# *A Synthesis of A Posteriori Error Estimation Techniques for Conforming, Non-Conforming, Mixed and Discontinuous FEM*

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Joint work with Richard Rankin

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# *Model Problem*

Consider

$$-\operatorname{div}(A \operatorname{grad} u) = f \text{ in } \Omega \quad \text{(Polygonal Domain)}$$

subject to

$$u = g_D \text{ on } \Gamma_D; \quad \boldsymbol{n} \cdot A \operatorname{grad} u = g_N \text{ on } \Gamma_N,$$

where $\Gamma_D \cap \Gamma_N = \partial\Omega$ are disjoint.

| | |
|---|---|
| Source Term: | $f \in L_2(\Omega)$; |
| Boundary Flux: | $g_N \in L_2(\Gamma_N)$; |
| Permeability Matrix: | $A \in L_\infty(\Omega; \mathbb{R}^{2\times 2})$symmetric positive definite |

Assume: $A$ piecewise constant on sub-domains

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# *Initial Finite Element Partition*

Initial mesh $\mathcal{P}_0$ consists of

- shape regular triangular elements (locally quasi-uniform);
- matches material interfaces;
- non-empty intersection of distinct elements is single common edge or single common vertex.

... usual assumptions.

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# *Refined Finite Element Partitions*

For $\ell \in \mathbb{N}$, $\mathcal{P}_\ell$ obtained from $\mathcal{P}_{\ell-1}$ by

- subdividing a marked set of elements $K$ into four congruent sub-triangles.

... generates *hanging nodes*.

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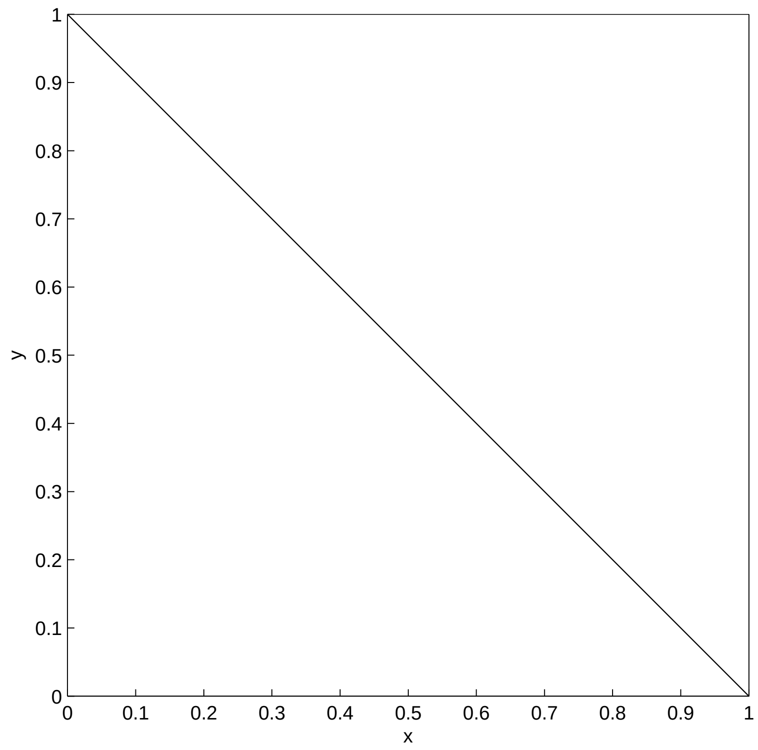
# *Finite Element Partitions—Typical Example*



