Equilibrated error estimator for contact problems

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Workshop on:
A posteriori estimates for adaptive mesh refinement and error control

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Outline

1. Equilibration techniques for error control
   - Equilibrated fluxes for Laplace operator
   - One-sided obstacle problem
   - Two-sided obstacle problem

2. A posteriori error estimates for contact problems
   - $H(\text{div})$-conforming approximations for symmetric tensors
   - Local definition of the estimator
   - Efficiency and reliability

3. AFEM strategy for one-body problems
   - Modified error estimator
   - Edge residuals
   - (Energy based error decay)
Prager–Synge theorem (Laplace operator)


Let \( u_h \) be a conforming finite element solution then

\[
\| \nabla u - \nabla u_h \|_0 \leq \| \nabla u_h - j \|_0 + C \| \text{div} j - f \|_1
\]

for all \( H(\text{div}) \)-conforming vector fields \( j \)

**Idea:** Construct a suitable \( H(\text{div}) \)-conforming finite element approximation \( j_h \)

**Remark:** This result can also be regarded as a hypercycle method

\( \implies \) asymptotically exact for postprocessed solution \( p_h := \frac{1}{2}(j_h + a \nabla u_h) \)
How to construct suitable $H(\text{div})$-approximations?

Idea: Use standard mixed finite elements, e.g., RT or BDM, such that

$$\text{div} j_h = P_h f,$$

where $P_h$ is locally defined and reproduces constants

But: Solution of a global mixed finite element problem to expensive
Need to recover $j_h$ locally from the conforming fe solution $u_h$

Possibility one: Define $j_h$ on a dual finite volume mesh and use a macro-element based Raviart–Thomas space of lowest order (jww Robert Luce, 04)

One macro-element associated with each vertex
Simplicial triangulation and finite volume boxes

$\mathcal{B}_h$: Finite volume boxes on $\Omega$ and $\mathcal{T}_h \prec \mathcal{K}_h$: Simplicial triangulations on $\Omega$

Local construction

Definition of the Raviart-Thomas space $S_h$:

$$S_h := \{ j \in H(\text{div}; \Omega); \ j|_K \in RT_0(K); \ K \in \mathcal{K}_h \ , \ \text{div} j|_B \in P_0(B); \ B \in \mathcal{B}_h \} \subset RT_h$$

Local basis of $S_h$: $S_h = \sum_{e \in \mathcal{E}_h} \text{span} \ \{ w_e \} \oplus \sum_{B \in \mathcal{B}_h} \text{span} \ \{ w_B \}$
Flux approximation in $S_h$

**Definition of $w_B$**

\[ w_B := \beta_B \text{curl } \phi_B, \]
\[ \beta_B^{-2} := (\text{curl } \phi_B, \text{curl} \phi_B)_0 \]

**Case I: $B$ is interior box**

\[ j_h := \sum_{e \in \mathcal{E}_B} \alpha_e w_e + \alpha_B w_B \]
\[ \alpha_e := \int_e a \nabla u_h n_e \, d\sigma \]
\[ \alpha_B := \int_B a \nabla u_h \, w_B \, dx \]

**Case II: $B$ is boundary box**

\[ j_h := \sum_{e \in \mathcal{E}_B} \alpha_e w_e + \sum_{e \in \partial \Omega} \alpha_e w_e \]
\[ \alpha_e := \int_e a \nabla u_h n_e \, d\sigma \quad e \text{ in } \Omega \]
\[ \alpha_e := \alpha_e + \frac{1}{2} \left( \int_{\Omega} -f \phi_B \, dx - \int_{\partial B} a \nabla u_h n \, d\sigma \right) \]

**Lemma:** \[ \text{div} j_h|_B = \frac{1}{|B|} \int_{\Omega} f \phi_B \, dx =: P_Q f|_B \]

**Definition of $w_e$**

\[ w_e n_e|_e := \frac{1}{h_e} \delta_{e e}, \]
\[ (w_e, w_B)_0 = 0 \]
Local error contributions (Laplace operator)

Triangulation, error in $u_h$, estimator and error for postprocessed solution (same scale)
Characteristic properties of the construction

