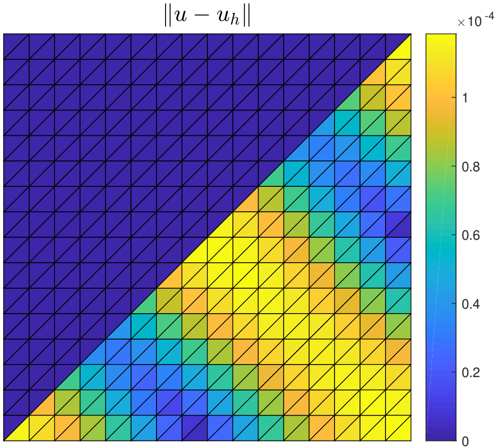
$k = 1, k' = 2, \boldsymbol{b} = (100, 100)^{\mathrm{t}}$						
# Elements	# DOF	$\|u - u_h\|$	η_{mod}	η_{NC}	$\eta_{\mathrm{R}}^{\mathrm{mod}}$	$I_{\mathrm{eff}}^{\mathrm{mod}}$
8	24	1.097e-01	2.284e-01	9.365e-02	2.083e-01	2.08
32	96	2.963e-02	4.894e-02	2.584e-02	4.156e-02	1.65
128	384	7.553e-03	1.101e-02	6.786e-03	8.666e-03	1.45
512	1536	1.897e-03	2.630e-03	1.727e-03	1.983e-03	1.38
2048	6144	4.749e-04	6.456e-04	4.347e-04	4.773e-04	1.35
8192	24576	1.187e-04	1.601e-04	1.088e-04	1.173e-04	1.34

$k = 1, k' = 2, \boldsymbol{b} = (y, x + 1)^{\mathrm{t}}$ $(\alpha = 1)$						
8	24	1.134e-01	2.435e-01	9.582e-02	2.239e-01	2.14
32	96	3.152e-02	5.787e-02	2.513e-02	5.212e-02	1.83
128	384	8.007e-03	1.393e-02	6.478e-03	1.233e-02	1.74
512	1536	2.013e-03	3.409e-03	1.636e-03	2.991e-03	1.69
2048	6144	5.053e-04	8.443e-04	4.103e-04	7.379e-04	1.67
8192	24576	1.267e-04	2.101e-04	1.027e-04	1.833e-04	1.65

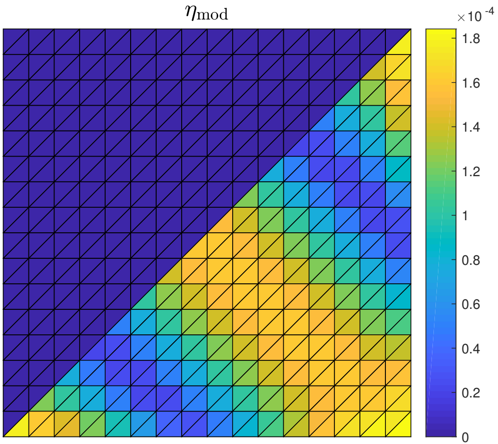
Discontinuous solution with aligned triangulation, $\boldsymbol{b} = (1, 1)^{\mathrm{t}}$, dG

# DOF	$\lVert u - u_h \rVert$	η_{mod}	η_{NC}	$\eta_{\mathrm{R}}^{\mathrm{mod}}$	$I_{\mathrm{eff}}^{\mathrm{mod}}$	I_{eff}
$k = 1, k' = 2$						
24	7.75e-02	1.61e-01	6.62e-02	1.47e-01	1.98	2.67
96	2.09e-02	3.46e-02	1.82e-02	2.94e-02	1.64	4.04
384	5.34e-03	7.78e-03	4.79e-03	6.12e-03	1.45	6.55
1536	1.34e-03	1.86e-03	1.22e-03	1.40e-03	1.38	11.8
6144	3.35e-04	4.56e-04	3.07e-04	3.37e-04	1.36	22.7
24576	8.39e-05	1.13e-04	7.70e-05	8.29e-05	1.35	44.7
$k = 2, k' = 3$						
48	1.33e-02	3.75e-02	1.61e-02	3.39e-02	2.82	3.81
192	1.75e-03	3.46e-03	2.19e-03	2.67e-03	1.97	4.50
768	2.21e-04	4.06e-04	2.81e-04	2.93e-04	1.83	7.58
3072	2.77e-05	5.00e-05	3.53e-05	3.54e-05	1.80	14.4
12288	3.48e-06	6.23e-06	4.42e-06	4.39e-06	1.78	28.5
49152	4.43e-07	7.83e-07	5.53e-07	5.54e-07	1.76	56.6