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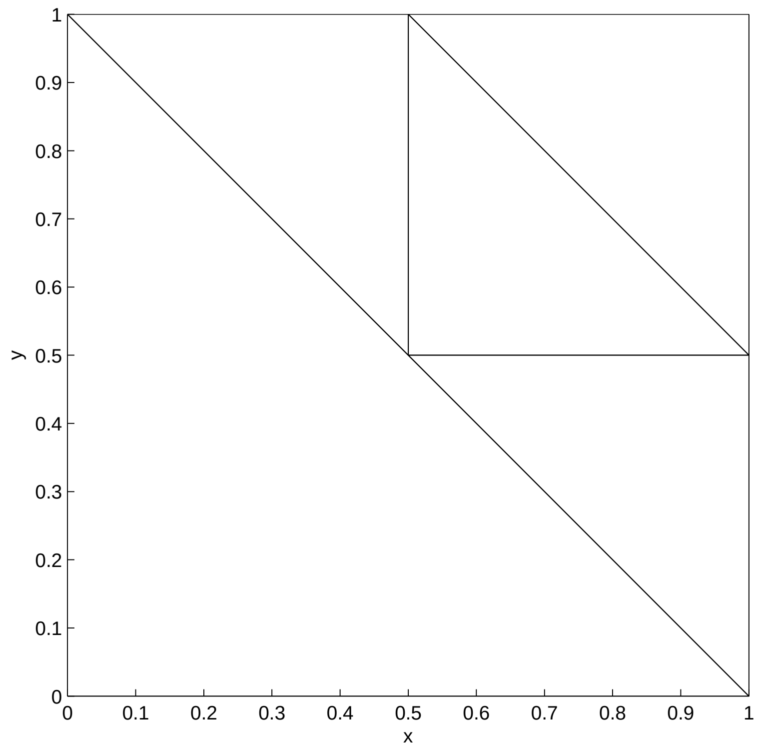


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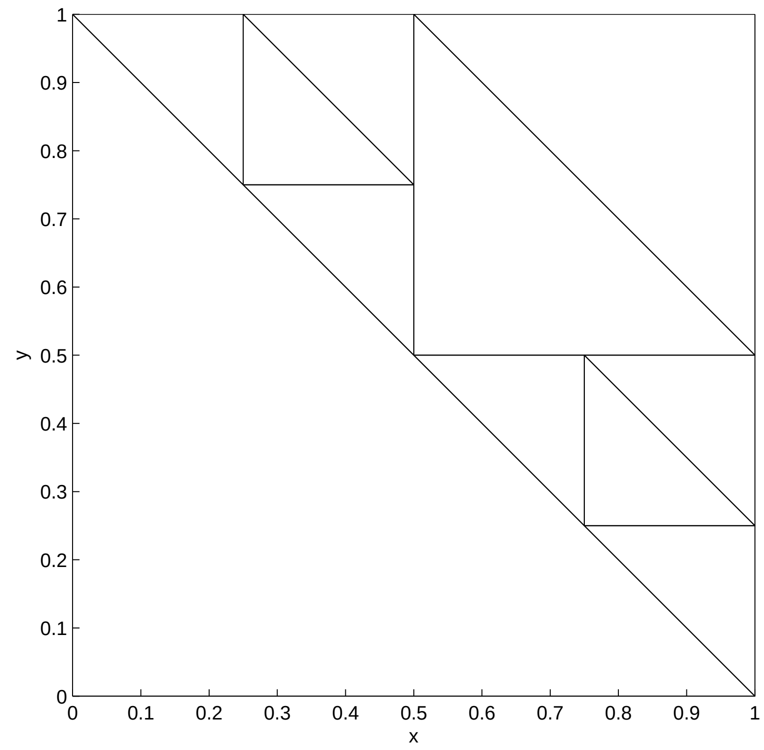

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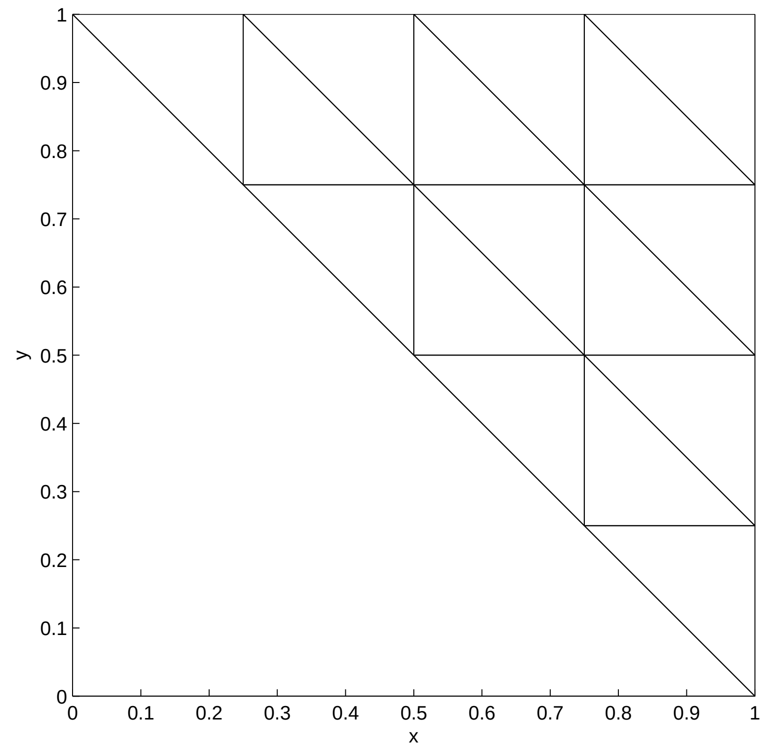


