Examination
May, 11, 2012

3 hours, no document authorized, no calculator, no computer, no phone

the scale of 20 points is indicative

please give all details and justify the answers rigorously

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal (polyhedral) domain (open, bounded, and connected set) and $\mathcal{T}_h$ its simplicial mesh.

Question 1. (Guaranteed a posteriori error estimate for the Stokes equation) (5 points)

The Stokes problem reads: for $\mathbf{f} \in [L^2(\Omega)]^d$, find $\mathbf{u}$ and p such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1a}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1b}$$
$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega. \tag{1c}$$

1) For $(\mathbf{v}, q), (\mathbf{z}, r) \in [H^1(\mathcal{T}_h)]^d \times L^2_0(\Omega)$, define the form

$$\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) := (\nabla \mathbf{v}, \nabla \mathbf{z}) - (\nabla \cdot \mathbf{z}, q) - (\nabla \cdot \mathbf{v}, r).$$

Recall the variational formulation of (1a)–(1c) using the form $\mathcal{B}$.

2) Let $\beta > 0$ be the constant from the inf–sup condition

$$\inf_{q \in L^2_0(\Omega)} \sup_{\mathbf{v} \in [H^1_0(\Omega)]^d} \frac{(q, \nabla \cdot \mathbf{v})}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta.$$

For $(\mathbf{v}, q) \in [H^1(\mathcal{T}_h)]^d \times L^2_0(\Omega)$, define the energy (semi-)norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2$$

and recall the stability equality

$$\inf_{(\mathbf{v},q) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)} \sup_{(\mathbf{z},r) \in [H^1_0(\Omega)]^d \times L^2_0(\Omega)} \frac{\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r))}{|||(\mathbf{z}, r)||| \, |||(\mathbf{v}, q)|||} = \frac{\sqrt{5} - 1}{2}.$$

Let $(\mathbf{u}, p)$ be the weak (variational) solution of (1a)–(1c). Let $(\mathbf{u}_h, p_h) \in [H^1(\mathcal{T}_h)]^d \times L^2_0(\Omega)$ be arbitrary. State and prove rigorously the a posteriori error estimate on the error $|||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||$.

Question 2. (Equilibrated flux reconstruction in the finite element method) (2.5 points)

Consider the finite element discretization of the Laplace equation

$$-\Delta u = f \qquad \text{in } \Omega, \tag{2a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2b}$$

$f \in L^2(\Omega)$, which consists in finding $u_h \in V_h := \mathbb{P}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$, $k \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{3}$$

Demonstrate how to obtain, by a local procedure, an equilibrated flux reconstruction $\boldsymbol{\sigma}_h \in \mathbf{RTN}_k(\mathcal{T}_h)$ which satisfies

$$(f - \nabla{\cdot}\boldsymbol{\sigma}_h, v_h)_K = 0 \qquad \forall v_h \in V_h(K) \qquad \forall K \in \mathcal{T}_h.$$

Question 3. (Efficiency of the residual-based a posteriori error indicators) (6 points)

Let $u_h, f \in \mathbb{P}_m(\mathcal{T}_h)$, $m \geq 1$, and let $\mathcal{T}_h$ be shape-regular. Let u be the weak solution of the Laplace equation (2a)–(2b).

1) Let $K \in \mathcal{T}_h$. Prove that there exists a constant C, only depending on the space dimension d, on the shape-regularity of the mesh $\mathcal{T}_h$, and on the polynomial degree m, such that

$$h_K \|f + \Delta u_h\|_K \leq C \|\nabla(u - u_h)\|_K.$$

2) Let $e \in \mathcal{E}_h^{\text{int}}$. Using the previous result, prove that there exists a constant C, only depending on the space dimension d, on the shape-regularity of the mesh $\mathcal{T}_h$, and on the polynomial degree m, such that

$$h_e^{1/2} \|[\![\nabla u_h]\!]{\cdot}\mathbf{n}_e\|_e \lesssim \|\nabla(u - u_h)\|_{\mathcal{T}_e},$$

where $\mathcal{T}_e$ stands for the two simplices that share the face e.

Question 4. (Advection–diffusion–reaction equation) (3.5 points)

Let $f \in L^2(\Omega)$, $r \in L^\infty(\Omega)$, $\mathbf{w} \in [W^{1,\infty}(\Omega)]^d$ such that $\frac{1}{2}\nabla{\cdot}\mathbf{w} + r \geq 0$, and $\underline{\mathbf{K}} \in [L^\infty(\Omega)]^{d\times d}$, symmetric with uniformly positive smallest eigenvalue. Consider the following problem: find u such that

$$-\nabla{\cdot}(\underline{\mathbf{K}}\nabla u) + \nabla{\cdot}(\mathbf{w}u) + ru = f \qquad \text{in } \Omega, \tag{4a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4b}$$

1) Recall the variational formulation of (4a)–(4b).

2) Define the flux by $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u + \mathbf{w}u$. Prove that $\boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega)$ with $\nabla{\cdot}\boldsymbol{\sigma} = f - ru$.

3) For $u, v \in H_0^1(\Omega)$, define the bilinear form $\mathcal{B}$ by

$$\mathcal{B}(u, v) := (\underline{\mathbf{K}}\nabla u, \nabla v) - (\mathbf{w}u, \nabla v) + (ru, v).$$

Identify an augmented norm $|\!|\!| \cdot |\!|\!|_\oplus$ which satisfies, for any $v \in H_0^1(\Omega)$,

$$\sup_{\varphi \in H_0^1(\Omega);\, |\!|\!|\varphi|\!|\!|=1} \mathcal{B}(v, \varphi) \leq |\!|\!|v|\!|\!|_\oplus \leq 3 \sup_{\varphi \in H_0^1(\Omega);\, |\!|\!|\varphi|\!|\!|=1} \mathcal{B}(v, \varphi),$$

where $|\!|\!| \cdot |\!|\!|$ is the energy norm of the problem (4a)–(4b).

Question 5. (Stopping criteria, balancing different error components, and adaptive strategies) (3 points)

1) Explain the principle of optimal stopping criteria for linear and nonlinear solvers.

2) On the example of the heat equation, explain the concept of identification of different error components and of their balancing.

3) Describe the principles of adaptive strategies for efficient numerical approximation of partial differential equations.

This is a conceptual question, no proofs are to be given here.