- Easy and simply construction for low order elements (✓)
- Decoupling of the global problem by “inner” boundaries (✓)
- Generalization to high order elements not straightforward (✗)
- Generalization to symmetric tensors not straightforward (✗)

Observation: Each edge of $\mathcal{T}_h$ is decomposed into two subedges with constant flux $\Rightarrow$ this motivates alternative approach in terms of equilibrated fluxes

Linear equilibrated fluxes per edge are decoupled by biorthogonality
Obstacle problem

- **Discrete primal formulation:** Find $u_h \in K_h$ such that

$$a(u_h, v - u_h) \leq f(v - u_h), \quad v \in K_h,$$

where $K_h$ is the discrete set of admissible elements, i.e.,

$$K_h := \{ v \in X_h, \int_\Omega v \mu_p \, dx \geq \int_\Omega \psi \mu_p \, dx \}$$

and $\{\mu_p\}_p$ forms a set of biorthogonal basis functions wrt $\{\phi_p\}_p$

- **Discrete hybrid formulation:** $(u_h, \lambda_h) \in (X_h, M_h^+)$, $M_h^+ := \{ \sum_p \alpha_p \mu_p, \alpha_p \geq 0 \}$

$$a(u_h, v_h) + b(\lambda_h, v_h) = f(v_h), \quad v_h \in X_h,$$

$$b(\mu_h - \lambda_h, u_h) \leq \langle \psi, \mu_h - \lambda_h \rangle, \quad \mu_h \in M_h^+.$$

$a(\cdot, \cdot)$ bilinear form, $b(\cdot, \cdot) := \langle \cdot, \cdot \rangle$ duality pairing between $H^{-1}$ and $H^1_0$.

$\lambda_h$ can be seen as an additional source term for the a posteriori analysis.
1 Equilibration techniques

Sinus-shaped obstacle

Problem Setting:
Obstacle:
\[ \psi = 3 \| x - (0.5, 0.5) \| - \sin(10\pi \| x - (0.5, 0.5) \|) \]
Rhs:
\[ f = 0 \]
Zero Dirichlet boundary conditions

Obstacle

Solution of contact problem (cut)
Grid and active set on different refinement levels
Non-smooth obstacle

Problem Setting:
Obstacle:
\[ \psi = \| x - (0.5, 0.5) \|_1 - 0.3 \]
Rhs:
\[ f = 0 \]
Zero Dirichlet boundary conditions
Adaptive meshes and active sets

Solution is not in $H(\text{div}) \implies$ Overestimation and no correct asymptotic
Adaptive meshes and active sets

Regularity of $j_h$ has to be weakened $\Rightarrow [j_h n]$ not necessarily non-zero
Obstacle problem between two membranes

Problem Setting (unconstrained):

\[ u_m = 0.5 \]
\[ u_s = e^{-100\|x-(0.45,0.57)\|} \]
\[ K_m = 3Id, \ K_s = Id \]
Dirichlet boundary conditions

Solution without restriction

Solution of contact problem
Non-matching adaptive meshes

Level 0  Level 3  Level 5  Level 8
Obstacle problem between two membranes

Problem Setting (unconstrained):
\[ u_m = 0.5 \]
\[ u_s = e^{-1000(\|x-(0.45,0.57)\|^2-0.1^2)^2} \]
\[ K_m = 3Id, K_s = Id \]
Dirichlet boundary conditions

Solution without restriction

Solution of contact problem (cut)
Non-matching meshes and active sets

Level 1  Level 2  Level 4  Level 6
Contact problem with Coulomb friction

Linear Elasticity:
\[-\text{div} \sigma = f \quad \text{in} \ \Omega,\]
\[u = 0 \quad \text{on} \ \Gamma_D, \quad \sigma n = p \quad \text{on} \ \Gamma_N\]

Non-penetration:
\[[u]_n - g \leq 0,\]
\[\sigma_n := \sigma_n(u_m) = \sigma_n(u_s) \leq 0,\]
\[\sigma_n([u]_n - g) = 0\]

Coulomb friction:
\[|\sigma_t| - \tilde{\psi}|\sigma_n| \leq 0,\]
\[[u]_t + \alpha^2 \sigma_t = 0,\]
\[[u]_t(|\sigma_t| - \tilde{\psi}|\sigma_n|) = 0\]