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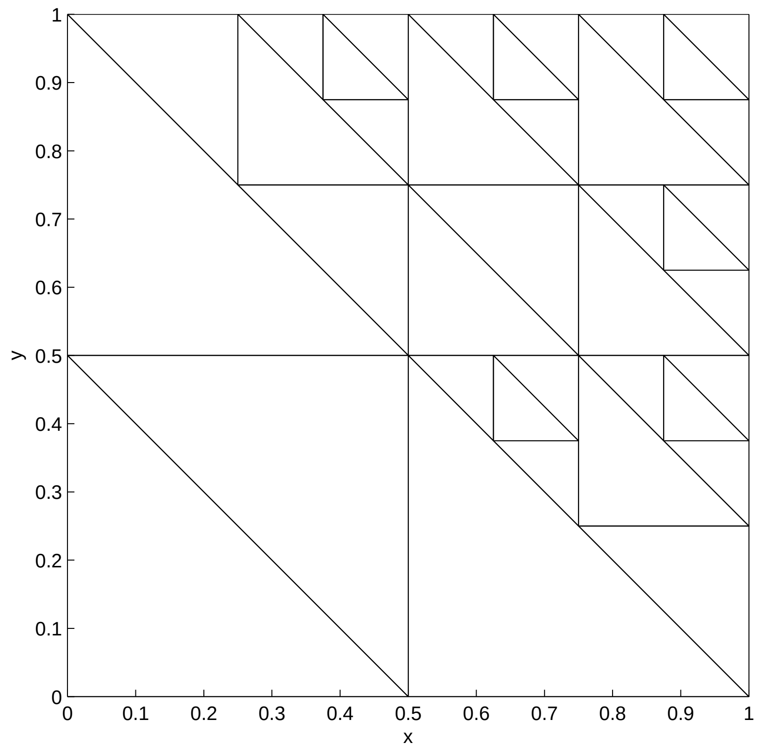

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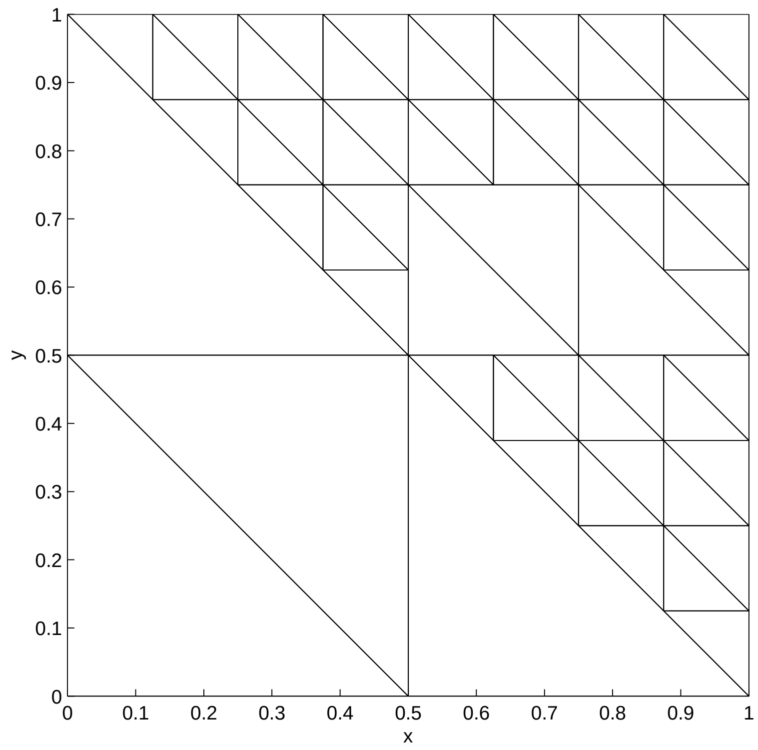


