Numerical methods, a priori and a posteriori error estimates, and *hp* finite elements

Martin Vohralík

Inria Paris & Ecole des Ponts

Prague, April 17, 2024

Outline

1. Numerical approximations of PDEs
2. A priori and a posteriori error analysis
3. A posteriori error estimates
4. Mesh adaptivity
5. *hp* finite elements: a priori error estimates
6. *hp* finite elements: mesh & polynomial degree adaptivity

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Numerical approximations of PDEs

Numerical methods

- mathematically-based algorithms evaluated by **computers**
- deliver **approximate solutions**
- conception: more effort $\Rightarrow$ closer to the unknown solution
- example: elastic string

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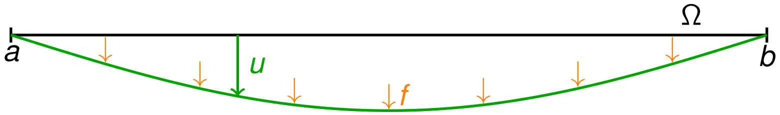

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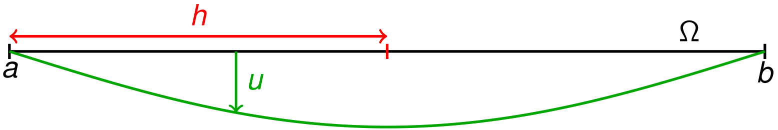

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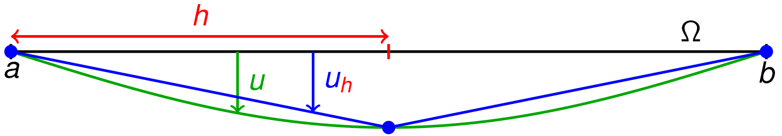

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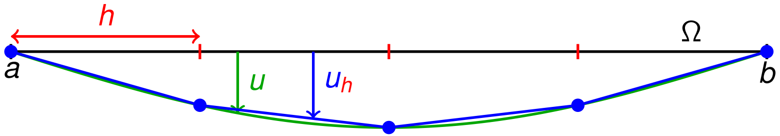

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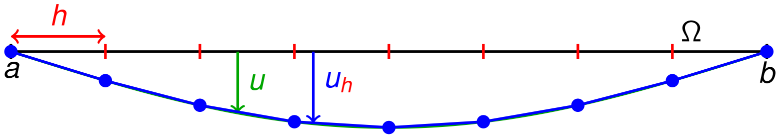

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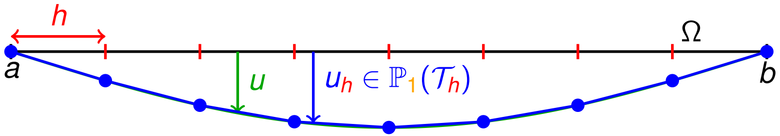

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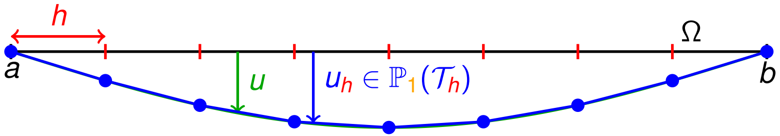

Numerical approximation u_h and its convergence to u

Error

$$\|\nabla(u-u_h)\| = \left\{\int_a^b |(u-u_h)'|^2\right\}^{\frac{1}{2}}$$

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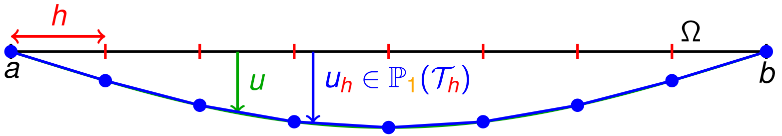

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Polynomial degree p

$$u_h \in \mathbb{P}_p(\mathcal{T}_h)$$

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A priori & a posteriori error estimates

Crucial questions

1. Does the method **converge**?
   $\|\nabla(u - u_h)\| \to 0$? For $h \searrow 0$? For $p \nearrow \infty$?
2. At which **speed**? $\|\nabla(u - u_h)\| \leq ?$
3. Is the **analysis optimal**? Is **uniform refinement** $h \searrow 0$ or $p \nearrow \infty$ **optimal**?

