University Pierre & Marie Curie
Master of Sciences & Technologies
M2 Mathematics & Applications

A posteriori error estimates for efficiency and error control in numerical simulations
ANEDP, NM497, 2010–2011

Examination
April, 27, 2011

3 hours, no document authorized, no calculator, no computer, no phone

the scale of 20 points is indicative

please give all details and justify the answers rigorously

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygonal (polyhedral) domain (open, bounded, and connected set) and $\mathcal{T}_h$ its simplicial mesh.

Question 1. (Guaranteed a posteriori error estimate for the Laplace equation) (5 points)

The Poisson problem for the Laplace equation reads: for $f \in L^2(\Omega)$, find u such that

$$-\Delta u = f \qquad \text{in } \Omega, \tag{1a}$$
$$u = 0 \qquad \text{on } \partial\Omega. \tag{1b}$$

1) Recall the variational formulation of (1a)–(1b).

2) Let u be the weak (variational) solution of (1a)–(1b). Let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. State and prove rigorously an a posteriori error estimate of the form

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{DF},K} + \eta_{\mathrm{R},K})^2 + \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2. \tag{2}$$

Specify in particular all the quantities $\eta_{\mathrm{DF},K}$, $\eta_{\mathrm{R},K}$, and $\eta_{\mathrm{NC},K}$.

Question 2. (Application to the vertex-centered finite volume method) (2.5 points)

1) Define the vertex-centered finite volume method for the problem (1a)–(1b).

2) Show in details how to apply the estimate (2) to this method (define all the quantities and give formulas how to compute them).

Question 3. (Efficiency of element residuals) (3.5 points)

Let $u_h, f \in \mathbb{P}_m(\mathcal{T}_h)$ and let $\mathcal{T}_h$ be shape-regular. Let u be the weak solution of (1a)–(1b). Let finally $K \in \mathcal{T}_h$. Prove that there exists a constant C, only depending on the space dimension d, on the shape-regularity of the mesh $\mathcal{T}_h$, and on the polynomial degree m, such that

$$h_K \|f + \Delta u_h\|_K \leq C \|\nabla(u - u_h)\|_K.$$

Question 4. (Optimal a posteriori error estimate) (2.5 points)

1) Recall the five optimal properties of an a posteriori error estimate.

2) Show whether and how these properties are satisfied in the previous questions for the Laplace equation (1a)–(1b) and the vertex-centered finite volume method.

This is a conceptual question, no proofs are to be given here.

Question 5. (Properties of the weak solution of the Stokes equation) (3 points)

Consider the Stokes problem: for $\mathbf{f} \in [L^2(\Omega)]^d$, find $\mathbf{u}$ and p such that

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{3a}$$

$$\nabla\cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{3b}$$

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega. \tag{3c}$$

1) Recall the variational formulation of (3a)–(3c).

2) From the variational formulation, define the stress by $\underline{\boldsymbol{\sigma}} := \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$. To which functional spaces $\mathbf{u}$ and $\underline{\boldsymbol{\sigma}}$ belong? Give a rigorous proof.

Question 6. (Nonlinear Laplace equation and linearization error) (3.5 points)

Let $a(x) := x^{p-2}$ for some real number $p \in (1, +\infty)$ and let

$$\boldsymbol{\sigma}(\boldsymbol{\xi}) = a(|\boldsymbol{\xi}|)\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

Define q by the relation $\frac{1}{p} + \frac{1}{q} = 1$ and consider the nonlinear Laplace equation: for $f \in L^q(\Omega)$, find u such that

$$-\nabla\cdot\boldsymbol{\sigma}(\nabla u) = f \qquad \text{in } \Omega, \tag{4a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4b}$$

1) Recall the variational formulation of (4a)–(4b).

2) Let u be the weak solution of (4a)–(4b), let $u_h \in W_0^{1,p}(\Omega)$ be arbitrary, and set

$$\mathcal{J}_u(u_h) = \sup_{\varphi \in W_0^{1,p}(\Omega);\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h), \nabla\varphi).$$

Derive an a posteriori error estimate of the form

$$\mathcal{J}_u(u_h) \leq \left\{ \sum_{K \in \mathcal{T}_h} (\eta_K)^q \right\}^{1/q}.$$

Specify in particular η_K.

3) Consider an approximation of the nonlinear function $\boldsymbol{\sigma}$ by a linear (affine) one $\boldsymbol{\sigma}_{\mathrm{L}}$. Within η_K, distinguish the error in linearization of $\boldsymbol{\sigma}$ by $\boldsymbol{\sigma}_{\mathrm{L}}$ (linearization error) and the discretization error.