Jump:
\[[u] := (u_s - P_m^s u_m)\]
\[[u]_n := [u] \cdot n, \quad [u]_t := [u] - [u]_n n\]

Stress:
\[\sigma_n := \sigma n \cdot n, \quad \sigma_t := \sigma n - \sigma_n n\]

\[\Rightarrow \text{Discretization in terms of mortar finite elements and dual Lagrange multipliers}\]
Discretization on non-matching meshes

- Constraints are **weakly** satisfied in terms of Lagrange multipliers:
  Displacement $u$: primal variable
  Contact stress $\lambda := -\sigma n$: dual variable

- **Discrete hybrid formulation:** $(u_h, \lambda_h) \in (X_h, M_h(\lambda_h))$
  
  $a(u_h, v_h) + b(\lambda_h, v_h) = f(v_h), \quad v_h \in X_h,$
  $b(\mu_h - \lambda_h, u_h) \leq \langle g, (\mu_h - \lambda_h) n \rangle, \quad \mu_h \in M_h(\lambda_h).$

  $a(u_h, \cdot)$ elasticity linear form, $b(\cdot, \cdot) := \langle [\cdot], \cdot \rangle$ contact bilinear form
  $(X_h, M_h)$ stable pair of mortar finite elements, $W_h$ trace space of $X_h$
  $M_h(\lambda_h) := \{ \mu \in M_h; \langle v, \mu \rangle \leq \langle |v_t|_h, \mathcal{F}(\lambda_h) n \rangle, \quad v \in W_h, v_n \leq 0 \}$

  Local static elimination of $\lambda_h$ possible due to biorthogonality

- **Coulomb friction:** **quasi** variational inequality

  No friction ($\mathcal{F} = 0$)/**Tresca friction** ($|n\lambda_h|_h \to g_f$): variational inequality
A posteriori error estimator for contact

Observation: Discrete displacement satisfies a variational equality for given Lagrange multiplier $\lambda_h$

Idea: Find a $H(\text{div})$-conforming approximation $\sigma_h$ for the stress such that

- the divergence satisfies
  \[ (\text{CD}) \quad \text{div} \sigma_h = -\Pi_1 f, \]
  where $\Pi_1$ is the $L^2$-projection onto piecewise affine functions.

- the surface traction satisfies
  \[ (\text{CS}) \quad (\sigma_h n^l)|_{\Gamma_N^l} = 0, \quad \text{and} \quad (\sigma_h n^l)|_{\Gamma_C^l} = -n^l \cdot n^s \Pi^*_l \lambda_h, \quad l \in \{m, s\} \]
  where $\Pi^*_l$ is the dual mortar projection onto the Lagrange multiplier space.

Definition of the error estimator

\[ \eta^2 := \sum_T \eta_T^2, \quad \eta_T^2 := \| C^{-1/2} (\sigma_h - \sigma(u_h)) \|_{0; T}^2. \]

Remark: The error estimator is elementwise defined.
How to obtain a suitable $\sigma_h$?

**Idea:** A posteriori error estimator based on equilibrated fluxes [Ainsworth-Oden 99, Ladeveze/Leguillon 83, Stein et al 97-01]

Let $u_h$ be the mortar finite element solution of the variational inequality, i.e.,

$$a(u_h, v_h) = (f, v_h)_{0;\Omega} - b(\lambda_h, v_h), \quad v_h \in X_h^i, \quad i \in \{s, m\}$$

$\implies$ $\lambda_h$ plays role of Neumann boundary condition

**Equilibrated fluxes**

Then there exists a $g_e \in [P_1(e)]^2$ such that $g_e = -n^l \cdot n^s \Pi_1 \Pi^*_1 \lambda_h$ on $\Gamma_C$

$$\int_{\partial T} (n_T \cdot n_e) g_e \cdot v \, ds = \Delta_T(v) := a_T(u_h, v) + b_T(\lambda_h, v) - (f, v)_{0;\Omega}, \quad v \in [P_1(T)]^2$$

Moreover, $g_e$ can be locally computed by rewriting

$$g_e = \mu_{e,p_1} \psi_{p_1} + \mu_{e,p_2} \psi_{p_2},$$

where $\int_e \psi_{p_j} \phi_{p_i} \, ds = \delta_{i,j}$. Then the moments $\mu_{e,p_i}$ are given by $\mu_{e,p_i} := \int_e g_e \phi_{p_i} \, ds$. 

