Examination
May, 15, 2015

3 hours, no document authorized, no calculator, no computer, no phone

the scale of 20 points is indicative

please give all details and justify the answers rigorously

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal (polyhedral) domain (open, bounded, and connected set), $\mathcal{T}_h$ its simplicial mesh, $\mathcal{E}_h$ the set of all its faces, and $\mathcal{E}_h^{\rm int}$ the set of its interior faces.

Question 1. (Equilibrated flux and potential reconstructions in nonconforming finite elements) (5 points)

Consider the Laplace equation

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1a}$$
$$u = 0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where $f \in L^2(\Omega)$, and its Crouzeix–Raviart nonconforming finite element discretization. This consists in finding $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h,$$

where $V_h \subset \mathbb{P}_k(\mathcal{T}_h)$ is the space of such piecewise polynomial functions of degree k that $\langle [\![v_h]\!], g_h \rangle_e = 0$ for all faces $e \in \mathcal{E}_h$ and all polynomials g_h on e of degree at most $k-1$.

1) Demonstrate how to obtain an equilibrated flux reconstruction $\boldsymbol{\sigma}_h \in \mathbf{RTN}_k(\mathcal{T}_h)$ which satisfies

$$(\nabla\cdot\boldsymbol{\sigma}_h, v_h)_K = (f, v_h)_K \qquad \forall v_h \in \mathbb{P}_k(K),\ \forall K \in \mathcal{T}_h, \tag{2}$$

by some local mixed finite element problems.

2) Demonstrate how to obtain a potential reconstruction $s_h \in \mathbb{P}_{k+1}(\mathcal{T}_h) \cap H_0^1(\Omega)$ by some local conforming finite element problems.

3) Show how $\boldsymbol{\sigma}_h$ and s_h can be used to obtain a guaranteed a posteriori estimate on the energy error $\|\nabla(u - u_h)\|$.

Question 2. (Properties of the weak solution of the Stokes equation) (3.5 points)

Consider the Stokes problem: for $\mathbf{f} \in [L^2(\Omega)]^d$, find $\mathbf{u}$ and p such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{3a}$$
$$\nabla\cdot\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{3b}$$
$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{3c}$$

1) Recall the weak (variational) formulation of (3a)–(3c).

2) From the variational formulation, define the stress by $\underline{\boldsymbol{\sigma}} := \nabla\mathbf{u} - p\underline{\boldsymbol{I}}$. To which functional spaces $\mathbf{u}$ and $\underline{\boldsymbol{\sigma}}$ belong? What do they satisfy? Give a rigorous proof.

Question 3. (Advection–diffusion–reaction equation) (5 points)

Let $f \in L^2(\Omega)$, $r \in L^\infty(\Omega)$, $\mathbf{w} \in [W^{1,\infty}(\Omega)]^d$ such that $\frac{1}{2}\nabla\cdot\mathbf{w} + r \geq 0$, and $\underline{\mathbf{K}} \in [L^\infty(\Omega)]^{d\times d}$, symmetric with uniformly positive smallest eigenvalue. Consider the following problem: find u such that

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla u) + \nabla\cdot(\mathbf{w}u) + ru = f \qquad \text{in } \Omega, \tag{4a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4b}$$

1) Recall the variational formulation of (4a)–(4b).

2) Define the flux by $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u + \mathbf{w}u$. Prove that $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{\sigma} = f - ru$.

3) For $u, v \in H_0^1(\Omega)$, define the bilinear form $\mathcal{B}$ by

$$\mathcal{B}(u, v) := (\underline{\mathbf{K}}\nabla u, \nabla v) - (\mathbf{w}u, \nabla v) + (ru, v).$$

Identify an augmented norm $|||\cdot|||_\oplus$ which satisfies, for any $v \in H_0^1(\Omega)$,

$$\sup_{\varphi\in H_0^1(\Omega);\, |||\varphi|||=1} \mathcal{B}(v, \varphi) \leq |||v|||_\oplus \leq 3 \sup_{\varphi\in H_0^1(\Omega);\, |||\varphi|||=1} \mathcal{B}(v, \varphi),$$

where $|||\cdot|||$ is the energy norm of the problem (4a)–(4b).

Question 4. (Nonlinear Laplace equation and linearization error) (4 points)

Let $a(x) := x^{p-2}$ for some real number $p \in (1, +\infty)$ and let

$$\overline{\boldsymbol{\sigma}}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

Define q by the relation $\frac{1}{p} + \frac{1}{q} = 1$ and consider the nonlinear Laplace equation: for $f \in L^q(\Omega)$, find u such that

$$-\nabla\cdot\overline{\boldsymbol{\sigma}}(\nabla u) = f \qquad \text{in } \Omega, \tag{5a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{5b}$$

1) Recall the variational formulation of (5a)–(5b).

2) Let u be the weak solution of (5a)–(5b), let $u_h \in W_0^{1,p}(\Omega)$ be arbitrary, and set

$$\mathcal{J}_u(u_h) = \sup_{\varphi\in W_0^{1,p}(\Omega);\, \|\nabla\varphi\|_p=1} (\overline{\boldsymbol{\sigma}}(\nabla u) - \overline{\boldsymbol{\sigma}}(\nabla u_h), \nabla\varphi).$$

Derive an a posteriori error estimate of the form

$$\mathcal{J}_u(u_h) \leq \left\{ \sum_{K\in\mathcal{T}_h} (\eta_K)^q \right\}^{1/q}.$$

3) Consider an approximation of the nonlinear function $\overline{\boldsymbol{\sigma}}$ by a linear (affine) one $\boldsymbol{\sigma}_{\mathrm{L}}$. Within η_K, distinguish the error in linearization of $\overline{\boldsymbol{\sigma}}$ by $\boldsymbol{\sigma}_{\mathrm{L}}$ (linearization error) and the discretization error.

Question 5. (Adaptive mesh refinement) (2.5 points)

Describe the principle of adaptive mesh refinement for efficient numerical approximation of linear elliptic (stationary) partial differential equations. This is a conceptual question, no proofs are to be given here.