A posteriori analysis
of the penalty method

Christine Bernardi
Laboratoire Jacques-Louis Lions
C.N.R.S. & Université Pierre et Marie Curie

A posteriori analysis has become the basic tool for automatic mesh adaptivity in finite elements and finite volumes. However many other applications have recently appeared.
• About multi-step discretizations

• Optimization of the penalty parameter for the Stokes equations
  with V. Girault, and F. Hecht

• Other applications of the penalty method
About multi-step discretizations

Let \( X \) be a Banach space, and assume that \( A \) is a continuous mapping from \( X \) into \( X \). For a given element \( f \) of \( X \), we are interested in the discretization of the equation:

\[
\text{Find } u \text{ in } X \text{ such that } A(u) = f.
\]

\( \delta \): positive parameter

We consider the problem:

\[
\text{Find } u_\delta \text{ in } X_\delta \text{ such that } A_\delta(u_\delta) = f_\delta,
\]

where \( X_\delta \) is a finite-dimensional subspace of \( X \), \( A_\delta \) denotes an approximation of \( A \) defined on \( X_\delta \) and \( f_\delta \) an approximation of \( f \) in \( X_\delta \).
A priori estimates

\[ \|u - u_\delta\|_X \leq F(\delta, u) + H(\delta, f). \]

The quantity \( F(\delta, u) \) is usually equal to some power of \( \delta \) times some norm of \( u \), so it involves the regularity of \( u \) (which is most often unknown).

\[ \implies \] proves the convergence of the method.

\[ \implies \] useful for the choice of the discretization.

A posteriori estimates

\[ \|u - u_\delta\|_X \leq G(\delta, f_\delta, u_\delta) + K(\delta, f). \]

The quantity \( G(\delta, f_\delta, u_\delta) \) can be computed explicitly once the discrete solution \( u_\delta \) is known.

First application: mesh adaptivity
A new application: multi-step discretizations

In a large number of cases, the discretization of partial differential equations involves one or several intermediate (non discrete) problems before the final discrete problem. However error indicators only depending on the discrete solution can be constructed for each step of the discretization.

Initial problem: Find $u$ in $X$ such that $A(u) = f$.

Intermediate problem: Find $u_\varepsilon$ in $X$ such that $A_\varepsilon(u_\varepsilon) = f$.

Discrete problem: Find $u_{\varepsilon\delta}$ in $X_\delta$ such that $A_{\varepsilon\delta}(u_{\varepsilon\delta}) = f_\delta$.

Several parameters $\varepsilon$ and $\delta$ are involved in the discretization. The aim of a posteriori analysis is to optimize simultaneously the choice of both $\varepsilon$ and $\delta$. 
• **Parabolic equation**

Intermediate problem: Time semi-discrete problem  
Discretization parameters: Time step and mesh size  
A. Bergam, C.B., Z. Mghazli

• **Penalization (or regularization) methods**

Intermediate problem: Continuous penalized problem  
Discretization parameters: Penalty parameter and mesh size  
C.B., V. Girault, F. Hecht

• **Automatic coupling of models**

M. Braack, A. Ern
Several independent parameters appear in the discretization. This leads to use several families of error indicators.

The aim is to uncouple as much as possible the errors linked to the different parameters. But each indicator only requires the knowledge of the fully discrete solution (in order that it can be computed explicitly). So this is not completely possible for the estimates.

The idea of local representation of the error must be given up (or at least modified) for some families of indicators. But the optimization of the other parameters must remain compatible with mesh adaptivity.

An optimization criterion: The error due to $\varepsilon$ must be of the same order as the discretization error due to $\delta$. 
Optimization of the penalty parameter for the Stokes equations

Let \( \Omega \) be a bounded open domain in \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \), with a Lipschitz-continuous boundary.

\[
-\Delta u + \text{grad } p = f \quad \text{in } \Omega, \\
\text{div } u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\]

The unknowns are the velocity \( u \) and the pressure \( p \).

Variational formulation

Find \( (u, p) \) in \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) satisfying

\[
\forall v \in H^1_0(\Omega)^d, \quad \int_{\Omega} \text{grad } u : \text{grad } v \, dx - \int_{\Omega} (\text{div } v)(x) p(x) \, dx = \int_{\Omega} f(x) \cdot v(x) \, dx, \\
\forall q \in L^2_0(\Omega), \quad -\int_{\Omega} (\text{div } u)(x) q(x) \, dx = 0.
\]

For any data \( f \) in \( H^{-1}(\Omega)^d \), this problem has a unique solution.
The penalized problem

Let $\varepsilon$ be the penalty parameter, $0 < \varepsilon \leq 1$.

Find $(u_\varepsilon, p_\varepsilon)$ in $H^1_0(\Omega)^d \times L^2_0(\Omega)$ satisfying

\[
\forall v \in H^1_0(\Omega)^d, \quad \int_\Omega \nabla u_\varepsilon : \nabla v \, dx - \int_\Omega (\text{div } v)(x) p_\varepsilon(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx,
\]

\[
\forall q \in L^2_0(\Omega), \quad -\int_\Omega (\text{div } u_\varepsilon)(x) q(x) \, dx - \varepsilon \int_\Omega p_\varepsilon(x) q(x) \, dx = 0.
\]

The second equation is equivalent to

\[
p_\varepsilon = -\varepsilon^{-1} \text{div } u_\varepsilon.
\]
As a consequence, the penalized problem can equivalently be written

\textit{Find } u_\varepsilon \textit{ dans } H^1_0(\Omega)^d \textit{ satisfying}

\[ \forall v \in H^1_0(\Omega)^d, \quad \int_\Omega \text{grad } u_\varepsilon : \text{grad } v \, dx + \varepsilon^{-1} \int_\Omega (\text{div } u_\varepsilon)(x)(\text{div } v)(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx, \]

\textit{Find } p_\varepsilon \textit{ dans } L^2_0(\Omega) \textit{ satisfying}

\[ \forall q \in L^2_0(\Omega), \quad \varepsilon \int_\Omega p_\varepsilon(x)q(x) \, dx = -\int_\Omega (\text{div } u_\varepsilon)(x)q(x) \, dx. \]

For any data } f \text{ in } H^{-1}(\Omega)^d, \text{ this problem has a unique solution.}

\textit{This does not require any inf-sup condition!}

A priori error estimates can be proven between the solutions \((u, p)\) and \((u_\varepsilon, p_\varepsilon)\). The error behaves like \(\varepsilon\).