Local postprocess to compute the moments

For each vertex \( p \) a local system has to be solved for the moments

\[
\begin{pmatrix}
-1 & 1 & & \\
& -1 & 1 & \\
& & -1 & 1 & \\
& & & \ddots & \ddots & \\
1 & & & & -1 & 1
\end{pmatrix}
\begin{pmatrix}
\mu_{e_1,p} \\
\mu_{e_2,p} \\
\vdots \\
\mu_{e_{N-1},p} \\
\mu_{e_N,p}
\end{pmatrix}
= \begin{pmatrix}
\Delta T_1(\phi_p) \\
\Delta T_2(\phi_p) \\
\vdots \\
\Delta T_{N-1}(\phi_p) \\
\Delta T_N(\phi_p)
\end{pmatrix}
\]

Singular system \((\text{interior vertex})\), but compatible rhs: \( \sum_i \Delta T_i(\phi_p) = 0 \)

Set e.g. \( \mu_{e_1,p} = 0 \Rightarrow \text{lower tridiagonal matrix} \)

\( g_e \) is conservative, i.e., \( \int_{\partial T} (n_T \cdot n_e) g_e \, ds = \int_T f \, dx \)
Alternative choice

Better approximation of the flux:

$$\min_{\mathbf{g}_e} \sum_{e} h_e \| \mathbf{g}_e - \{ \sigma(u_h) \} \mathbf{n}_e \|_{0;e}^2$$

$$\implies$$ minimization problem (Lagrange multipliers on elements)

New local system (interior vertex):

$$\begin{pmatrix}
    \text{Id} & -\frac{1}{2}\text{Id} & 0 & \ldots & 0 & -\frac{1}{2}\text{Id} \\
    -\frac{1}{2}\text{Id} & \text{Id} & -\frac{1}{2}\text{Id} & 0 & \ldots & 0 \\
    0 & \ldots & -\frac{1}{2}\text{Id} & 0 & \ldots & \ldots \\
    \vdots & \ddots & \ldots & \ldots & \ddots & \ldots \\
    0 & \ldots & 0 & -\frac{1}{2}\text{Id} & \text{Id} & -\frac{1}{2}\text{Id} \\
    -\frac{1}{2}\text{Id} & 0 & \ldots & 0 & -\frac{1}{2}\text{Id} & \text{Id}
\end{pmatrix}
\begin{pmatrix}
    \tilde{\mu}_1 \\
    \vdots \\
    \tilde{\mu}_N
\end{pmatrix}
= 
\begin{pmatrix}
    \Delta_1 \\
    \vdots \\
    \Delta_N
\end{pmatrix}
$$

Similar systems for boundary nodes depending on boundary cond. (D-D, D-N, N-N)

$\mathbf{g}_e$ is uniquely defined [Ainsworth-Oden, Ladeveze, Stein et al]
Arnold–Winther [02] elements

The Arnold–Winther element is locally defined on each $T$ by the 24-dimensional space

$$X_T := \{ \mathbf{\tau}_h \in [P_3(T)]^2 \times 2, \ (\mathbf{\tau}_h)_{12} = (\mathbf{\tau}_h)_{21}, \ \text{div} \mathbf{\tau}_h \in [P_1(T)]^2 \},$$

and a global finite element space $X_h := X_s \times X_m$ which is on each body $H(\text{div})$-conforming can be obtained using

- the **nodal values** (3 dof) at each node $p$,
- the **zero and first order moments** of $\mathbf{\tau}_h e$ (4 dof) on each edge $e$,
- the **mean value** (3 dof) on each element $T$

as degrees of freedom on each of the two bodies.