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3. **Adaptivity**, focusing, h & p refined **non uniformly**.

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Case Studies in Engineering Failure Analysis 3 (2015) 88–95

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A posteriori error estimates: **certify** the error

Poisson equation

$$\begin{aligned} -\Delta u = f & \quad \text{in} \quad \Omega, \\ u = 0 & \quad \text{on} \quad \partial\Omega \end{aligned}$$

Guaranteed error upper bound (reliability)

$$\underbrace{\|\nabla(u - u_h)\|}_{\text{unknown error}} \leq \underbrace{\eta(u_h)}_{\text{computable estimator}}$$

Global error lower bound (global efficiency; mathematical equivalence of the error and estimator)

$$\eta(u_h) \leq C\|\nabla(u - u_h)\|$$

Local error lower bound (local efficiency; if the estimator predicts error in an element K, then it is in K and around)

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A posteriori error estimates: reconstructions

Theorem (Error characterization)

Let $u \in H_0^1(\Omega)$ be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$\|\nabla(u - u_h)\|^2 = \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div},\Omega) \\ \nabla\cdot\mathbf{v} = f}} \|\nabla u_h + \mathbf{v}\|^2 + \min_{v \in H_0^1(\Omega)} \|\nabla(u_h - v)\|^2.$$

Comments

- It is now enough to choose suitable $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div},\Omega)$ and $s_h \in H_0^1(\Omega)$.
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h	p	$\eta(u_h)$	rel. error estimate $\frac{\eta(u_h)}{\|\nabla u_h\|}$	$\|\nabla(u-u_h)\|$	rel. error $\frac{\|\nabla(u-u_h)\|}{\|\nabla u_h\|}$	$I^{\rm eff} = \frac{\eta(u_h)}{\|\nabla(u-u_h)\|}$
h_0	1	1.25	28%	1.07	24%	1.17
$\approx h_0/2$						
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$\approx h_0/4$	3	2.62×10^{-4}	5.9×10^{-3}%	2.60×10^{-4}	5.9×10^{-3}%	
$\approx h_0/8$	4	2.60×10^{-7}	5.9×10^{-6}%	2.58×10^{-7}	5.8×10^{-6}%	

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$\approx h_0/2$		6.07×10^{-1}	14%	5.56×10^{-1}	13%	**1.09**
$\approx h_0/4$		3.10×10^{-1}	7.0%	2.92×10^{-1}	6.6%	**1.06**
$\approx h_0/8$		1.45×10^{-1}	3.3%	1.39×10^{-1}	3.1%	**1.04**
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A. Ern, M. Vohralík, SIAM Journal on Numerical Analysis (2015)
V. Dolejší, A. Ern, M. Vohralík, SIAM Journal on Scientific Computing (2016)

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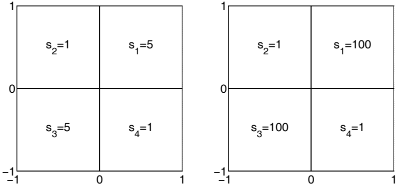
6. *hp* finite elements: mesh & polynomial degree adaptivity

Problem with singular solution

- consider the pure diffusion equation

$$-\nabla\cdot \boldsymbol{S}\nabla u = 0 \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

- discontinuous and inhomogeneous $\boldsymbol{S}$, two cases:

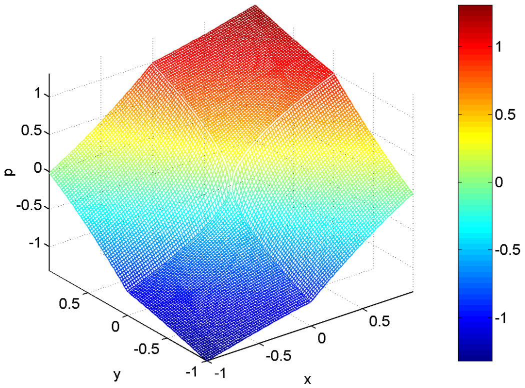


- analytical solution: **singularity** at the origin

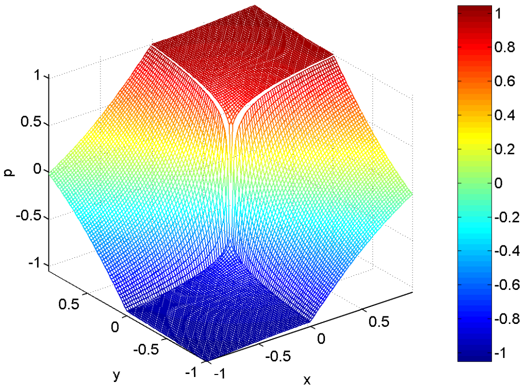
$$u(r,\theta) = r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution, $u \in H^{1+\alpha}(\Omega)$

Analytical solutions

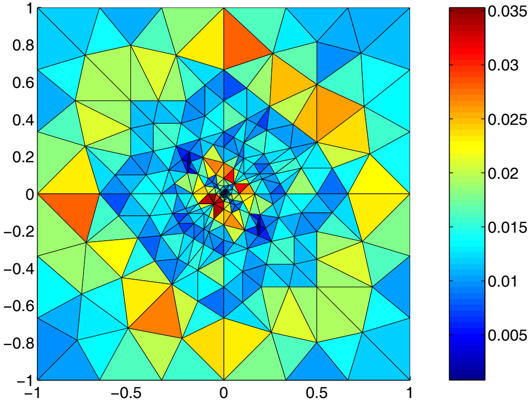


Case 1 ($\alpha \approx 0.54$)

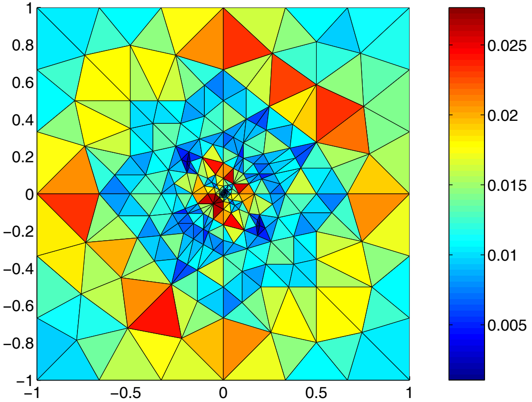


Case 2 ($\alpha \approx 0.13$)

Where is the error **localized**?



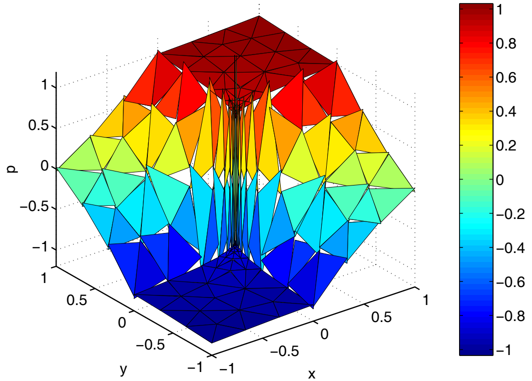
Estimated error distribution $\eta_K(u_h)$



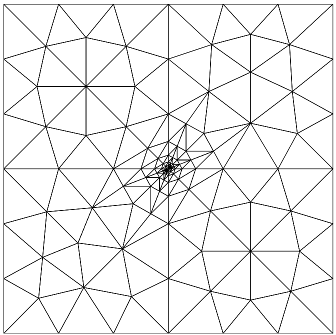
Exact error distribution $\|\nabla(u - u_h)\|_K$

M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Can we adapt the mesh to better approximate the solution?