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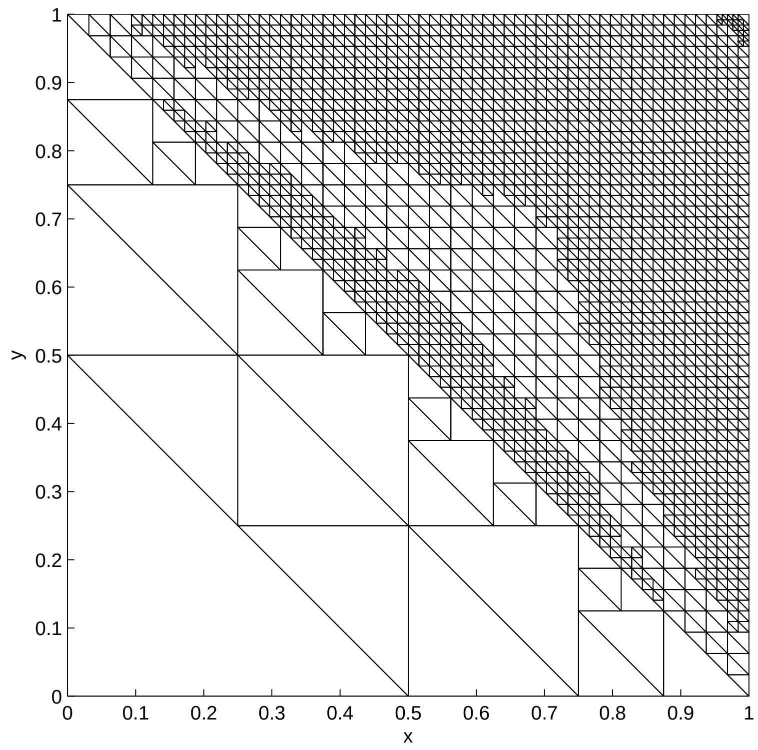
# *Finite Element Partitions—Typical Example*

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# *Discontinuous Galerkin FE Approximation*

Let $\mathcal{P}$ denote a particular mesh $\mathcal{P}_\ell$ for $\ell \in \mathbb{N}$, and denote

$$X_{\mathcal{P}} = \{v \in H^1(\mathcal{P}) : v_{|K} \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{P}\}.$$

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# *Discontinuous Galerkin FE Approximation*

For fixed $\tau \in [-1,1]$, define bilinear form on $\mathcal{B}_{\mathcal{P}_\tau} : X_{\mathcal{P}} \times X_{\mathcal{P}} \to \mathbb{R}$ by

$$\mathcal{B}_\tau(v,w) = \sum_{K\in\mathcal{P}} (a\,\mathbf{grad}_{\mathcal{P}}\, v, \mathbf{grad}_{\mathcal{P}}\, w)_K$$

$$- \sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D} \int_\gamma \left(\langle\sigma_\nu(v)\rangle\,[w] - \tau\,[v]\,\langle\sigma_\nu(w)\rangle\right)\,\mathsf{d}s$$

$$+ \sum_{\gamma\in\mathcal{E}_I\cup\mathcal{E}_D} \frac{\kappa}{h_\gamma}\int_\gamma [v]\,[w]\,\mathsf{d}s$$

where $\langle\sigma_\nu(v)\rangle$ is average value of normal derivative, and $[v]$ is the value of jump on element boundary.

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# *Discontinuous Galerkin FE Approximation*

Define linear form $\mathfrak{L}_\tau : X_{\mathcal{P}} \to \mathbb{R}$ by

$$\mathfrak{L}_\tau(w) = \sum_{K\in\mathcal{P}} (f, w_K)_K + \sum_{\gamma\in\mathcal{E}_N} \int_\gamma g_N w \, \mathsf{d}s$$

$$-\tau \sum_{\gamma\in\mathcal{E}_D} \int_\gamma g_D \left\langle \sigma_\nu(w) \right\rangle \, \mathsf{d}s$$

$$+ \sum_{\gamma\in\mathcal{E}_D} \frac{\kappa}{h_\gamma} \int_\gamma g_D w \, \mathsf{d}s.$$

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# *Discontinuous Galerkin FE Approximation*

Seek $U_{\mathcal{P}} \in X_{\mathcal{P}}$:

$$\mathcal{B}_{\tau}(U_{\mathcal{P}}, v) = \mathcal{L}_{\tau}(v) \quad \forall v \in X_{\mathcal{P}}$$

Special Cases:

- $\tau = 1$ : Symmetric Interior Penalty Galerkin (Arnold, Wheeler, ...)
- $\tau = -1$ : Non-symmetric (Babuška, Oden and Baumann)
- $\tau = 0$ : Incomplete (Girault, Wheeler, ...)