\textit{V. Girault, P.-A. Raviart}
The discrete problem

\((T_h)_h\) : regular family of triangulations of \(\Omega\) by triangles or tetrahedra.

\[ X_h \subset H_0^1(\Omega)^d, \quad M_h \subset L_0^2(\Omega). \]

The spaces \(X_h\) et \(M_h\) are made of piecewise polynomial functions on the triangulation \(T_h\).

\[
\text{Find } (u_{\varepsilon h}, p_{\varepsilon h}) \text{ in } X_h \times M_h \text{ satisfying }
\]

\(\forall v_h \in X_h, \quad \int_{\Omega} \text{grad } u_{\varepsilon h} : \text{grad } v_h \, dx - \int_{\Omega} (\text{div } v_h)(x) p_{\varepsilon h}(x) \, dx = \int_{\Omega} f(x) \cdot v_h(x) \, dx,\)

\(\forall q_h \in M_h, \quad -\int_{\Omega} (\text{div } u_{\varepsilon h})(x) q_h(x) \, dx - \varepsilon \int_{\Omega} p_{\varepsilon h}(x) q_h(x) \, dx = 0.\)

Let \(\Pi_h\) denote the orthogonal projection operator from \(L_0^2(\Omega)\) onto \(M_h\). The second equation is equivalent to

\[ p_{\varepsilon h} = -\varepsilon^{-1} \Pi_h(\text{div } u_{\varepsilon h}). \]
The discrete problem can equivalently be written

\[ \text{Find } u_{\varepsilon h} \text{ in } X_h \text{ satisfying} \]
\[ \forall v_h \in X_h, \quad \int_{\Omega} \text{grad } u_{\varepsilon h} \cdot \text{grad } v_h \, dx + \varepsilon^{-1} \int_{\Omega} \Pi_h(\text{div } u_{\varepsilon h})(x) \Pi_h(\text{div } v_h)(x) \, dx = \int_{\Omega} f(x) \cdot v_h(x) \, dx, \]

\[ \text{Find } p_{\varepsilon h} \text{ in } M_h \text{ satisfying} \]
\[ \forall q_h \in M_h, \quad \varepsilon \int_{\Omega} p_{\varepsilon h}(x) q_h(x) \, dx = -\int_{\Omega} (\text{div } u_{\varepsilon h})(x) q_h(x) \, dx. \]

For any \( f \) in \( H^{-1}(\Omega)^d \), this problem has a unique solution.

This does not require any inf-sup condition!

Provides an efficient algorithm for solving the discrete problem, since the two unknowns are now uncoupled!

If an inf-sup condition exists between the spaces \( X_h \) and \( M_h \), a priori error estimates can be proven between the solutions \((u, p)\) and \((u_{\varepsilon h}, p_{\varepsilon h})\).
Two families of error indicators

- **Error indicator related to the penalty term**

  \[ \eta_\varepsilon = \varepsilon \| p_{\varepsilon h} \|_{L^2(\Omega)}. \]

- **Error indicators related to the finite elements**

  \[ \mathcal{E}_K : \text{set of edges or faces of } K \text{ which are not contained in } \partial \Omega. \]

For any element \( K \) of \( T_h \),

\[
\eta_K = h_K \| f_h + \Delta u_{\varepsilon h} - \text{grad} p_{\varepsilon h} \|_{L^2(K)^d} \\
+ \frac{1}{2} \sum_{e \in \mathcal{E}_K} h_e^\frac{1}{2} \| [\partial_n u_{\varepsilon h} - p_{\varepsilon h} n] \|_{L^2(e)^d} + \| \text{div } u_{\varepsilon h} \|_{L^2(K)},
\]

where \( f_h \) denotes a piecewise polynomial approximation of \( f \).

C.B., B. Métivet, R. Verfürth
An a posteriori error estimate

The aim is to bound the error between \((u, p)\) and \((u_{\varepsilon h}, p_{\varepsilon h})\) as a function of \(\eta_{\varepsilon}\), the \(\eta_K\) and the data.

\[
|u - u_{\varepsilon h}|_{H^1(\Omega)} \leq |u - u_{\varepsilon}|_{H^1(\Omega)} + |u_{\varepsilon} - u_{\varepsilon h}|_{H^1(\Omega)},
\]

\[
\|p - p_{\varepsilon h}\|_{L^2(\Omega)} \leq \|p - p_{\varepsilon}\|_{L^2(\Omega)} + \|p_{\varepsilon} - p_{\varepsilon h}\|_{L^2(\Omega)}.
\]

Idea: A residual equation to bound each term.
Residual equation for the first term:

\[ \forall v \in H^1_0(\Omega)^d, \int_{\Omega} \nabla (u - u_\varepsilon) : \nabla v \, dx - \int_{\Omega} \nabla \cdot v(x) (p - p_\varepsilon)(x) \, dx = 0, \]

\[ \forall q \in L^2_0(\Omega), \quad -\int_{\Omega} \left( \nabla \cdot (u - u_\varepsilon) \right)(x) q(x) \, dx = \varepsilon \int_{\Omega} p_\varepsilon(x) q(x) \, dx. \]

The estimate results from the stability properties of the Stokes operator and a triangle inequality.

Proposition: The following a posteriori estimate holds

\[ |u - u_\varepsilon|_{H^1(\Omega)^d} + \|p - p_\varepsilon\|_{L^2(\Omega)} \leq c \left( \eta_\varepsilon + \varepsilon \|p_\varepsilon - p_\varepsilon h\|_{L^2(\Omega)} \right). \]
Residual equation for the second term:

\[
\forall v \in H_0^1(\Omega)^d, \quad \int_{\Omega} \nabla (u_\varepsilon - u_{\varepsilon h}) : \nabla v \, dx + \ldots
\]

The estimate is derived via the standard arguments for the Stokes problem (and requires the introduction of an approximation \(v_h\) of the function \(v\) in \(X_h\)).