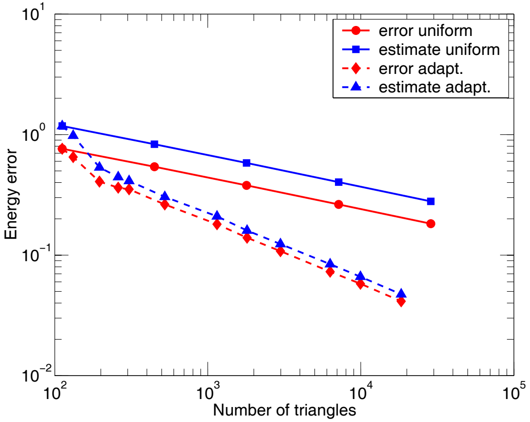
Nonconforming finite elements



Adaptively refined mesh

M. Vohralík, SIAM Journal on Numerical Analysis (2007)

Does this lead to a better error decrease?



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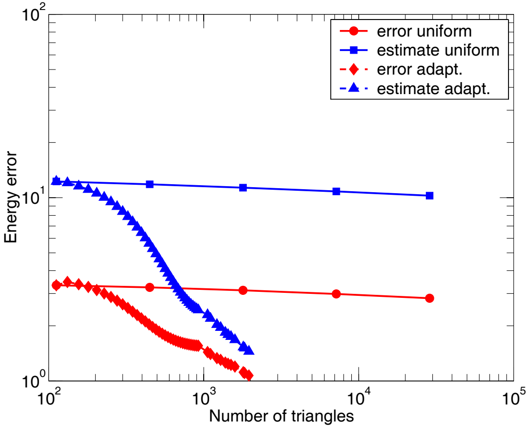

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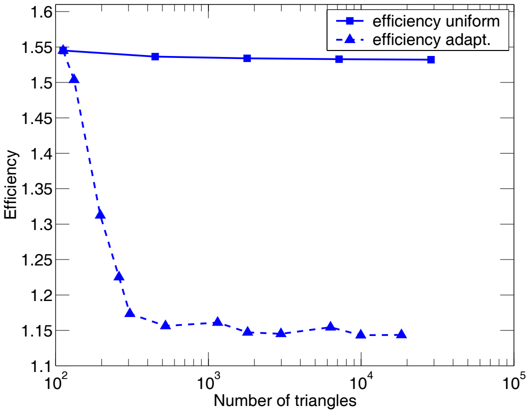
Quality of the estimates for a singular solution



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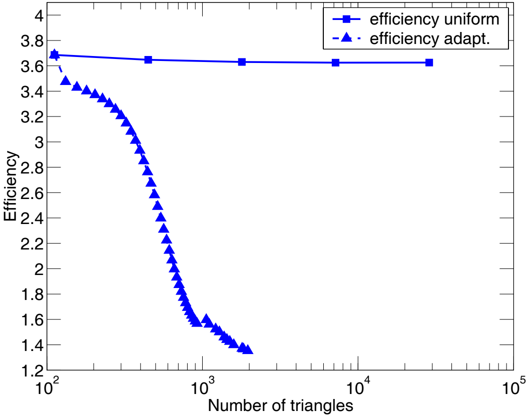

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Adaptive mesh refinement

Mesh adaptivity

- Dörfler marking: subset $\mathcal{M}_\ell$ containing θ-fraction of the estimates

$$\sum_{K\in\mathcal{M}_\ell} \eta_K(u_\ell)^2 \geq \theta^2 \sum_{K\in\mathcal{T}_\ell} \eta_K(u_\ell)^2$$

- refine the elements in $\mathcal{M}_\ell$

Convergence on a sequence of adaptively refined meshes $\mathcal{T}_\ell$

- $$\|\nabla(u-u_\ell)\| \to 0 \text{ for } \ell \to \infty$$
- some mesh elements may not be refined at all: ~~$h \searrow 0$ **uniformly**~~

Optimal error decay rate wrt degrees of freedom

- $$\|\nabla(u-u_\ell)\| \leq C|\mathrm{DoF}_\ell|^{-p/d} \qquad \text{(replaces } h^p\text{)}$$
- same for smooth & singular solutions: ~~higher-order only pays-off for sm. sol.~~
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h vs. *hp* a priori analysis

Theorem (Deny–Lions/Bramble–Hilbert)