# *Choice of $\kappa$*

Let $\boldsymbol{S}_K$ denote element stiffness matrix with entries

$$[\boldsymbol{S}_K]_{jk} = \int_K (\operatorname{grad} \lambda_k)^\top \boldsymbol{A} (\operatorname{grad} \lambda_j) \; \mathsf{d}\boldsymbol{x}$$

where $\{\lambda_j\}_{j=1}^3$ denote barycentric coordinates on $K$.

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# *Choice of $\kappa$*

**Theorem 2** *(MA & Rankin, 2008)*
*Let $\rho(\boldsymbol{S}_K)$ denote spectral radius of $\boldsymbol{S}_K$. If $\kappa > (1+\tau)^2 \max_{K\in\mathcal{P}} \rho(\boldsymbol{S}_K)$, then there exists a unique discontinuous Galerkin FE approximation.*

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# *Choice of $\kappa$*

**Theorem 3** *(MA & Rankin, 2008)*
*Let $\rho(\boldsymbol{S}_K)$ denote spectral radius of $\boldsymbol{S}_K$. If $\kappa > (1+\tau)^2 \max_{K\in\mathcal{P}} \rho(\boldsymbol{S}_K)$, then there exists a unique discontinuous Galerkin FE approximation.*

- fully explicit, computable bound on value of interior penalty parameter;
- bound *independent* of number of levels of hanging nodes;
- improves on bound obtained by Shabazzi (2005);
- different bound obtained by Epshteyn and Rivière (2007) ... sometimes better, sometimes worse.

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# *'Choose* $\kappa = 10$*'*

Often hear advice to 'choose $\kappa = 10$' to ensure that SIPG is stable.

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# *'Choose* $\kappa = 10$*'*

Let $\boldsymbol{A}_{|K} = a\boldsymbol{I}$, and

$$Q = \frac{1}{4|K|} \sum_{\gamma \subset \partial K} |h_\gamma|^2$$

then $Q \geq \sqrt{3}$, and spectral radius given by

$$\rho(\boldsymbol{S}_K) = \frac{1}{2} a \left( Q + \sqrt{Q^2 - 3} \right).$$

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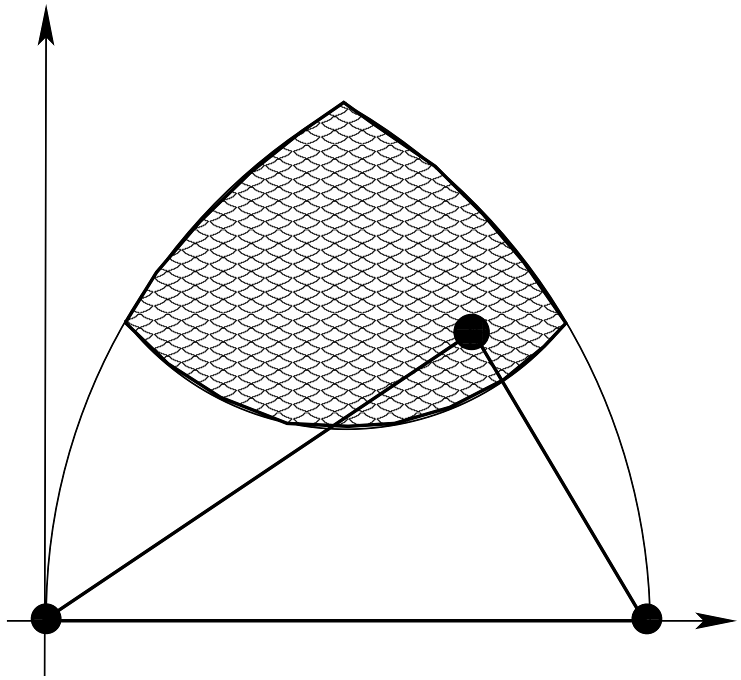
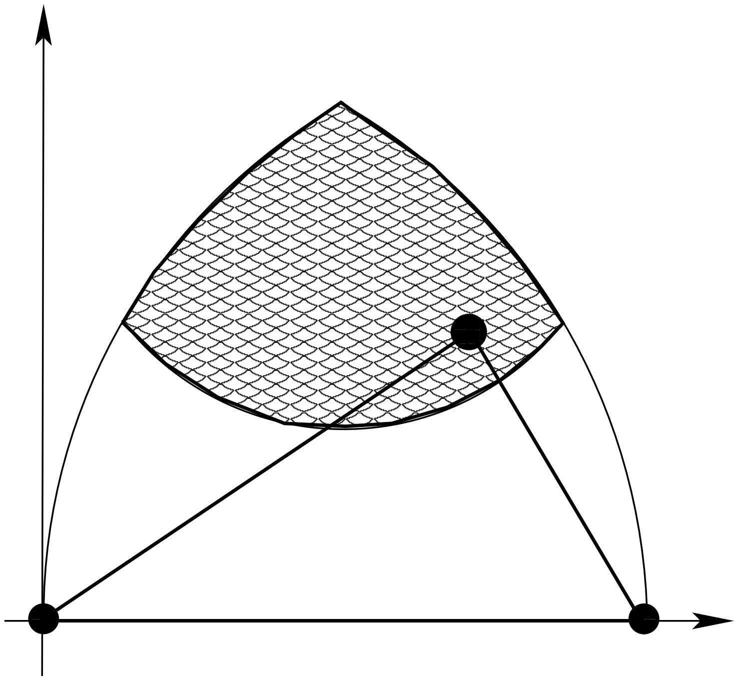
# ‘Choose $\kappa = 10$’

