

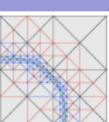
# Robust A Posteriori Error Estimates for Non-Stationary Convection-Diffusion Problems

R. Verfürth

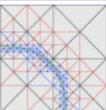
Fakultät für Mathematik  
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[www.ruhr-uni-bochum.de/num1](http://www.ruhr-uni-bochum.de/num1)

Paris / July 7th, 2010

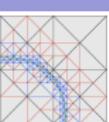


# Goal



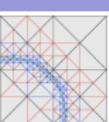
# Goal

Establish residual a posteriori error estimates for SUPG-discretizations of non-stationary convection-diffusion problems which yield upper and lower bounds for the energy norm of the error that are uniform with respect to all possible relative sizes of convection to diffusion.



# Outline

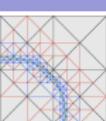
## Variational Problem



# Outline

Variational Problem

Discretization

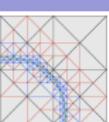


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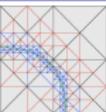
Variational Problem

Discretization

A Posteriori Error Analysis

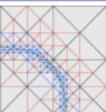


# Differential Equation



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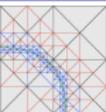
$$\begin{aligned}\partial_t u - \operatorname{div}(d \nabla u) + \mathbf{a} \cdot \nabla u + ru &= f && \text{in } \Omega \times (0, T] \\ u &= 0 && \text{on } \Gamma \times (0, T] \\ u &= u_0 && \text{in } \Omega\end{aligned}$$



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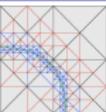
- $d > 0$



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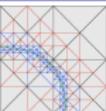
- ▶  $d > 0$
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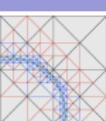
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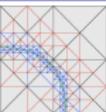
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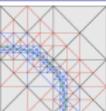
# Norms



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- ▶ Energy norm

$$\|v\| = \{d\|\nabla v\|^2 + r\|v\|^2\}^{\frac{1}{2}}$$



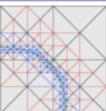
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$$\|\varphi\|_* = \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\langle \varphi, v \rangle}{\|v\|}$$



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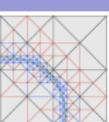
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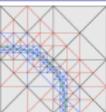
$$\|\varphi\|_* = \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\langle \varphi, v \rangle}{\|v\|}$$

- ▶ Error norm

$$\begin{aligned} \|u\|_{X(a,b)} &= \left\{ \text{ess. sup}_{t \in (a,b)} \|u(\cdot, t)\|^2 + \int_a^b \|u(\cdot, t)\|^2 dt \right. \\ &\quad \left. + \int_a^b \|(\partial_t u + \mathbf{a} \cdot \nabla u)(\cdot, t)\|_*^2 dt \right\}^{\frac{1}{2}} \end{aligned}$$

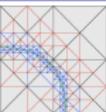


# Meshes and Spaces



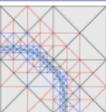
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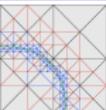
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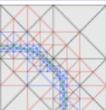
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- ▶  $\mathcal{T}_n$ ,  $0 \leq n \leq N_{\mathcal{I}}$ , affine equivalent, admissible, shape regular partitions of  $\Omega$ .



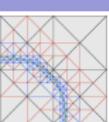
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- ▶ **Transition condition:** There is a common refinement  $\tilde{\mathcal{T}}_n$  of  $\mathcal{T}_n$  and  $\mathcal{T}_{n-1}$  such that  $h_K \leq ch_{K'}$  for all  $K \in \mathcal{T}_n$  and all  $K' \in \tilde{\mathcal{T}}_n$  with  $K' \subset K$ .

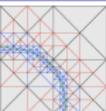


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- ▶  $V_n \subset H_0^1(\Omega)$  finite element space corresponding to  $\mathcal{T}_n$ .

