A fair of energy norm error estimators

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Outline

Tasks of aposteriori error estimators

Aposteriori error estimators are used in differential problems to

- quantify the global error in terms of a given approximate solution and data
- provide the problem-specific information for the decisions in adaptivity

There are several aposteriori error estimators on the market. The question thus arises: Which one do you buy?

Previous work and goal

Theory provides (generic) constants for ratio error-estimator affected by worst cases.

Babuska et al '94: under certain conditions, the local asymptotic error-estimator ratio is determined via an eigenvalue problem
Mitchell '90: in 10 benchmark pb's, number of nodes for a given error about the same for 4 refinement techniques and 7 estimators
Carstensen/Funken/Klose '02: in 3 benchmark pb's, the global effectivity indices of 7 estimators vary by one order

Here: benchmark pb's emphasize robustness within problem class

Outline

Problem class and error notion

Linear symmetric ‘positive definite’ boundary value problems of the form

$$-\operatorname{div}\left(A\nabla u\right) + cu = f \qquad \text{in } \Omega \subset \mathbb{R}^2,$$
$$u = g \qquad \text{on } \Gamma$$

The quality of an approximate solution is measured in the so-called energy norm:

$$\|v\| := \left(\int_\Omega A\nabla v \cdot \nabla v + c|v|^2\right)^{1/2}$$

Discretization class and approximate solutions

Continuous linear finite elements on isotropic triangulations: given a triangulation $\mathcal{T}$ of Ω (and $g = 0$), let $u_{\mathcal{T}} \in V(\mathcal{T})$ such that

$$\forall V \in V(\mathcal{T}) \quad \int_\Omega A\nabla u_{\mathcal{T}} \cdot \nabla V + c u_{\mathcal{T}} V = \int_\Omega f V$$

where

$$V(\mathcal{T}) := \{v \in C^0(\bar{\Omega}) \mid \forall T \in \mathcal{T}\ v_{|T} \text{ affine},\ v_{|\Gamma} = 0\}$$

Recall

$$\|u - u_{\mathcal{T}}\| = \inf\left\{ \|u - V\| \mid V \in V(\mathcal{T})\right\}$$

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   - Classes of interest
   - Examined error estimators
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   - About our tests
   - Error quantification
   - Performance for adaptivity

Error estimator structure and selection

Each of the following error estimator stands for an ‘estimation idea’ and satisfies

$$\mathcal{E} = \Big(\sum_{i \in \mathcal{I}} \mathcal{E}_i^2 \Big)^{1/2},$$

where

- $\mathcal{I}$ is an index set (e.g. $\mathcal{T}$ itself)
- each indicator $\mathcal{E}_i$ can be locally computed in terms of the finite element solution $u_\mathcal{T}$ and data
- $\mathcal{E} \approx \|u_\mathcal{T} - u\|$ within a subclass

Error, residual and residual structure

Introducing the residual

$$R := f - cu_{\mathcal{T}} + \operatorname{div}(A\nabla u_{\mathcal{T}}) \in H^{-1}(\Omega)$$

there holds

$$\|u - u_{\mathcal{T}}\| = \|R\|_* = \sup\left\{\langle R, \varphi\rangle \mid \|\varphi\| \leq 1\right\}$$

Moreover,

$$\forall \Phi \in V(\mathcal{T}) \quad \langle R, \Phi\rangle = 0$$

$$\langle R, \varphi\rangle = \int_\Omega r\varphi + \int_\Sigma j\varphi, \qquad r, j \text{ asymp pw discrete}$$

Weighted residual estimator

Estimate the dual energy norm of the residual R in terms of local integral norms and appropriate weights (Verfürth '98):

$$\mathcal{E}_{\mathrm{wR},v}^2 = C_1 \mathrm{w}_{1,v} \int_{\sigma_v} |j|^2 \phi_v + C_2 \mathrm{w}_{2,v} \int_{\omega_v} |r - r_v|^2 \phi_v$$

where

- v is any vertex of $\mathcal{T}$ and ϕ_v the corresponding hat function
- $\mathrm{w}_{1,v}$ and $\mathrm{w}_{2,v}$ depend on h_v, λ and c around v
- r_v is a mean value of r around v