**Proposition:** Let us assume that the space \(X_h\) contains

\[
Y_h = \{v_h \in H_0^1(\Omega)^d; \forall K \in T_h, v_h|_K \in \mathcal{P}_1(K)^d\}.
\]

The following a posteriori estimate holds

\[
|u_\varepsilon - u_{\varepsilon h}|_{H^1(\Omega)^d} + \|p_\varepsilon - p_{\varepsilon h}\|_{L^2(\Omega)} \leq c \left( \eta_\varepsilon + \left( \sum_{K \in T_h} (\eta_K^2 + h_K^2 \|f - f_h\|_{L^2(K)^d}^2)^{\frac{1}{2}} \right) \right).
\]
An upper bound for the indicators

Idea: In the first residual equation,

$$\forall q \in L^2_0(\Omega),$$

$$\varepsilon \int_{\Omega} p_\varepsilon h(x)q(x) \, dx = -\varepsilon \int_{\Omega} (p_\varepsilon - p_\varepsilon h)(x)q(x) \, dx - \int_{\Omega} (\text{div} (u - u_\varepsilon))(x) q(x) \, dx,$$

we take $q$ equal to $p_\varepsilon h$.

Proposition: The following estimate holds

$$\eta_\varepsilon \leq c |u - u_\varepsilon|_{H^1(\Omega)} + \varepsilon \|p_\varepsilon - p_\varepsilon h\|_{L^2(\Omega)}.$$
Idea: In the second residual equation, we take \( v \) successively equal to

\[
v_K = (f_h + \Delta u_{\epsilon h} - \text{grad} p_{\epsilon h}) \psi_K \quad \text{on } K, \quad 0 \quad \text{elsewhere},
\]

\[
v_e = \mathcal{L}(\partial \nu u_{\epsilon h} - p_{\epsilon h} n) \psi_e \quad \text{on } K \cup K', \quad 0 \quad \text{elsewhere},
\]

next \( q \) equal to

\[
q_K = \text{div} u_{\epsilon h} \quad \text{on } K, \quad 0 \quad \text{elsewhere}.
\]

Proposition: The following estimate holds, for any element \( K \) of \( T_h \),

\[
\eta_K \leq c \left( |u - u_{\epsilon}|_{H^1(K)^d} + |u_{\epsilon} - u_{\epsilon h}|_{H^1(\omega_K)^d} \right.
\]

\[
+ \|p_{\epsilon} - p_{\epsilon h}\|_{L^2(\omega_K)} + h_K \|f - f_h\|_{L^2(\omega_K)^d},
\]

where \( \omega_K \) stands for the union of elements of \( T_h \) that share at least an edge or a face with \( K \).

Optimal estimates, the second one is local in space!
Some numerical experiments

realized with the code FreeFem++

F. Hecht, O. Pironneau

Taylor–Hood finite element, uniform mesh

Given smooth solution in a square

- Influence of the penalty parameter
- Influence of the mesh

\[ \eta_h = \left( \sum_{K \in T_h} \eta_K^2 \right)^{\frac{1}{2}}. \]
\[ h = \sqrt{2} \times 10^{-1} \]
\[ \varepsilon = 10^{-2} \]
Extensions

• Local penalization

We have tried to replace \( \varepsilon \int_{\Omega} p(x)q(x) \, dx \) by

\[
\sum_{K \in \mathcal{T}_h} \varepsilon_K \int_K p(x)q(x) \, dx.
\]

+ two types of local error indicators.

Some restrictions on the discretization, no special interest.

• Navier–Stokes equations

J. Pousin, J. Rappaz
R. Verfürth

No further difficulty.
Data: viscosity $\nu = 10^{-2}$ and non zero tangential velocity on the left boundary.

$$\varepsilon = 0.14 \times 10^{-2}$$
Other applications of the penalty method

- Spectral element discretization of the Stokes problem
  with A. Blouza, N. Chorfi, and N. Kharrat

Let $\Omega$ be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitz–continuous boundary.

\[-\Delta u + \text{grad } p = f \quad \text{in } \Omega,\]
\[
\text{div } u = 0 \quad \text{in } \Omega,\]
\[
u = 0 \quad \text{on } \partial \Omega.
\]

The unknowns are the velocity $u$ and the pressure $p$. 
The penalized problem

Let \( \varepsilon \) be the penalty parameter, \( 0 < \varepsilon \leq 1 \).

Find \((u, p)\) in \(H^1_0(\Omega)^d \times L^2_0(\Omega)\) such that

\[
\forall v \in H^1_0(\Omega)^d, \quad \int_{\Omega} \text{grad } u : \text{grad } v \, dx - \int_{\Omega} (\text{div } v(x)) p(x) \, dx = \int_{\Omega} f(x) \cdot v(x) \, dx,
\]

\[
\forall q \in L^2_0(\Omega), \quad -\int_{\Omega} (\text{div } u(x)) q(x) \, dx - \varepsilon \int_{\Omega} p_\varepsilon(x) q(x) \, dx = 0.
\]
We assume that $\Omega$ admits a decomposition without overlap

$$\bar{\Omega} = \bigcup_{k=1}^{K} \Omega_k \text{ et } \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k < k' \leq K,$$

where:

- Each $\Omega_k$ is a rectangle ($d = 2$) or a rectangular parallelepiped ($d = 3$),
- The intersection of two different $\Omega_k$ is either empty or a vertex or a whole edge or a whole face of both subdomains.

**The discrete spaces**

$P_n(\Omega_k)$: space of restrictions to $\Omega_k$ of polynomials with degree $\leq n$ with respect to each variable.