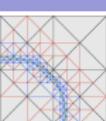


# Discrete Problem



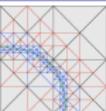
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Find  $\textcolor{red}{u}_{T_n}^n \in X_n$ ,  $0 \leq n \leq N_{\mathcal{I}}$ , such that



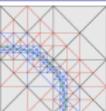
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and, for  $n = 1, \dots, N_{\mathcal{I}}$  and all  $v_{\mathcal{T}_n} \in X_n$



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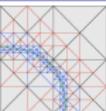
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$$\int_{\Omega} \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} v_{\mathcal{T}_n} + \textcolor{blue}{a}(\theta \nabla u_{\mathcal{T}_n}^n + (1-\theta) \nabla u_{\mathcal{T}_{n-1}}^{n-1}, v_{\mathcal{T}_n})$$

$$= \int_{\Omega} f v_{\mathcal{T}_n}$$

with

$$\textcolor{blue}{a}(u, v) = d(\nabla u, \nabla v) + (\mathbf{a} \cdot \nabla u, v) + r(u, v),$$



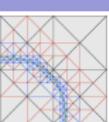
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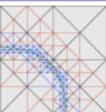
$$\begin{aligned} & \int_{\Omega} \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} v_{\mathcal{T}_n} + \textcolor{blue}{a}(\theta \nabla u_{\mathcal{T}_n}^n + (1-\theta) \nabla u_{\mathcal{T}_{n-1}}^{n-1}, v_{\mathcal{T}_n}) \\ & + \sum_{K \in \tilde{\mathcal{T}}_n} \delta_K \int_K \left( \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} + \textcolor{magenta}{L}(\theta u_{\mathcal{T}_n}^n + (1-\theta) u_{\mathcal{T}_{n-1}}^{n-1}) \right) \mathbf{a} \cdot \nabla v_{\mathcal{T}_n} \\ & = \int_{\Omega} f v_{\mathcal{T}_n} + \sum_{K \in \tilde{\mathcal{T}}_n} \delta_K \int_K f \mathbf{a} \cdot \nabla v_{\mathcal{T}_n} \end{aligned}$$

with

$$\begin{aligned} \textcolor{blue}{a}(u, v) &= d(\nabla u, \nabla v) + (\mathbf{a} \cdot \nabla u, v) + r(u, v), \\ \textcolor{magenta}{L}v &= -\operatorname{div}(d \nabla u) + \mathbf{a} \cdot \nabla u + ru \end{aligned}$$

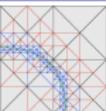


# Basic Steps



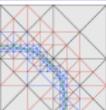
# Basic Steps

- ▶ Error and residual are equivalent.



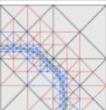
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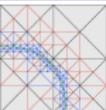
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- ▶ The norm of the sum of these is equivalent to the sum of their norms.



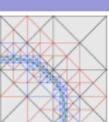
# Basic Steps

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- ▶ Derive a reliable, efficient and robust error indicator for the spatial residual.

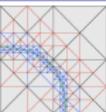


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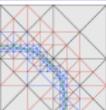


# Equivalence of Error and Residual



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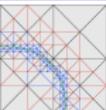
- $u_{\mathcal{T}}$  continuous piece-wise affine, equals  $u_{T_n}^n$  at  $t_n$ .



# Equivalence of Error and Residual

- $u_{\mathcal{I}}$  continuous piece-wise affine, equals  $u_{\mathcal{T}_n}^n$  at  $t_n$ .
- Residual:

$$\begin{aligned}\langle R(u_{\mathcal{I}}), v \rangle &= (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d \nabla u_{\mathcal{I}}, \nabla v) \\ &\quad - (\mathbf{a} \cdot \nabla u_{\mathcal{I}}, v) - (r u_{\mathcal{I}}, v)\end{aligned}$$



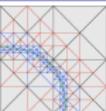
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- Lower bound:

$$\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega))} \leq \sqrt{2} \|u - u_{\mathcal{I}}\|_{X(t_{n-1}, t_n)}$$