Hierarchical estimator

Since the residual R is asymptotically discrete, test it with basis functions of an appropriate hierarchical extension:

$$\mathcal{E}_{\mathrm{H},S} = \left| \int_\Omega f\varphi_S - A\nabla u_{\mathcal{T}} \cdot \nabla\varphi_S - c u_{\mathcal{T}} \varphi_S \right|$$

where

- S is any interior side of $\mathcal{T}$
- φ_S is the quadratic bubble associated with S, normalized such that $|||\varphi_S||| = 1$

Bank-Weiser estimator

Hoping to incorporate properties of the differential operator, locally lift the residual to the 'error side':

$$\mathcal{E}_{\mathrm{BW},T} = \|e_T\| \quad \text{with } e_T \in Q_T^0 \text{ verifying}$$

$$\forall \varphi \in Q_T^0 \quad \int_T A\nabla e_T \cdot \nabla\varphi + c e_T \varphi = \int_T f\varphi - A\nabla u_{\mathcal{T}} \cdot \nabla\varphi - c u_{\mathcal{T}}\varphi + \int_{\partial T\setminus\Gamma} \overline{F}\varphi,$$

where

- Q_T^0 denote the span of the quadratic bubbles associated to the sides of T with zero bdry vals on Γ,
- $\overline{F}$ is the average of the normal flux $A\nabla u_{\mathcal{T}} \cdot \nu$ on each side

ZZ estimator

Recover a continuous approximation for the gradient and, motivated by superconvergence results, compute the difference to the original discontinuous gradient:

$$\mathcal{E}^2_{ZZ,T} = \int_T A(Gu_\mathcal{T} - \nabla u_\mathcal{T}) \cdot (Gu_\mathcal{T} - \nabla u_\mathcal{T})$$

where

- T is a triangle of $\mathcal{T}$
- $Gu_\mathcal{T}$ is pw linear and $Gu_\mathcal{T}(v)$ is an average of $\nabla u_\mathcal{T}$ around each vertex v

. . . and others

also variants of the above estimators have been examined, e.g.

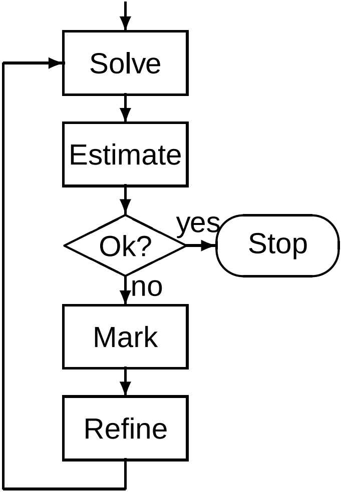
- standard residual estimator
- variants of hierarchical estimator, in particular in 3d

planned:

- local problems with equilibrated fluxes
- estimation based upon flux reconstruction in H(div)
- explicit standard residual error estimator

Outline

Adaptive algorithm



Courant elements with interpolated boundary values

StdRes, WeiRes, Hier2, BW, ZZ, TrueErr

suppressed

maximum strategy using corresponding indicators

bisection

Benchmark problem features

Test bed consists of 9 problems in 2d (and others in 3d), some with parameters, featuring

- second derivatives that are square-integrable or not
- oscillatory data
- dominating reaction
- ellipticity that depends on space moderately, strongly and discontinuously
- anisotropic ellipticity

Structured and unstructured initial triangulations.

In total 612 test runs in 2d.

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Global effectivity index

To evalutate the quality of error quantification, we consider

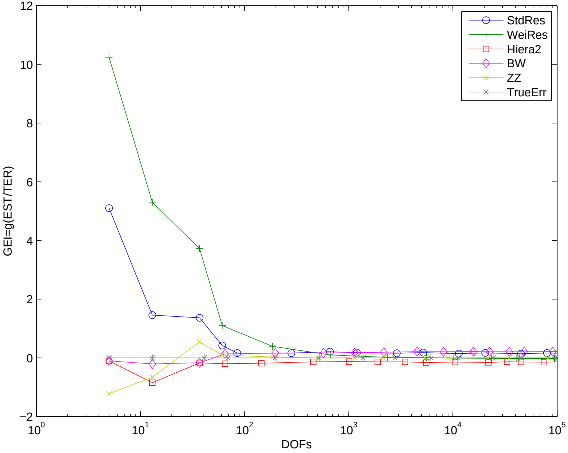
$$\mathrm{GEI} = g\left(\frac{\mathcal{E}}{\|u_{\mathcal{T}} - u\|}\right)$$

with $g(\frac{1}{q}) = -g(q)$ and such that

$$\begin{aligned} &\mathrm{GEI} > 0 \quad \text{overestimation} \\ &\mathrm{GEI} < 0 \quad \text{underestimation} \end{aligned}$$