**Norm equivalence**

$$\| \mathbf{\tau} \|_0^2 = \sum_p |T| \| \mathbf{\tau}(p) \|_0^2 + \sum_e \| \int_e \mathbf{\tau} n_e \, ds \|_0^2 + \| \int_e \mathbf{\tau} n_e \phi_e \, ds \|_0^2 + \frac{1}{|T|} \| \int_T \mathbf{\tau} \, dx \|_0^2$$

$$=: m_p(\mathbf{\tau})$$

$$=: m_e(\mathbf{\tau})$$

$$=: m_i(\mathbf{\tau})$$
Definition of $\sigma_h$

$$\sigma_h(p) := \frac{1}{N^p} \sum_{T \in T_p} \sigma(u_h)|_T(p) + \alpha(p), \quad (1)$$

$$\int_e \sigma_h n_e \cdot q ds := \int_e g_e \cdot q ds, \quad q \in [P_1(e)]^2, \quad (2)$$

$$\int_T \sigma_h : \nabla v ds := a_T(u_h, v), \quad v \in [P_1(T)]^2. \quad (3)$$

$\alpha(p)$ depends on the type of the node, e.g. $\alpha(p) = 0$ if $p$ is an interior node

**Lemma:**

i) Let $\sigma_h \in X_h$ be defined such that (2) and (3) hold. Then,

$$\text{div}\sigma_h = -\Pi_1 f.$$ 

ii) Let $\sigma_h \in X_h$ be defined such that (1) and (2) hold. Then,

$$\left(\sigma_h n^l\right)|_{\Gamma_N^l} = 0, \quad \text{and} \quad \left(\sigma_h n^l\right)|_{\Gamma_C^l} = -n^l \cdot n^s \Pi^*_l \lambda_h.$$
Reliability of the error estimator

**Theorem:** Under suitable regularity assumptions the error estimator $\eta$ yields a global upper bound for the discretization error

$$a(u - u_h, u - u_h)^{\frac{1}{2}} \leq \eta + O(h^{\frac{3}{2}})$$

The definition of the error estimator yields

$$\|u - u_h\|^2_a \leq \eta \|u - u_h\|_a + \sum_{l \in \{m,s\}} \int_{\Omega^l} (\sigma(u) - \sigma_h) : (\epsilon(u) - \epsilon(u_h)) \, dx$$

To bound $I$ one has to exploit:

- the properties (CD) and (CS) of $\sigma_h$
- the a priori results for $b(u - u_h, \lambda_h - \lambda)$
- the approximation property of $\Pi^*_h$, i.e., $\int_{\Gamma^m} (\lambda_h - \Pi^*_m \lambda_h) \cdot (u^m - u^m_h) \, ds$

**Remark:** There is no constant in the upper bound

No additional terms enter due to the contact
Efficiency of the error estimator

**Theorem:** Under suitable regularity assumptions the error estimator \( \eta_T \) yields a local lower bound for the discretization error

\[
\eta_T \leq C a_{\bar{T}}(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h)^{1/2} + O(h^{3/2})
\]

Proof is based on the norm equivalence and \( C^{-1/2}(\sigma_h - \sigma(\mathbf{u}_h)) \in X_T : \)

- \( m_i(C^{-1/2}(\sigma_h - \sigma(\mathbf{u}_h))) = 0 \)
- \( m_e(C^{-1/2}(\sigma_h - \sigma(\mathbf{u}_h))) \leq C \sum_e h_e (\|g_e - \sigma(\mathbf{u}_h)\mathbf{n}_e\|^2_e + \|\Pi^* \lambda_h - \sigma(\mathbf{u}_h)\mathbf{n}_m\|^2_e) \)
- \( m_p(C^{-1/2}(\sigma_h - \sigma(\mathbf{u}_h))) \leq C \sum_e h_e \|\sigma(\mathbf{u}_h)\mathbf{n}_e\|^2_e \)

These terms can be found in the analysis of

- the residual based error estimator for a variational equality
- the equilibrated error estimator for a variational equality
- the a priori estimates for the variational inequality
Influence of the material parameters $E_i$

Poisson number: $\nu_1 = \nu_2 = 0.3$, Coulomb friction coefficient: $\mathcal{F} = 0.4$;

$E_1 = 500$
$E_2 = 10^6$
$E_1 = 10^3$
$E_2 = 10^5$
$E_1 = 10^4$
$E_2 = 10^4$
$E_1 = 10^5$
$E_2 = 10^3$
$E_1 = 10^6$
$E_2 = 500$

Solution after 4 refinement steps; top: deformed mesh, bottom: effective stress
Adaptivity preserves optimality

Low-regularity problem:
soft ball on hard diamond

height $l$ determines regularity of problem
Why is the high order term non standard?