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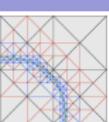
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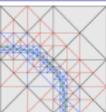
$$\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega))} \leq \sqrt{2} \|u - u_{\mathcal{I}}\|_{X(t_{n-1}, t_n)}$$

- Upper bound:

$$\|u - u_{\mathcal{I}}\|_{X(0, t_n)} \leq \left\{ 4 \|u_0 - \pi_0 u_0\|^2 + 6 \|R(u_{\mathcal{I}})\|_{L^2(0, t_n; H^{-1}(\Omega))}^2 \right\}^{\frac{1}{2}}$$



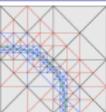
# Proof of the Equivalence



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- ▶ Relation of residual and error:

$$\langle R(u_I), v \rangle = (\partial_t e, v) - (\mathbf{a} \cdot \nabla e, v) - (d \nabla e, \nabla v) - (r e, v)$$

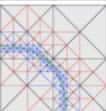


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- ▶ Lower bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.

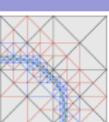


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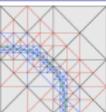
- ▶ Relation of residual and error:

$$\langle R(u_{\mathcal{I}}), v \rangle = (\partial_t e, v) - (\mathbf{a} \cdot \nabla e, v) - (d \nabla e, \nabla v) - (r e, v)$$

- ▶ Lower bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- ▶ Upper bound: Parabolic energy estimate with  $v = e$  as test-function.



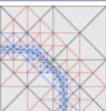
# Decomposition of the Residual



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- ▶ Temporal residual:

$$\begin{aligned}\langle R_\tau(u_{\mathcal{I}}), v \rangle &= (d\nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), \nabla v) + (\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), v) \\ &\quad + (r(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), v)\end{aligned}$$



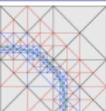
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- ▶ Temporal residual:

$$\begin{aligned}\langle R_{\tau}(u_{\mathcal{I}}), v \rangle &= (d\nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), \nabla v) + (\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), v) \\ &\quad + (r(u_{\mathcal{T}_n}^n - u_{\mathcal{I}}), v)\end{aligned}$$

- ▶ Spatial residual:

$$\begin{aligned}\langle R_h(u_{\mathcal{I}}), v \rangle &= (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u_{\mathcal{T}_n}^n, \nabla v) \\ &\quad - (\mathbf{a} \cdot \nabla u_{\mathcal{T}_n}^n, v) - (r u_{\mathcal{T}_n}^n, v)\end{aligned}$$



# Decomposition of the Residual

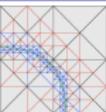
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- ▶ Splitting:  $R(u_{\mathcal{I}}) = R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})$



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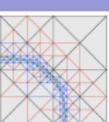
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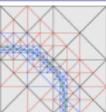
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- Splitting:  $R(u_I) = R_\tau(u_I) + R_h(u_I)$
- Estimate for  $L^2(t_{n-1}, t_n; H^{-1}(\Omega))$ -norms:

$$\begin{aligned}\frac{1}{5} \{ \|R_\tau(u_I)\|^2 + \|R_h(u_I)\|^2 \}^{\frac{1}{2}} &\leq \|R_\tau(u_I) + R_h(u_I)\| \\ &\leq \|R_\tau(u_I)\| + \|R_h(u_I)\|\end{aligned}$$



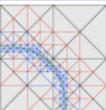
# Motivation of the Lower Bound



# Motivation of the Lower Bound

- Strengthened Cauchy-Schwarz inequality for  $v = c$  and  $w = \frac{b-t}{b-a}$ :

$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2} \|v\|_{(a,b)} \|w\|_{(a,b)}$$



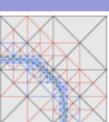
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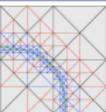
$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2} \|v\|_{(a,b)} \|w\|_{(a,b)}$$

- Hence:

$$\|v + w\|_{(a,b)}^2 \geq \left(1 - \frac{\sqrt{3}}{2}\right) \left\{ \|v\|_{(a,b)}^2 + \|w\|_{(a,b)}^2 \right\}$$

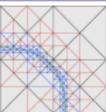


# Proof of the Lower Bound



# Proof of the Lower Bound

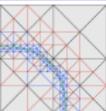
- ▶  $R_h(u_{\mathcal{I}})$  is piece-wise constant.