Let $\lambda$ be a fixed parameter, $0 < \lambda \leq 1$. For any integer $N \geq 2$, we define the spaces

$$X_N = \left\{ v_N \in H_0^1(\Omega)^d; \; v_N|_{\Omega_k} \in P_N(\Omega_k)^d, \; 1 \leq k \leq K \right\},$$

$$M_N = \left\{ q_N \in L_0^2(\Omega); \; q_N|_{\Omega_k} \in P_{N-2}(\Omega_k) \cap P_{\lambda N}(\Omega_k), \; 1 \leq k \leq K \right\}.$$
Numerical integration

\[ \xi_0 = -1, \quad \xi_N = 1. \]

Gauss-Lobatto quadrature formula: there exist nodes \( \xi_j, 1 \leq j \leq N - 1 \), and weights \( \rho_j, 0 \leq j \leq N \), such that

\[ \forall \Phi \in P_{2N-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta) \, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j) \rho_j. \]

If \( F_k \) denotes one of the affine mappings that maps the reference domain \([-1,1]^d \) onto \( \Omega_k \), we define a discrete product, for instance in dimension \( d = 2 \) by

\[ (u,v)_N = \sum_{k=1}^{K} \frac{\operatorname{meas}(\Omega_k)}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} u \circ F_k(\xi_i,\xi_j) v \circ F_k(\xi_i,\xi_j) \rho_i \rho_j, \]
The discrete problem

Find \((u_{\varepsilon N}, p_{\varepsilon N})\) in \(X_N \times M_N\) such that

\[
\forall v_N \in X_N, \quad \left(\text{grad } u_{\varepsilon N}, \text{grad } v_N\right)_N - \left(\text{div } v_N, p_{\varepsilon N}\right)_N = (f, v_N)_N,
\]

\[
\forall q_N \in M_N, \quad -\left(\text{div } u_{\varepsilon N}, q_N\right)_N - \varepsilon \left(p_{\varepsilon N}, q_N\right)_N = 0.
\]

Let \(\Pi_N\) be the orthogonal projection operator from \(L^2_0(\Omega)\) onto \(M_N\). The second equation reads

\[
p_{\varepsilon N} = -\varepsilon^{-1} \Pi_N(\text{div } u_{\varepsilon N}).
\]
The discrete problem can equivalently be written as

\[
\begin{align*}
\text{Find } u_{\varepsilon N} \text{ in } X_N \text{ such that } \\
\forall v_N \in X_N, \quad \left( \nabla u_{\varepsilon N}, \nabla v_N \right)_N + \varepsilon^{-1} \left( \Pi_N(\nabla u_{\varepsilon N}), \Pi_N(\nabla v_N) \right)_N \\
= (f, v_N)_N,
\end{align*}
\]

\[
\begin{align*}
\text{Find } p_{\varepsilon N} \text{ in } M_N \text{ such that } \\
\forall q_N \in M_N, \quad \varepsilon \left( p_{\varepsilon N}, q_N \right)_N = -\left( \nabla u_{\varepsilon N}, q_N \right)_N.
\end{align*}
\]

For any continuous data \( f \) on \( \overline{\Omega} \), this problem has a unique solution.
\[
\text{This does not require any inf-sup condition!}
\]

Provides an efficient algorithm for solving the discrete problem, since the two unknowns are now uncoupled!

If an inf-sup condition exists between the spaces \( X_N \) and \( M_N \), a priori error estimates can be proven between the solutions \( (u, p) \) and \( (u_{\varepsilon N}, p_{\varepsilon N}) \).
Two families of error indicators

- **Error indicator related to the penalty term**

  \[ \eta_\varepsilon = \varepsilon \| p_{\varepsilon N} \|_{L^2(\Omega)}. \]

- **Error indicators related to the spectral discretization**

  \( \mathcal{E}_k : \) set of edges or faces of \( \Omega_k \) which are not contained in \( \partial \Omega \).

For \( 1 \leq k \leq K \),

\[ \eta_k = N^{-1} \| I_N f + \Delta u_{\varepsilon N} - \text{grad} p_{\varepsilon N} \|_{L^2(K)^d} \]

\[ + \sum_{e \in \mathcal{E}_k} N^{-\frac{1}{2}} \| [\partial_\nu u_{\varepsilon N} - p_{\varepsilon N} n] \|_{L^2(e)^d} + \| \text{div} u_{\varepsilon N} \|_{L^2(K)}, \]

where \( I_N \) stands for the interpolation operator at all nodes \( F_k(\xi_i, \xi_j) \) or \( F_k(\xi_i, \xi_j, \xi_p) \).
A posteriori error estimate

Proposition: The following a posteriori error estimate holds

\[ |u - u_\varepsilon|_{H^1(\Omega)^d} + \|p - p_\varepsilon\|_{L^2(\Omega)} \leq c \left( \eta_\varepsilon + \varepsilon \|p_\varepsilon - p_\varepsilon N\|_{L^2(\Omega)} \right). \]

Proposition: The following a posteriori error estimate holds

\[ |u_\varepsilon - u_\varepsilon N|_{H^1(\Omega)^d} + \|p_\varepsilon - p_\varepsilon N\|_{L^2(\Omega)} \leq c \left( \eta_\varepsilon + \rho_\Omega \left( \sum_{k=1}^{K} \eta_k^2 \right)^{\frac{1}{2}} + \|f - \mathcal{I}_N f\|_{L^2(K)^d}^{\frac{1}{2}} \right), \]

where \( \rho_\Omega \) is equal to 1 in dimension \( d = 2 \) or if \( \Omega \) is convex, to \( N^{-\frac{1}{2}} \) otherwise.
An upper bound for the indicators

**Proposition:** The following estimate holds

\[ \eta_\varepsilon \leq |u - u_\varepsilon|_{H^1(\Omega)} + \varepsilon \|p_\varepsilon - p_\varepsilon N\|_{L^2(\Omega)}. \]

We do not state an upper bound for the \( \eta_k \) (which would be nonoptimal) since we do not intend to adapt the \( N \).
Some numerical experiments

Adaptivity strategy: Let us choose $\varepsilon^0 \leq 1$ and a parameter $\mu$, $0 < \mu < 1$. Next, assuming that $\varepsilon^m$ is known,
(i) we compute the solution $(u_{\varepsilon N}, p_{\varepsilon N})$ for $\varepsilon = \varepsilon^m$, the indicator $\eta^{\varepsilon^m}$ and the Hilbertian sum $\eta(N) = \left( \sum_{k=1}^{K} \eta_k^2 \right)^{1/2}$.
(ii) if
$$\mu \eta(N) \leq \eta^{\varepsilon^m} \leq \frac{1}{\mu} \eta(N),$$
we stop the process.
(iii) otherwise, we take $\varepsilon^{m+1}$ equal to $\varepsilon^m \eta(N)/\eta^{\varepsilon^m}$ and we go back to step (i).