**Observation:** Higher order term $O(h^{3/2})$ depends not only on given data but also on unknown solution

- **Primal nonconformity:** Non-matching meshes
  $\implies$ weak but no strong non-penetration

- **Dual nonconformity:** Biorthogonality
  $\implies$ LM is weakly but not in a strong sense non-negative

**Remedy:** Postprocessing of the discrete LM $\lambda_h$
Observation: Discrete Lagrange multiplier $\lambda_h$ is not non-negative.

Orthogonality between the normal components of $u_h$ and $\lambda_h$ (left) and $P_{u_h} \lambda_h$ (right)

$$P_{u_h} \mu_h := \begin{cases} 0 & e \in \mathcal{E}_h^s \\ \mu_h & e \in \mathcal{E}_h^i \text{ and if } (\mu_h)_n \geq 0 \text{ on } e \\ (\alpha^1_1 \phi^1_e + \alpha^2_2 \phi^2_e) n & e \in \mathcal{E}_h^i \text{ and otherwise} \\ (\alpha^1_1 w^1_e \phi^1_e + \alpha^2_2 w^2_e \phi^2_e) n & e \in \mathcal{E}_h^b \end{cases}$$

where $\phi^1_e, \phi^2_e$ are the local nodal Lagrange basis functions, and

$$w^i_e := \begin{cases} \frac{\text{meas}(\text{supp} \psi_{p ge(i)})}{\text{meas}(e)} & \text{if supp } \psi_{p ge(i)} \subset \text{supp}_h u_h, \\ 1 & \text{otherwise} \end{cases}$$
Modified error estimator

In addition to $\eta$, we define the quantity

$$
\eta^2_C := \sum_{e \in \mathcal{E}_h^C} \eta^2_e, \quad \eta^2_e := \frac{h_e}{\sqrt{2\mu}} \| \lambda_h - P_{u_h} \lambda_h \|^2_{0;e}.
$$

**Assumption:** For each edge $e \subset \text{supp } \lambda_h \cap \text{supp } u_h$, we assume that there exists an adjacent edge $\hat{e}$ such that $\hat{e} \subset \Gamma_C \setminus \text{supp } u_h$.

This assumption excludes isolated points such as
A posteriori error estimator for a one-body problem

As it is standard for a posteriori estimates, we define a higher order term which only depends on the given data

\[
\xi^2 := \sum_{T \in T_h} \xi_T^2, \quad \xi_T^2 := \frac{h_T^2}{2\mu} \| f - \Pi_1 f \|_{0;T}^2.
\]

**Theorem:** Under the Assumption, there exist constants \( C_1, C_2 < \infty \) independent of the mesh-size such that

\[
\| u - u_h \|_a \leq \eta + C_1 \eta_C + C_2 \xi.
\]

**Theorem:** Under the Assumption, there exists a constant \( C < \infty \) independent of the mesh-size such that \( \beta(h) \leq C \) and

\[
\| u - u_h \|_a \leq (1 + C_1 \beta(h)) \eta + C_2 \xi.
\]

**Remark:** The numerical results show that \( \beta(h) \) tends asymptotically to zero and the upper bound \( 1 + C_1 \beta(h) \) tends to one.
Hertz-problem