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- ▶  $R_h(u_{\mathcal{I}})$  is piece-wise constant.
- ▶  $R_{\tau}(u_{\mathcal{I}})$  is piece-wise affine:  $R_{\tau}(u_{\mathcal{I}}) = \frac{t_n - t}{\tau_n} \rho^n$  with

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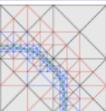
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- ▶ Choose  $v, w \in H_0^1(\Omega)$  such that

$$\|v\| = \|R_h(u_{\mathcal{I}})\|_*, \quad \langle R_h(u_{\mathcal{I}}), v \rangle = \|R_h(u_{\mathcal{I}})\|_*^2,$$

$$\|w\| = \|\rho^n\|_*, \quad \langle \rho^n, w \rangle = \|\rho^n\|_*^2.$$



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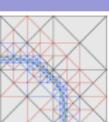
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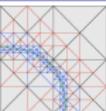
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- ▶ Insert  $3\left(\frac{t-t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n-t}{\tau_n} w$  as test-function in representation of  $R(u_{\mathcal{I}})$ .



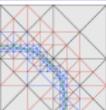
# Estimation of the Spatial Residual



# Estimation of the Spatial Residual

- Spatial error indicator  $\eta_{\tilde{\mathcal{T}}_n}^n$ :

$$\eta_{\tilde{\mathcal{T}}_n}^n = \left\{ \sum_{K \in \tilde{\mathcal{T}}_n} \alpha_K^2 \|R_K\|_K^2 + \sum_{E \in \mathcal{E}_{\tilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}} \alpha_E \|R_E\|_E^2 \right\}^{\frac{1}{2}}.$$

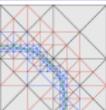


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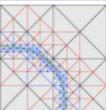
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- ▶ Standard arguments for stationary problems yield:

$$\|R_h(u_{\mathcal{I}})\|_* \leq c^\dagger \eta_{\mathcal{T}_n}^n,$$

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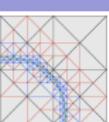
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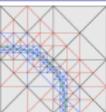
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- ▶  $c^\dagger c_\dagger$  only depend on the polynomial degrees and on the shape parameters of the partitions  $\tilde{\mathcal{T}}_n$ .



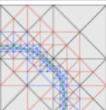
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- ▶ Recall  $R_\tau(u_I) = \frac{t_n - t}{\tau_n} \rho^n$  with

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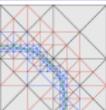
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- ▶ Upper bound:

$$\|\rho^n\|_* \leq \left\{ \|u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}\| + \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \right\}$$



# Estimation of the Temporal Residual

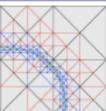
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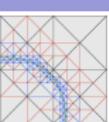
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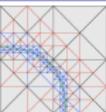
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- ▶ Follows from definition of  $\rho^n$  and  $\|\cdot\|_*$ .
- ▶ Lower bound:

$$\frac{1}{3} \{\|u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}\| + \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*\} \leq \|\rho^n\|_*$$

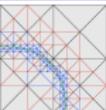


# Proof of the Lower Bound



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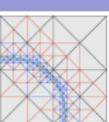
- ▶ Set  $w^n = u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}$  and choose  $v \in H_0^1(\Omega)$  with  $\|v\|_* = \|\mathbf{a} \cdot \nabla w^n\|_*$  and  $(\mathbf{a} \cdot \nabla w^n, v) = \|\mathbf{a} \cdot \nabla w^n\|_*^2$