For a given solution in a $L$-shaped domain divided into three squares,

<table>
<thead>
<tr>
<th>$N$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{opt}$</td>
<td>0.0375</td>
<td>0.0069</td>
<td>0.0024</td>
<td>0.0007</td>
<td>0.0006</td>
</tr>
</tbody>
</table>
$N = 20$
Domain decomposition methods
with T. Chacón Rebollo, E. Chacón Vera, and D. Franco Coronil

When a decomposition of the domain without overlap is considered, the matching conditions on the interfaces are most often handled via the introduction of a Lagrange multiplier. And of course it is possible to use a penalty method for the corresponding mixed problem.

Let $\Omega$ be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitz-continuous boundary.

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

We assume that $\Omega$ admits a decomposition without overlap

$$\Omega = \Omega_1 \cup \Omega_2 \quad \text{and} \quad \Omega_1 \cap \Omega_2 = \emptyset,$$

and also that, if $\Gamma$ denotes the interface $\partial\Omega_1 \cap \partial\Omega_2$, $\Gamma$ is Lipschitz-continuous and $t\partial\Gamma$ is contained in $\partial\Omega$. 
Variational formulation

\[ H^1_*(\Omega_i) = \{ v \in H^1(\Omega_i); \ v = 0 \ \text{on} \ \partial\Omega_i \setminus \Gamma \}. \]

Find \( (u_1, u_2, \lambda) \) in \( H^1_*(\Omega_1) \times H^1_*(\Omega_2) \times H^{\frac{1}{2}}_{00}(\Gamma) \) such that

\[ \forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2), \]

\[ \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx + [[v_1 - v_2, \lambda]]_{\Gamma} = \sum_{i=1}^{2} \int_{\Omega_i} f(x) \cdot v_i(x) \, dx, \]

\[ \forall \mu \in H^{\frac{1}{2}}_{00}(\Gamma), \quad [[u_1 - u_2, \mu]]_{\Gamma} = 0, \]

where \([[,\cdot]]_{\Gamma}\) denotes the scalar product of \( H^{\frac{1}{2}}_{00}(\Gamma) \).

The equivalence of this formulation with the initial problem is readily checked.
The penalized problem

Find \((u_1^\varepsilon, u_2^\varepsilon, \lambda^\varepsilon)\) in \(H^1_*(\Omega_1) \times H^1_*(\Omega_2) \times H^{\frac{1}{2}}_{00}(\Gamma)\) such that

\[
\forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2), \\
\sum_{i=1}^{2} \int_{\Omega_i} \text{grad } u_i^\varepsilon \cdot \text{grad } v_i \, dx + [[v_1 - v_2, \lambda^\varepsilon]]_{\Gamma} = \sum_{i=1}^{2} \int_{\Omega_i} f(x) \cdot v_i(x) \, dx,
\]

\[
\forall \mu \in H^{\frac{1}{2}}_{00}(\Gamma), \quad [[u_1^\varepsilon - u_2^\varepsilon, \mu]]_{\Gamma} = \varepsilon [[\lambda^\varepsilon, \mu]]_{\Gamma}.
\]

A priori error estimates of order \(\varepsilon\) between the solutions \((u_1, u_2, \lambda)\) and \((u_1^\varepsilon, u_2^\varepsilon, \lambda^\varepsilon)\) are easily derived.
But when replacing the scalar product $\left[[\cdot, \cdot]\right]_\Gamma$ by the scalar product of $L^2(\Gamma)$, the convergence is only of order $\sqrt{\varepsilon}$.

T. Chacón Rebollo, E. Chacón Vera

From now on, we only consider the reduced problem

Find $(u_1^\varepsilon, u_2^\varepsilon)$ in $H^1_*(\Omega_1) \times H^1_*(\Omega_2)$ such that

$$\forall (v_1, v_2) \in H^1_*(\Omega_1) \times H^1_*(\Omega_2),$$

$$\sum_{i=1}^{2} \int_{\Omega_i} \text{grad } u_i^\varepsilon \cdot \text{grad } v_i \, dx + \varepsilon^{-1} \left[[u_1^\varepsilon - u_2^\varepsilon, v_1 - v_2]\right]_\Gamma = \sum_{i=1}^{2} \int_{\Omega_i} f(\mathbf{x}) \cdot v_i(\mathbf{x}) \, d\mathbf{x},$$
The discrete problem

\((T_h)_h: \) regular family of triangulations of \(\Omega\) by triangles or tetrahedra such that \(\Gamma\) is contained in edges \((d = 2)\) or faces \((d = 3)\) of elements of \(T_h\).

We consider discrete spaces built from triangulation \(T_h\) such that

\[ X_{ih} \subset H^{1}(\Omega_i). \]

Find \((u^{\varepsilon}_{1h}, u^{\varepsilon}_{2h})\) in \(X_{1h} \times X_{2h}\) such that

\[ \forall (v_{1h}, v_{2h}) \in X_{1h} \times X_{2h}, \]

\[ \sum_{i=1}^{2} \int_{\Omega_i} \text{grad} u_{ih}^{\varepsilon} \cdot \text{grad} v_{ih} \, dx + \varepsilon^{-1} [u_{1h}^{\varepsilon} - u_{2h}^{\varepsilon}, v_{1h} - v_{2h}]_{\Gamma} = \sum_{i=1}^{2} \int_{\Omega_i} f(x) \cdot v_{ih}(x) \, dx. \]