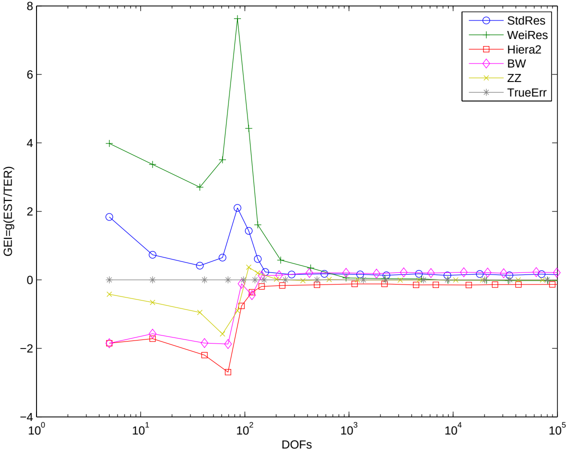
In what follows: GEI versus DOFs in semilogx scale

A basic example



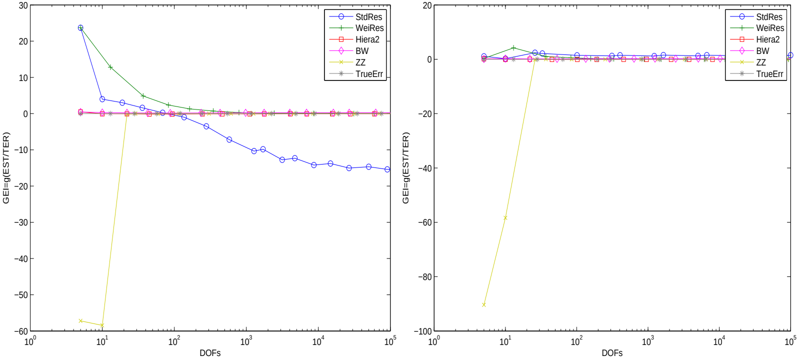
solution is an exponential peak

A basic example



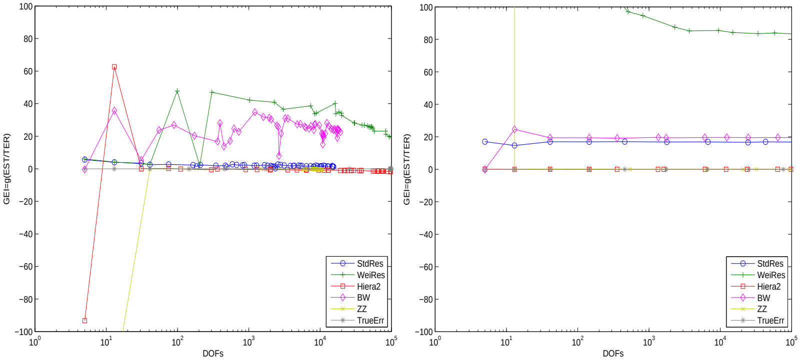
solution is an exponential peak – more stressed

Need of dual energy norm



small (left) and big (right) isotropic ellipticity

Challenging anisotropy - I



anisotropic ellipticity, direction of min (left) and max (right) eigenvalue

Outline

Error versus DOFs

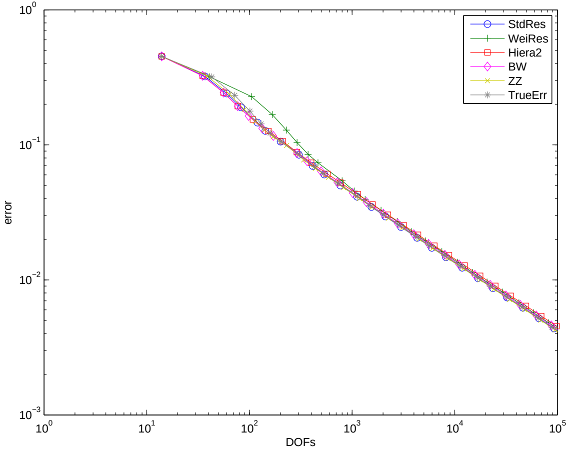
To measure the performance for adaptivity, we consider the graph of the error $\|u_{\mathcal{T}} - u\|$ depending on $\#$DOFs, a measure of the complexity of the finite element solution.

The lower the graph, the better the estimator.

Recall: marking with maximum strategy.