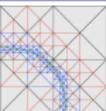
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- ▶ Insert  $\frac{1}{2}w^n + \frac{1}{2}v$  in the definition of  $\rho^n$ :

$$\begin{aligned}& \langle \rho^n, \frac{1}{2}w^n + \frac{1}{2}v \rangle \\&= \underbrace{\frac{1}{2}(d\nabla w^n, \nabla w^n) + \frac{1}{2}(rw^n, w^n)}_{=\frac{1}{2}\|w^n\|^2} + \underbrace{\frac{1}{2}(\mathbf{a} \cdot \nabla w^n, w^n)}_{=0} \\&\quad + \underbrace{\frac{1}{2}(d\nabla w^n, \nabla v) + \frac{1}{2}(rw^n, v)}_{\geq -\frac{1}{2}\|w^n\|\|\mathbf{a} \cdot \nabla w^n\|_*} + \underbrace{\frac{1}{2}(\mathbf{a} \cdot \nabla w^n, v)}_{=\frac{1}{2}\|\mathbf{a} \cdot \nabla w^n\|_*^2}\end{aligned}$$

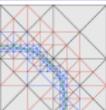


# Estimation of the Convective Derivative I



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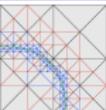
- ▶ Assume that  $\|\mathbf{a}\|_\infty \leq c_c d$ .



# Estimation of the Convective Derivative I

- ▶ Assume that  $\|\mathbf{a}\|_\infty \leq c_c d$ .
- ▶ Friedrichs' inequality implies

$$\begin{aligned} (\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}), v) &\leq \|\mathbf{a}\|_\infty \|\nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\| \|v\| \\ &\leq \|\mathbf{a}\|_\infty \|\nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\| c_\Omega \|\nabla v\| \end{aligned}$$

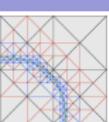


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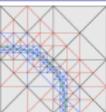
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- ▶ Hence  $\|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \leq c_c c_\Omega \|u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}\|$  and  $\|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$  is equivalent to  $\|u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}\|$ .

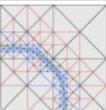


# Estimation of the Convective Derivative II



# Estimation of the Convective Derivative II

- ▶ Assume that  $\|\mathbf{a}\|_\infty \gg d$ .

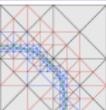


# Estimation of the Convective Derivative II

- ▶ Assume that  $\|\mathbf{a}\|_\infty \gg d$ .
- ▶ Consider the auxiliary problem

$$d(\nabla v_{\mathcal{T}_n}^n, \nabla \varphi) + r(v_{\mathcal{T}_n}^n, \varphi) = (\mathbf{a} \cdot \nabla (u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}), \varphi) \quad (*)$$

with variational and discrete solutions  $\Phi$  and  $\Phi_{\mathcal{T}_n}$ .



# Estimation of the Convective Derivative II

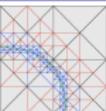
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with variational and discrete solutions  $\Phi$  and  $\Phi_{\mathcal{T}_n}$ .

- ▶ Then  $\|\Phi\| = \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$ ,  
 $\|\Phi_{\mathcal{T}_n}\| \leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$  and therefore

$$\begin{aligned} \frac{1}{3} \{ \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\| \} &\leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \\ &\leq \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\|. \end{aligned}$$



## Estimation of the Convective Derivative II

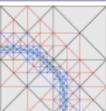
- ▶ Assume that  $\|\mathbf{a}\|_\infty \gg d$ .
- ▶ Consider the auxiliary problem

$$d(\nabla v_{\mathcal{T}_n}^n, \nabla \varphi) + r(v_{\mathcal{T}_n}^n, \varphi) = (\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}), \varphi) \quad (*)$$

with variational and discrete solutions  $\Phi$  and  $\Phi_{\mathcal{T}_n}$ .

- ▶ Then  $\|\Phi\| = \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$ ,  
 $\|\Phi_{\mathcal{T}_n}\| \leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$  and therefore
- $$\begin{aligned} \frac{1}{3} \{ \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\| \} &\leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \\ &\leq \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\|. \end{aligned}$$

- ▶ Transition condition implies that  $\|\Phi - \Phi_{\mathcal{T}_n}\|$  is equivalent to every robust, e.g. residual, error indicator  $\eta_{\mathcal{T}}^n$  for  $(*)$ .