This problem has a unique solution.
Two families of error indicators

- **Error indicator related to the penalty term**
  \[
  \eta_\varepsilon = \varepsilon \| u_{1h}^\varepsilon - u_{2h}^\varepsilon \|_{H_{00}^2(\Gamma)}.
  \]

- **Error indicators related to the finite element discretization**
  For each element \( K \) of \( \mathcal{T}_h \) contained in \( \Omega_i, i = 1, 2, \)
  \[
  \eta_K = h_K \| f_h + \Delta u_{ih}^\varepsilon \|_{L^2(K)} + \frac{1}{2} \sum_{e \in \mathcal{E}_K} h_e^{\frac{1}{2}} \| [\partial_n u_{ih}^\varepsilon] \|_{L^2(e)},
  \]

  where \( f_h \) is a piecewise polynomial approximation of \( f \).
A posteriori error estimate

\[ h_m = \min_{K \cap \Gamma \neq \emptyset} h_K. \]

Proposition: Assume that each space \( X_{ih} \) contains

\[ Y_{ih} = \{ v_h \in H^1_*(\Omega_i); \forall K \in T_h, v_h|_K \in P_1(K) \}. \]

The following a posteriori error estimate holds

\[ \sum_{i=1}^{2} |u - u_{ih}|_{H^1(\Omega_f)} \leq c \left( \eta_\epsilon + \left( \sum_{K \in T_h} \mu_K^2 (\eta_K^2 + \| f - f_h \|_{L^2(K)}^2) \right)^{\frac{1}{2}} \right), \]

where \( \mu_K \) is equal to 1 if \( K \cap \Gamma = \emptyset \), \( \mu_K^{-\frac{1}{2}} \) otherwise.

The loss of optimality is local. It is due to the nonconformity of the discretization.
An upper bound for the indicators

**Proposition:** The following estimate holds

\[ \eta_{\epsilon} \leq c \sum_{i=1}^{2} |u - u_{i\epsilon}^h|_{H^1(\omega_i)}, \]

and, for any element \( K \) of \( T_h \),

\[ \eta_K \leq c \left( |u - u_h^\epsilon|_{H^1(\omega_K)} + h_K \|f - f_h\|_{L^2(\omega_K)} \right), \]

where \( \omega_K \) is the union of elements of \( T_{nh} \) that share at least an edge \((d = 2)\) or a face \((d = 3)\) with \( K \).
Key numerical difficulty:

Computing the scalar product $[[\cdot, \cdot]]_\Gamma$ is very expensive, even when using an extension of

E. Casas, J.-P. Raymond

So the method is not so efficient as it could be thought from the estimates.
Some numerical experiments

Smooth solution in a square divided into two rectangles.

Influence of the penalty parameter (left part), of the mesh (right part)

\[ h = 3 \cdot 10^{-2} \quad \varepsilon = 10^{-2} \]
Smooth solution in a $L$-shaped domain divided into two trapezia.

Influence of the penalty parameter (left part), of the mesh (right part)

$$h = 3 \times 10^{-2}$$

$$\varepsilon = 10^{-2}$$
The adaptivity algorithm in the $L$-shaped domain

Comparison of the methods with $H_{00}^{1/2}(\Gamma)$- and $L^2(\Gamma)$-matching

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon_{opt}$</th>
<th>Iterations</th>
<th>CPU time</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{00}^{1/2}$</td>
<td>0.0503</td>
<td>165</td>
<td>126.3s</td>
</tr>
<tr>
<td>$L^2$</td>
<td>0.0029</td>
<td>157</td>
<td>99.0s</td>
</tr>
</tbody>
</table>

$h = 1/64$
Finite element discretization of a shell model
with A. Blouza, F. Hecht, and H. Le Dret

The Naghdi model and its variational formulations

\( \omega : \) polygon in \( \mathbb{R}^2 \).

The midsurface of the shell is given by \( S = \varphi(\bar{\omega}) \) where \( \varphi \) is a one-to-one mapping in \( W^{2,\infty}(\omega)^3 \) such that the two vectors

\[ a_\alpha(x) = (\partial_\alpha \varphi)(x) \]

are linearly independent at each point \( x \) of \( \bar{\omega} \). The thickness of the shell is denoted by \( e \).

Unit normal vector to \( S \) at point \( \varphi(x) \): \( a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|} \).

Local contravariant basis: \( a_i(x) \cdot a^j(x) = \delta_i^j \).

Area element of the midsurface: \( \sqrt{a}(x) = |a_1(x) \wedge a_2(x)| \).

First fundamental form of the surface: \( a^{\alpha \beta} = a^\alpha \cdot a^\beta \).
In the case of homogeneous, isotropic material with Young modulus $E > 0$ and Poisson coefficient $ν$, $0 \leq ν < \frac{1}{2}$, the coefficients of the elasticity tensor are given by

$$a^{αβρσ} = \frac{E}{2(1 + ν)}(a^{αρ}a^{βσ} + a^{ασ}a^{βρ}) + \frac{Eν}{1 - ν^2}a^{αβ}a^{ρσ}.$$  

This tensor is symmetric and uniformly strictly positive.

Covariant components of the change of metric tensor:

$$γ_{αβ}(u) = \frac{1}{2}(∂_αu · a_β + ∂_βu · a_α).$$

Covariant components of the change of transverse shear tensor:

$$δ_{α3}(u, r) = \frac{1}{2}(∂_αu · a_3 + r · a_α).$$

Covariant components of the change of curvature tensor:

$$χ_{αβ}(u, r) = \frac{1}{2}(∂_αu · ∂_βa_3 + ∂_βu · ∂_αa_3 + ∂_αr · a_β + ∂_βr · a_α).$$
Boundary conditions:

The boundary $\partial \omega$ is divided into two parts: $\gamma_0$ on which the shell is clamped and its complementary part $\gamma_1 = \partial \omega \setminus \gamma_0$

$$H_{\gamma_0}(\omega) = \{ \mu \in H^1(\omega); \mu = 0 \text{ on } \gamma_0 \}.$$

The unknowns are the midsurface displacement $u$ of the shell and its rotation $r$ (so that $r$ is tangential to the midsurface).

