# Goal-oriented a posteriori error estimation for conforming and nonconforming approximations with inexact solvers

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DanieTack

#### Model problem

Find  $u: \Omega \to \mathbb{R}$  such that

# $\begin{aligned} -\nabla \cdot (\underline{\boldsymbol{K}} \nabla \boldsymbol{u}) &= \boldsymbol{f} & \text{in } \Omega, \\ -\underline{\boldsymbol{K}} \nabla \boldsymbol{u} \cdot \boldsymbol{n} &= \sigma_{\mathrm{N}} & \text{on } \Gamma_{\mathrm{N}}, \\ \boldsymbol{u} &= \boldsymbol{u}_{\mathrm{D}} & \text{on } \Gamma_{\mathrm{D}} \end{aligned}$

Numerical approximation

 $u_h^i \in \mathbb{P}_p(\mathcal{T}_h)$  arbitrary: nonconforming, high-order, on iteration *i* of an algebraic iterative solver (corresponding dual approximation  $\tilde{u}_h^i \in \mathbb{P}_p(\mathcal{T}_h)$  arbitrary as well) Goals For goal functional  $Q(v) := (\tilde{t}, v) - (\underline{K} \nabla v \cdot n, \tilde{u}_D)_{\Gamma_D}, v \in H^1(\mathcal{T}_h)$ , design

#### Model problem

Find  $\mu: \Omega \to \mathbb{R}$  such that  $-\nabla \cdot (\mathbf{K} \nabla u) = f$ in  $\Omega$ .  $-\mathbf{K}\nabla u \cdot \mathbf{n} = \sigma_{\mathrm{N}}$ on  $\Gamma_N$ , on  $\Gamma_{\rm D}$  $u = u_{\rm D}$ 

#### Numerical approximation

 $u'_{P} \in \mathbb{P}_{P}(\mathcal{T}_{P})$  arbitrary: nonconforming, high-order, on iteration *i* of an algebraic

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a posteriori estimate

$$\left| \mathcal{Q}(u) - \mathcal{Q}(u_h^l) 
ight| \leq rac{\eta_h^l \ ilde \eta_h^l}{2}$$

dual discrete problems (stop: on it: i)
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M. Vohralík

- inexactly solve both primal and dual discrete problems (stop. on it. 7)
- estimate spatial error distribution
- adaptively refine mesh

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Case 1 Case 2 Case 3

## 2D regular solution and uniform mesh refinement





Exact solution

Goal functional  $Q(v) = \frac{1}{|\omega|} (1, v)_{\omega}$ 

Case 1 Case 2 Case 3

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Case 1 Case 2 Case 3

## 2D regular solution and uniform mesh refinement





Case 1 Case 2 Case 3

## 3D singular solution and adaptive mesh refinement





Case 1 Case 2 Case 3

## 2D heterogeneous media and uniform mesh refinement





# $$\begin{split} & \text{Outflow goal functional} \\ & \text{Q}(\nu) = - \big(\underline{\textit{K}} \nabla \nu \cdot \textit{\textbf{n}}, \tilde{\textit{u}}_{\mathrm{D}} \big)_{\Gamma_{\mathrm{D}}} \\ & \tilde{\textit{u}}_{\mathrm{D}}|_{\{y=0\}} = 0, \; \tilde{\textit{u}}_{\mathrm{D}}|_{\{y=2200\}} = 1 \end{split}$$



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Case 1 Case 2 Case 3

## 2D heterogeneous media and uniform mesh refinement