Disc. sol. with aligned triangulation, $\boldsymbol{b} = (1,1)^{\mathrm{t}}$, dG, $k = 1$, $k' = 2$



L^2 error



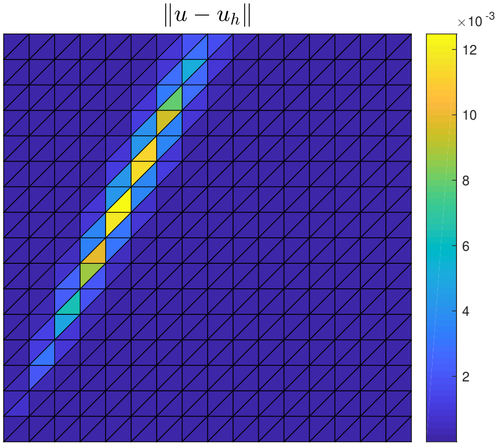
estimate

Discontinuous sol. with non-aligned triangulation, $\boldsymbol{b} = (1,2)^{\mathrm{t}}$, dG

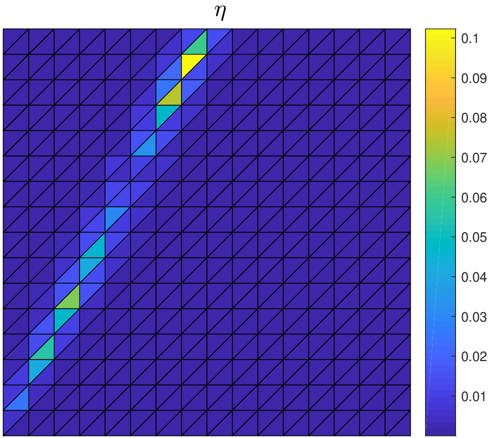
$k = 1, k' = 2$					
# DOF	$\|u - u_h\|$	η	η_{NC}	η_{R}	I_{eff}
24	1.41e-01	5.70e-01	7.60e-02	5.65e-01	4.03
96	8.36e-02 (0.76)	4.02e-01 (0.50)	3.11e-02 (1.29)	4.01e-01 (0.50)	4.80
384	5.34e-02 (0.65)	2.89e-01 (0.48)	1.17e-02 (1.41)	2.89e-01 (0.47)	5.42
1536	4.08e-02 (0.39)	2.31e-01 (0.32)	5.51e-03 (1.09)	2.31e-01 (0.32)	5.67
6144	3.16e-02 (0.37)	1.93e-01 (0.26)	2.93e-03 (0.91)	1.94e-01 (0.26)	6.13
24576	2.45e-02 (0.37)	1.70e-01 (0.18)	1.62e-03 (0.86)	1.71e-01 (0.18)	6.97

$k = 2, k' = 3$					
48	4.31e-02	4.17e-01	1.28e-01	5.65e-01	3.24
192	1.12e-02 (1.94)	2.82e-01 (0.56)	7.08e-02 (0.85)	3.76e-01 (0.59)	3.99
768	5.59e-03 (1.00)	2.29e-01 (0.30)	4.75e-02 (0.58)	2.80e-01 (0.43)	4.83
3072	2.83e-03 (0.98)	1.84e-01 (0.32)	3.50e-02 (0.44)	2.13e-01 (0.39)	5.26
12288	1.50e-03 (0.92)	1.45e-01 (0.33)	2.54e-02 (0.46)	1.61e-01 (0.40)	5.73
49152	8.41e-04 (0.83)	1.20e-01 (0.28)	1.85e-02 (0.46)	1.21e-01 (0.41)	6.47

Discontinuous sol. with non-aligned triangulation, $\boldsymbol{b} = (1, 2)^{\mathrm{t}}$, dG