$$V(\omega) = \{(v, s) \in H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3; s \cdot a_3 = 0 \text{ in } \omega \}.$$
First variational problem

Find \((u, r)\) in \(V(\omega)\) such that

\[
\forall (v, s) \in V(\omega), \quad a((u, r); (v, s)) = L((v, s)),
\]

where the bilinear form \(a(\cdot; \cdot)\) is defined by

\[
a((u, r); (v, s)) = \int_\omega \left\{ e\alpha\beta\rho\sigma \left[ \gamma\alpha\beta(u)\gamma\rho\sigma(v) + \frac{e^2}{12} \chi\alpha\beta(u, r)\chi\rho\sigma(v, s) \right] \\
+ 2e \frac{E}{1 + \nu} a\alpha\beta \delta\alpha\beta(u, r)\delta\beta\beta(v, s) \right\} \sqrt{a} \, dx,
\]

and the linear form \(L(\cdot)\) is given by

\[
L((v, s)) = \int_\omega f \cdot v \sqrt{a} \, dx + \int_{\gamma_1} (M \cdot v + N \cdot s) \, d\tau.
\]

The data \(f\), \(M\) and \(N\) represent a given resultant force density, an applied moment density and an applied traction density density, respectively.
For any data \((f, M, N)\) in \(H_{\gamma_0}(\omega)^3 \times H_{\gamma_1}^2(\omega)^3 \times H_{\gamma_0}^1(\omega)^3\), this problem admits a unique solution \((u, r)\) in \(\mathbb{V}(\omega)\).

But, in view of the discretization, a Lagrange multiplier must be introduced to handle the tangency constraint \(r \cdot a_3 = 0\).

Second (mixed) variational problem

\[
X(\omega) = H_{\gamma_0}^1(\omega)^3 \times H_{\gamma_0}^1(\omega)^3, \quad M(\omega) = H_{\gamma_0}^1(\omega).
\]

From now on, we set : \(U = (u, r), V = (v, s)\).
Find \((U, \psi)\) in \(X(\omega) \times M(\omega)\) such that
\[
\forall V \in X(\omega), \quad a(U; V) + b(V; \psi) = L(V),
\]
\[
\forall \chi \in M(\omega), \quad b(U; \chi) = 0,
\]
where the new bilinear form \(b(\cdot; \cdot)\) is defined by
\[
b(V; \chi) = \int_\omega \partial_\alpha (s \cdot a_3) \partial_\alpha \chi \, dx.
\]

The penalized problem

Find \((U, \psi)\) in \(X(\omega) \times M(\omega)\) such that
\[
\forall V \in X(\omega), \quad a(U; V) + b(V; \psi) = L(V),
\]
\[
\forall \chi \in M(\omega), \quad b(U; \chi) = \varepsilon c(\psi, \chi),
\]
where the new bilinear form \(c(\cdot; \cdot)\) is defined by
\[
c(\psi; \chi) = \int_\omega \partial_\alpha \psi \partial_\alpha \chi \, dx.
\]

All these problems are well-posed.
The discrete problem

\((T_h)_h\) : regular family of triangulations of \(\omega\) by triangles such that \(\gamma_0\) is the union of whole edges of elements of \(T_h\).

We thus define the basic discrete space

\[ M_h = \{ \chi_h \in H^1(\omega); \ \forall K \in T_h, \chi_h|_K \in P_1(K) \}, \]

next the spaces that are involved in the discrete problem

\[ M_{\gamma_0} = M_h \cap H^1_{\gamma_0}(\omega), \quad X_h = (M_{\gamma_0}^\gamma)^3 \times (M_{\gamma_0}^\gamma)^3. \]

**Find \((U_h, \psi_h)\) in \(X_h \times M_{\gamma_0}^\gamma\) such that**

\[ \forall V_h \in X_h, \quad a(U_h, V_h) + b(V_h, \psi_h) = \mathcal{L}(V_h), \]

\[ \forall \chi_h \in M_{\gamma_0}^\gamma, \quad b(U_h, \chi_h) = \varepsilon c(\psi_h, \chi_h). \]

**This problem has a unique solution.**
Approximation of the data: We consider an approximation $f_h$ of $f$ in $\mathbb{Z}_h$ and approximations $N_h$ and $M_h$ of $N$ and $M$ in $\mathbb{Z}_h^1$, where the spaces $\mathbb{Z}_h$ and $\mathbb{Z}_h^1$ are defined by

$$\mathbb{Z}_h = \{ g_h \in L^2(\omega)^3; \forall K \in T_h, g_h|_K \in \mathcal{P}_0(K)^3 \},$$

$$\mathbb{Z}_h^1 = \{ P_h \in L^2(\gamma_1)^3; \forall e \in \mathcal{E}_h^1, P_h|_e \in \mathcal{P}_0(e)^3 \}.$$

$\mathcal{E}_h^1$; set of edges of elements of $T_h$ which are contained in $\gamma_1$.

Approximation of the coefficients: We introduce approximations $a_{\alpha\beta}^h$, $a_{\alpha\beta\rho\sigma}^h$, $(\sqrt{a})_h$ and $\ell_h$ of the scalar coefficients $a^{\alpha\beta}$, $a^{\alpha\beta\rho\sigma}$, $\sqrt{a}$ and $\ell$ in the space $M_h$. Similarly, we consider approximations $a_k^h$ of the vectors $a_k$ and $d_\alpha^h$ of the $\partial_\alpha a_3$ in the space $(M_h)^3$. 
We also agree to denote by $\gamma^h_{\alpha\beta}(\cdot)$, $\delta^h_{\alpha3}(\cdot)$ and $\chi^h_{\alpha\beta}(\cdot)$ the coefficients of the tensors introduced above where all coefficients are replaced by their approximations.