Error reduction

- Uniform: $O(h^{0.9})$
- Adaptive: $O(h)$

Error reduction

- $\eta$: $O(h)$
- $\eta_C$: $O(h^2)$
Square on triangle

$\alpha = \pi / 6$

Mesh at level 12

error reduction

$\eta$ = $O(h)$

$\eta$ = $O(h^2)$

$\eta$ contact

$\eta$ C

$\eta$ C

uniform

adaptive

$O(h)$

$O(h^{0.3})$

$O(h^{0.5})$

$O(h)$
A posteriori estimator for the Lagrange multiplier

Standard a priori estimate:

\[ \| \lambda - \lambda_h \|_{-\frac{1}{2}; \Gamma_C} \leq C \left( \inf_{\mu_h \in \mathcal{M}_h} \| \lambda - \mu_h \|_{-\frac{1}{2}; \Gamma_C} + \| u - u_h \|_a \right) \mathcal{O}(h^{\frac{3}{2}}) \]

**But:** \( \lambda \) is not a given data

Data oscillation term:

\[ \tilde{\xi}^2 := \sum_{T \in \mathcal{T}_h} \tilde{\xi}_T^2, \quad \tilde{\xi}_T^2 := \frac{h_T^2}{2 \mu} \| f - Q^* f \|_{0; T}^2 \]

\( \tilde{\xi} = 0 \) if \( f \) is constant on \( \Omega \), \( Q^* \) Scott–Zhang type operator

**Theorem:** There exists a constant \( C < \infty \) independent of the mesh-size such that

\[ \| \lambda - \lambda_h \|_{-\frac{1}{2}; \Gamma_C} \leq C \left( \| u - u_h \|_a + \tilde{\xi} \right) . \]
Conclusion

- Equilibration techniques can be generalized to elasticity
- Error bound for the LM in terms of the primal bound
- Variational inequality does not bring in extra terms
- Sign controlling terms are of higher order
- Non-matching meshes are problematic (theory)
AFEM based strategies

**AFEM for standard fe estimates:** guaranteed error decay


**AFEM for obstacle problems:**
No Galerkin orthogonality but minimization property on convex set


These results can be applied for one-body contact problems

**In the case of a two-body contact problem on non-matching meshes:**

- Convex sets are non-nested, i.e., $K_l \not\subset K_{l+1} \not\subset K$
- Higher order term cannot be controlled by given data
- No classical inverse estimate for the discrete trace, i.e.,

$$
\| [v_h] \|_{1/2; \Gamma_C} \not\lesssim C \frac{1}{h} \| [v_h] \|_{0; \Gamma_C}
$$

**BUT**

$$
\| [v_h] \|_{1/2; \Gamma_C} \leq C \frac{\ln \epsilon + 1}{h} \| [v_h] \|_{0; \Gamma_C}
$$

$\epsilon$ minimal relative shift
Strong monotonicity in the energy

**Corollary:** There exists a constant independent of the mesh-size such that

\[ J(u_h) - J(u) \leq C(\eta^2 + \xi^2), \]

where the energy \( J(v) \) is given by \( J(v) := \frac{1}{2}a(v, v) - f(v) \).

The variational inequality is equivalent to a constrained minimization problem, i.e., \( J(u) \leq J(v), v \in K \), and in terms of \( K_{l} \subset K_{l+1} \), we have

\[ 0 \leq \delta_{l+1} \leq \delta_{l} := J(u_{l}) - J(u). \]

**Theorem:** There exist constants \( \rho_1, \rho_2 < 1 \) and \( c_\xi, C_\xi < \infty \) such that

\[
\begin{align*}
\delta_{l+1} & \leq \rho_1 \delta_l + c_\xi \hat{\xi}^2_l , \\
\delta_{l+1} + C_\xi \hat{\xi}^2_{l+1} & \leq \rho_2 (\delta_l + C_\xi \hat{\xi}^2_l ) .
\end{align*}
\]

**Remark:** We observe that \( \hat{\xi}_l = 0 \) for a constant \( \mathbf{f} \). In that case, the energy term \( \delta_l \) is a strictly decreasing function with respect to the refinement level \( l \).
AFEM strategy for example 3

Energy difference $\delta_l := J(u_l) - J(u)$

$\alpha = \pi/6$

energy reduction

$\alpha = \pi/4$

energy reduction

$\alpha = \pi/3$

energy reduction
Comparison of different refinement strategies

Example 3, $\alpha = \pi/3$: no interior point (above) and with interior point (below)
Comparison of different error estimators

Example 2:

Example 3: \((\alpha = \pi/3)\)