For all $K \in \mathcal{T}_h$ *and* $v \in H^{p+1}(K)$,

$$\min_{v_h \in \mathcal{P}_p(K)} \|\nabla(v - v_h)\|_K \leq \sqrt{(p+1)!}\left(\frac{h_K}{\pi}\right)^p |v|_{H^{p+1}(K)}.$$

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- C does not depend on p
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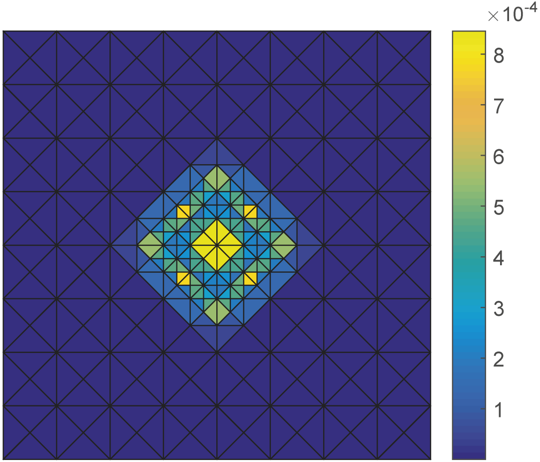

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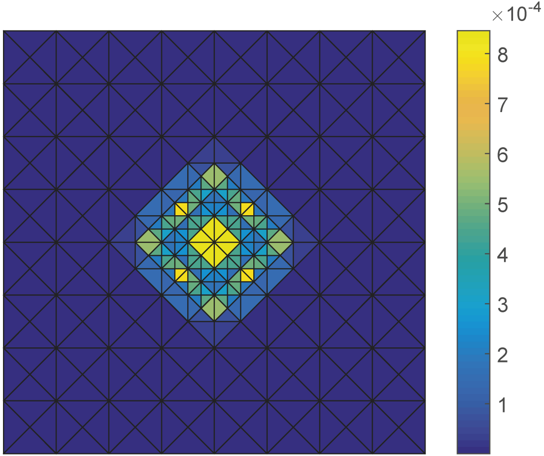
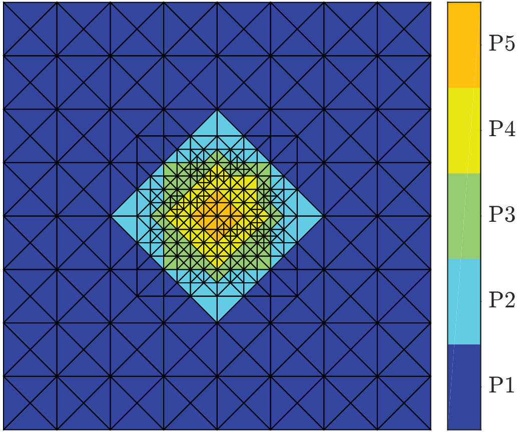


Exact error distribution $\|\nabla(u - u_h)\|_K$

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Can we decrease the error efficiently? *hp* adaptivity, (**smooth** solution)



Mesh $\mathcal{T}_\ell$ and pol. degrees p_K

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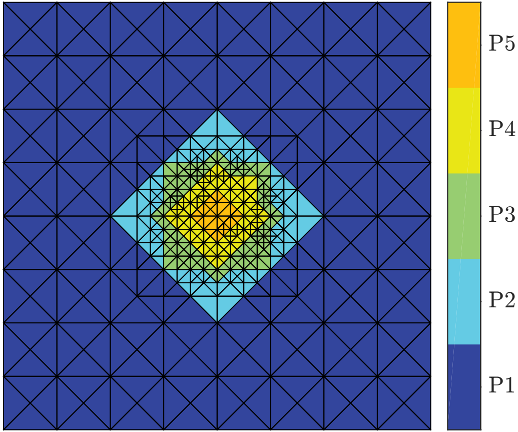