# Estimation of the Convective Derivative II

- ▶ Assume that  $\|\mathbf{a}\|_\infty \gg d$ .
- ▶ Consider the auxiliary problem

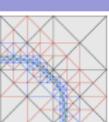
$$d(\nabla v_{\mathcal{T}_n}^n, \nabla \varphi) + r(v_{\mathcal{T}_n}^n, \varphi) = (\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}), \varphi) \quad (*)$$

with variational and discrete solutions  $\Phi$  and  $\Phi_{\mathcal{T}_n}$ .

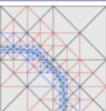
- ▶ Then  $\|\Phi\| = \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$ ,  
 $\|\Phi_{\mathcal{T}_n}\| \leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$  and therefore

$$\begin{aligned} \frac{1}{3} \{ \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\| \} &\leq \|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_* \\ &\leq \|\Phi_{\mathcal{T}_n}\| + \|\Phi - \Phi_{\mathcal{T}_n}\|. \end{aligned}$$

- ▶ Transition condition implies that  $\|\Phi - \Phi_{\mathcal{T}_n}\|$  is equivalent to every robust, e.g. residual, error indicator  $\eta_{\tau}^n$  for (\*).
- ▶ Hence  $\|\mathbf{a} \cdot \nabla(u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1})\|_*$  is equivalent to  $\|\Phi_{\mathcal{T}_n}\| + \eta_{\tau}^n$ .



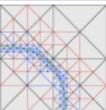
# A Posteriori Error Estimate



# A Posteriori Error Estimate

- ▶ Define the space-time error estimator by:

$$\eta^n = \tau_n^{\frac{1}{2}} \left[ \underbrace{\left( \eta_{\mathcal{T}_n}^n \right)^2}_{\text{spatial}} + \underbrace{\| u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1} \|^2}_{\text{temporal}} + \left( \eta_{\tau}^n \right)^2 \right]^{\frac{1}{2}}.$$



# A Posteriori Error Estimate

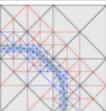
- ▶ Define the space-time error estimator by:

$$\eta^n = \tau_n^{\frac{1}{2}} \left[ \underbrace{\left( \eta_{\mathcal{T}_n}^n \right)^2}_{\text{spatial}} + \underbrace{\| u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1} \|^2}_{\text{temporal}} + \left( \eta_{\tau}^n \right)^2 \right]^{\frac{1}{2}}.$$

- ▶ Then

$$\| e \|_{X(0,T)} \leq c^* \left\{ \| u_0 - \pi_0 u_0 \|^2 + \sum_{n=1}^{N_{\mathcal{T}}} (\eta^n)^2 \right\}^{\frac{1}{2}},$$

$$\eta^n \leq c_* \| e \|_{X(t_{n-1}, t_n)}.$$



# A Posteriori Error Estimate

- ▶ Define the space-time error estimator by:

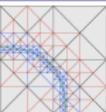
$$\eta^n = \tau_n^{\frac{1}{2}} \left[ \underbrace{\left( \eta_{\mathcal{T}_n}^n \right)^2}_{\text{spatial}} + \underbrace{\| u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1} \|^2}_{\text{temporal}} + \left( \eta_{\tau}^n \right)^2 \right]^{\frac{1}{2}}.$$

- ▶ Then

$$\|e\|_{X(0,T)} \leq c^* \left\{ \|u_0 - \pi_0 u_0\|^2 + \sum_{n=1}^{N_{\mathcal{T}}} (\eta^n)^2 \right\}^{\frac{1}{2}},$$

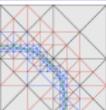
$$\eta^n \leq c_* \|e\|_{X(t_{n-1}, t_n)}.$$

- ▶  $c_* c^*$  only depends on the polynomial degrees and the shape parameters of the partitions  $\tilde{\mathcal{T}}_n$ .



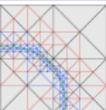
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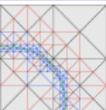
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