**Some new notation:** We define the contravariant components of the stress resultant

$$n^\rho_\sigma(u) = e a^\alpha_\beta^\rho_\sigma \gamma^h_{\alpha\beta}(u),$$

of the stress couple

$$m^\rho_\sigma(U) = \frac{e^3}{12} a^\alpha_\beta^\rho_\sigma \chi_{\alpha\beta}(U),$$

and of the transverse shear force

$$t^{\beta}(U) = e \frac{E}{1 + \nu} a^\alpha_\beta \delta_{\alpha3}(U).$$

We also observe that

$$\chi^\rho_\sigma(V) = \theta^\rho_\sigma(v) + \gamma^\rho_\sigma(s), \quad \text{with} \quad \theta^\rho_\sigma(v) = \frac{1}{2}(\partial^\rho v \cdot \partial^\sigma a_3 + \partial^\sigma v \cdot \partial^\rho a_3).$$
Two families of error indicators

• Error indicator related to the penalty term
  \[ \eta_\varepsilon = \varepsilon |\psi_h|_{H^1(\omega)} \]

• Error indicators related to the finite element discretization
  For each element \( K \) of \( \mathcal{T}_h \),
  \[ \eta_K = \eta_{K1} + \eta_{K2} + \eta_{K3}, \]
  with
  \[
  \eta_{K1} = h_K \| f_h(\sqrt{a})_h + \partial_\rho \left((n^\rho_\sigma(u_h)a_\sigma^h + m^\rho_\sigma(U_h)d_\sigma^h + t^\rho(U_h)a_3^h)(\sqrt{a})_h\right)\|_{L^2(K)}^3 \\
  + \sum_{e \in \mathcal{E}_K \setminus \mathcal{E}_K^1} \frac{1}{h_e^2} \| \left[ \nu_\rho(n^\rho_\sigma(u_h)a_\sigma^h + m^\rho_\sigma(U_h)d_\sigma^h + t^\rho(U_h)a_3^h)(\sqrt{a})_h \right]_e \|_{L^2(e)}^3 \\
  + \sum_{e \in \mathcal{E}_K^1} \frac{1}{h_e^2} \| N_h \ell_h - \nu_\rho(n^\rho_\sigma(u_h)a_\sigma^h + m^\rho_\sigma(U_h)d_\sigma^h + t^\rho(U_h)a_3^h)(\sqrt{a})_h \|_{L^2(e)}^3,
  \]
\[
\eta_{K2} = h_K \| \partial_\rho \left( m_h^\rho \sigma (U_h) a_{\sigma h}^h (\sqrt{a})_h - t^\beta_h (U_h) a_{\beta h}^h (\sqrt{a})_h + \partial_\rho \left( a_3^h \partial_\rho \psi_h \right) - d_\rho^h \partial_\rho \psi_h \right) \|_{L^2(K)}^3 \\
+ \sum_{e \in \mathcal{E}_K \setminus \mathcal{E}_K^1} h_{\ell e}^2 \left\| \left[ \nu_\rho m_h^\rho \sigma (U_h) a_{\sigma h}^h (\sqrt{a})_h + \nu_\rho \partial_\rho \psi_h a_3^h \right]_e \right\|_{L^2(e)}^3 \\
+ \sum_{e \in \mathcal{E}_K^1} h_{\ell e}^2 \left\| M_h \ell_h - \nu_\rho m_h^\rho \sigma (U_h) a_{\sigma h}^h (\sqrt{a})_h - \nu_\rho \partial_\rho \psi_h a_3^h \right\|_{L^2(e)}^3,
\]

and

\[
\eta_{K3} = h_K \| \partial_\alpha (\partial ar_h^p \cdot a_{3h} + r_h^p \cdot d_a^h - \varepsilon_p \partial_\alpha \psi_h^p) \|_{L^2(K)} \\
+ \sum_{e \in \mathcal{E}_K \setminus \mathcal{E}_K^1} h_{\ell e}^2 \left\| \left[ \partial_\nu (r_h \cdot a_{3h}^h - \varepsilon_p \psi_h^p) \right]_e \right\|_{L^2(e)} + \sum_{e \in \mathcal{E}_K^1} h_{\ell e}^2 \left\| \partial_\nu (r_h \cdot a_{3h}^h - \varepsilon_p \psi_h^p) \right\|_{L^2(e)}.
\]

\( \mathcal{E}_K \): set of edges of \( K \) not contained in \( \gamma_0 \),  
\( \mathcal{E}_K^1 \): set of edges of \( K \) contained in \( \gamma_1 \).
A posteriori error estimates

Despite the slightly alarming aspect of the indicators, all a posteriori error estimates and upper bounds for these indicators are fully optimal, up to the terms involving the data

\[ \varepsilon_{K}^{(d)} = h_K \| f - f_h \|_{L^2(K)}^3 + \sum_{e \in E_1} h_e^2 \left( \| N - N_h \|_{L^2(e)}^3 + \| M - M_h \|_{L^2(e)}^3 \right), \]

and the coefficients

\[ \varepsilon_{\ell}^{(c)} = \| \sqrt{a} - (\sqrt{a})_h \|_{L^\infty(\omega)} + h^\frac{1}{2} \| \ell - \ell_h \|_{L^\infty(\gamma_1)} + \ldots \]
Key numerical difficulty:

- The main part of the solution \((u_h, r_h - (r_h \cdot a_3) a_3)\) seems fully independent of \(\varepsilon\).
- The error indicator \(\eta_\varepsilon\) is much smaller than the \(\eta_K\).

So, no adaptivity procedure with respect to \(\varepsilon\) seems possible. We only choose an initial \(\varepsilon\) such that

\[
\eta_\varepsilon \leq 10^{-3} \left( \sum_{K \in T_h} \eta_K^2 \right)^{\frac{1}{2}}.
\]
Case of a hyperbolic paraboloid shell clamped on the whole boundary and subjected to a uniform pressure.

\[ \omega = \{(x, y); |x| + |y| \leq b\sqrt{2}\}, \quad \varphi(x, y) = \left(x, y, \frac{c}{2b^2}(x^2 - y^2)\right)^T, \]

with \( b = 50 \text{ cm}, \ c = 10 \text{ cm}, \) and \( e = 0.8 \text{ cm}. \)

The “over-deformed” shell \( \varphi(x) + 1000u(x) \)
Thank you for your attention