L^2 error



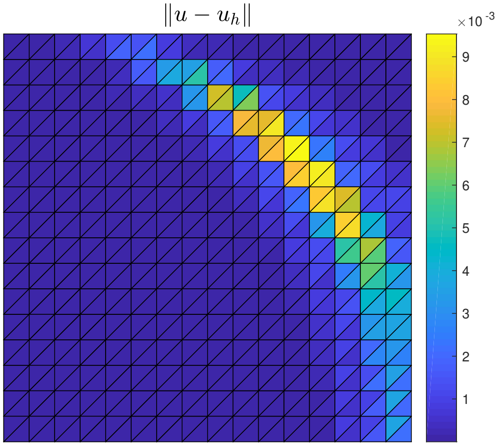
estimate

Discontinuous sol. with non-aligned triangulation, $\mathbf{b} = (y, -x)^{\mathrm{t}}$, dG

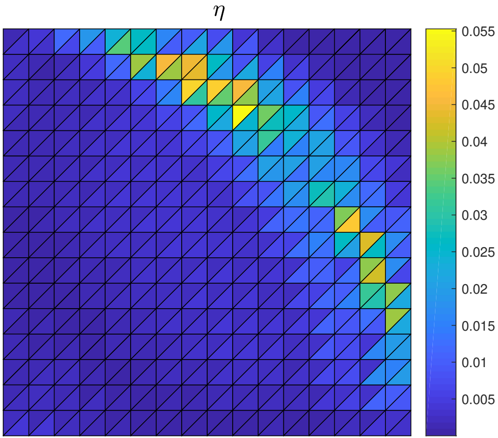
$k = 1, k' = 2$					
# DOF	$\|u - u_h\|$	η	η_{NC}	η_{R}	I_{eff}
24	1.70e-01	6.14e-01	7.30e-02	6.09e-01	3.60
96	9.31e-02 (0.87)	4.42e-01 (0.47)	2.99e-02 (1.29)	4.41e-01 (0.47)	4.75
384	6.01e-02 (0.63)	3.24e-01 (0.45)	1.16e-02 (1.37)	3.24e-01 (0.44)	5.39
1536	4.62e-02 (0.38)	2.67e-01 (0.28)	5.31e-03 (1.13)	2.68e-01 (0.27)	5.79
6144	3.57e-02 (0.37)	2.36e-01 (0.18)	2.79e-03 (0.93)	2.37e-01 (0.18)	6.61
24576	2.78e-02 (0.36)	2.29e-01 (0.04)	1.54e-03 (0.86)	2.29e-01 (0.05)	8.26

$k = 2, k' = 3$					
48	9.83e-02	4.31e-01	3.72e-02	4.29e-01	4.38
192	5.72e-02 (0.78)	2.85e-01 (0.59)	1.06e-02 (1.81)	2.85e-01 (0.59)	4.98
768	4.64e-02 (0.30)	2.34e-01 (0.29)	5.14e-03 (1.04)	2.34e-01 (0.28)	5.03
3072	3.31e-02 (0.48)	1.90e-01 (0.29)	2.78e-03 (0.89)	1.90e-01 (0.30)	5.75
12288	2.59e-02 (0.35)	1.72e-01 (0.14)	1.55e-03 (0.84)	1.72e-01 (0.14)	6.63
49152	1.92e-02 (0.43)	1.58e-01 (0.12)	8.44e-04 (0.88)	1.58e-01 (0.12)	8.27

Discontinuous sol. with non-aligned triangulation, $\boldsymbol{b} = (y, -x)^{\mathrm{t}}$, dG



L^2 error



estimate

Outline

Extension to advection–reaction problems in 1D

The advection problem

Find $u : \Omega \subset \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \boldsymbol{b}\cdot\nabla u + cu &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial_-\Omega. \end{aligned}$$

- $\boldsymbol{b} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R})$: divergence-free (constant since $d = 1$) velocity field
- $f \in L^2(\Omega)$: source term
- $c \in L^\infty(\Omega)$, $c \geq 0$: reaction coefficient

Results

Estimator η such that

$$\underbrace{\|u - u_h\|}_{\text{unknown error}} \leq \underbrace{\eta}_{\text{estimator computable from } u_h} \leq C\|u - u_h\| + \text{data oscillation},$$

where C is independent of sizes of $\boldsymbol{b}$ and c.

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- a posteriori error estimates **robust** with respect to the advective filed $\mathbf{b}$ and the polynomial degree k in **1D**
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Papers

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VOHRALÍK, M. Guaranteed and robust L^2-norm a posteriori error estimates for 1D linear advection–reaction problems. In preparation, 2024.

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Thank you for your attention!