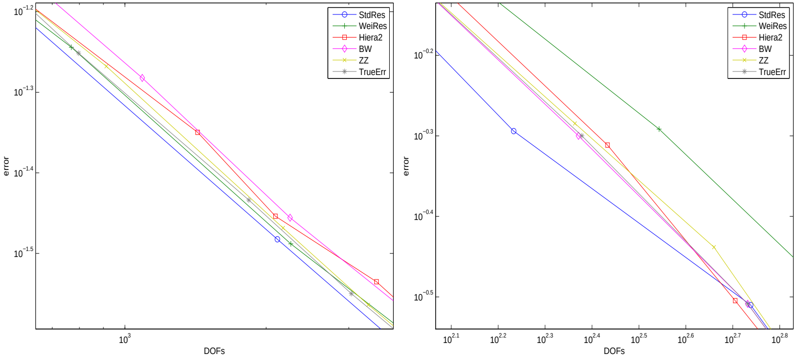
In what follows: $\|u_{\mathcal{T}} - u\|$ versus $\#$DOFs in loglog scale

A basic example



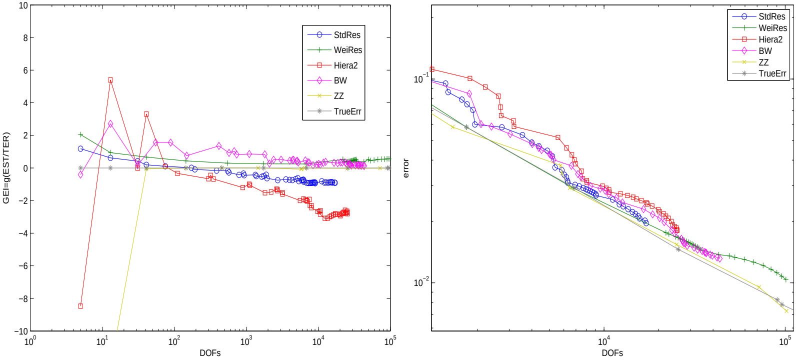
geometric singularity for slit

Is the local true error the best indicator?



polynomial solution (left) and dominating reaction (right)

Challenging anisotropy - II



anisotropic ellipticity, direction of minimum eigenvalue

Some conclusions

All estimators may suffer on relatively coarse meshes.

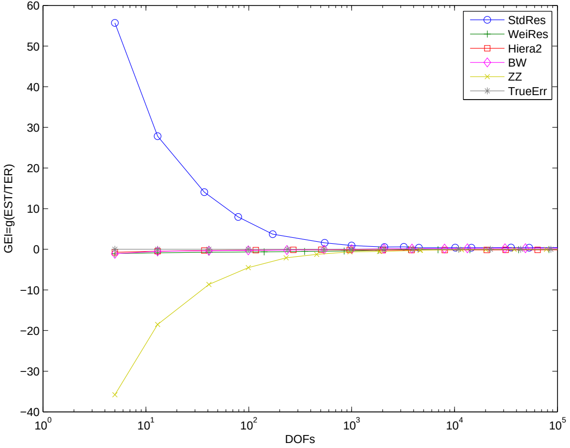
No examined estimator performs well in the whole problem class.

The relevant properties for adaptivity (and their measurement) seem to be unclear.

(Solving local problems seems not to pay off.)

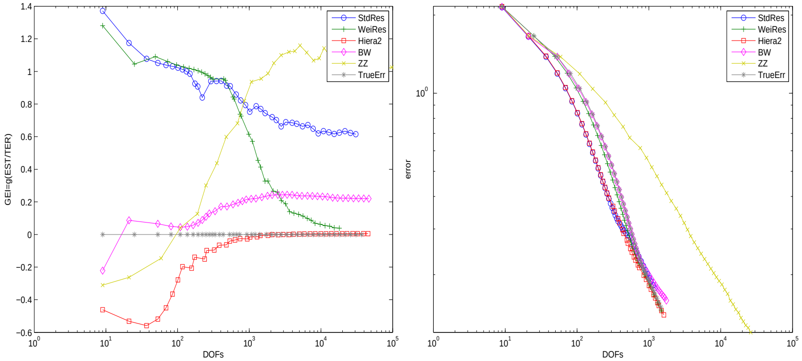
Strong anisotropy is still challenging.

Error quantification for dominating reaction



dominating reaction

Principal failure of ZZ



discontinuous isotropic ellipticity