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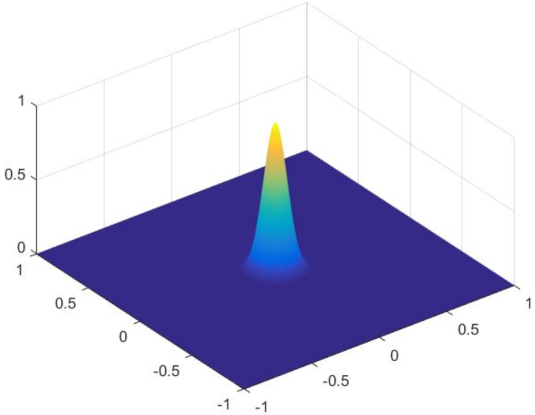

Exact solution

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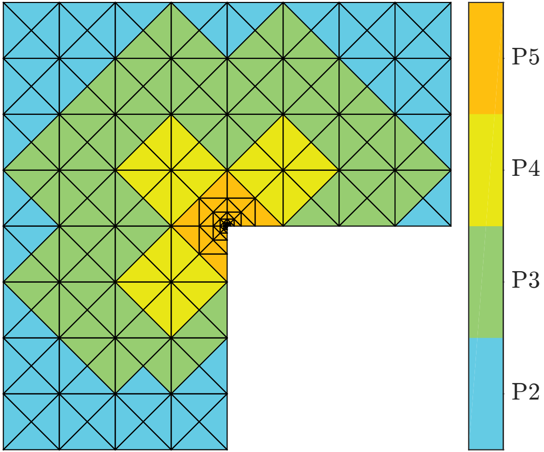

Mesh $\mathcal{T}_\ell$ and polynomial degrees p_K

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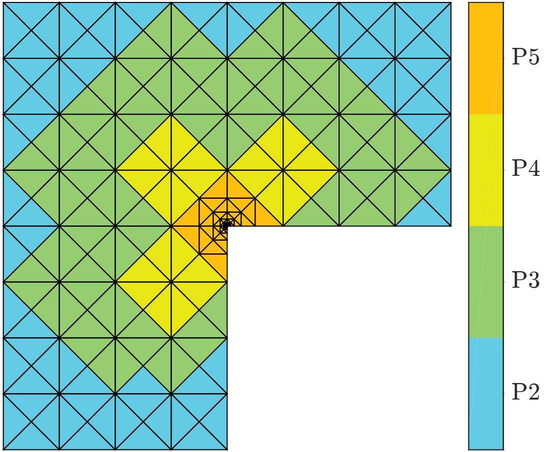

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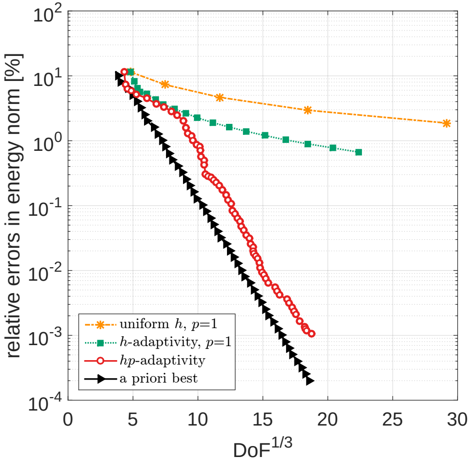

Relative error as a function of DoF

P. Daniel, A. Ern, I. Smears, M. Vohralík, Computers & Mathematics with Applications (2018)

Adaptive mesh & polynomial degree refinement

Mesh & polynomial degree adaptivity

- **decision** between h or p refinement needs to be done
- much harder than just h-adaptivity

Convergence on a sequence of adaptively refined hp spaces V_ℓ

- $$\|\nabla(u-u_\ell)\| \to 0 \text{ for } \ell \to \infty$$
- ~~$h \searrow 0$ **uniformly**~~, ~~$p \nearrow 0$ **uniformly**~~

Optimal error decay rate wrt degrees of freedom

- for $d=2$, hp refinement gives

$$\|\nabla(u-u_\ell)\| \leq C_1 \frac{1}{e^{C_2 \mathrm{DoF}^{1/3}}}$$

- **exponential** convergence **rate**
- for $d=2$ and $p=1$ fixed, adaptive mesh h refinement only gives

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