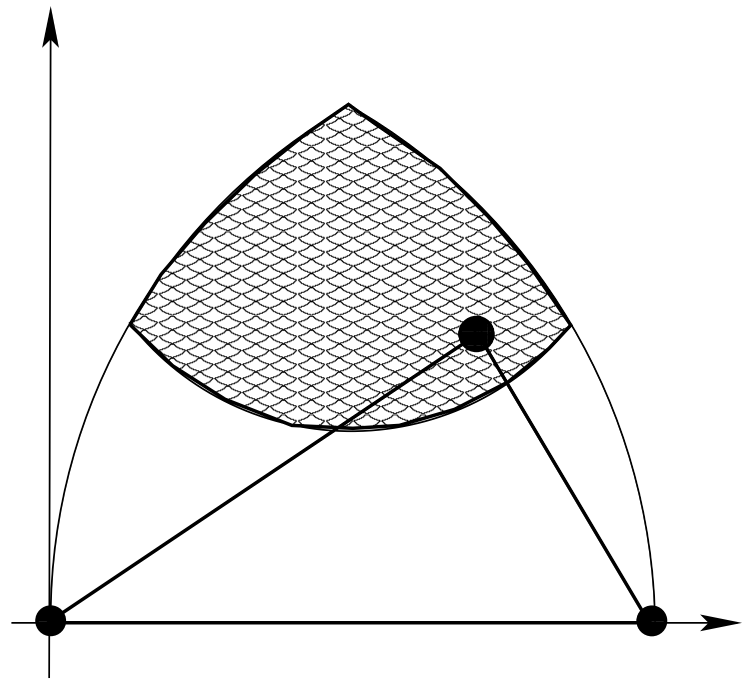


# *'Choose* $\kappa = 10$*'*



For SIPG, probability that $\kappa = 10$ is large enough for *random* triangle is 48%.

# *'Choose* $\kappa = 10$'



For SIPG, probability that $\kappa = 10$ is large enough for *random* triangle is 48%.

Alternatively, if ratio $h/\rho \leq 3.1$, then $\kappa = 10$ will be stable for SIPG.

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# *A Posteriori Error Bounds*

# *A Posteriori Error Estimation*

Aim–To derive computable upper bound for error $e = u - u_{\mathsf{nc}}$ in energy norm

$$\|e\|^2 = \sum_{K \in \mathcal{P}} \int_K (\mathbf{grad}_{\mathcal{P}}\, e)^\top A\, \mathbf{grad}_{\mathcal{P}}\, e$$

and/or DG-Norm

$$\|e\|_{DG}^2 = \|e\|^2 + \sum_{\gamma} \frac{\kappa}{h_\gamma} \int_\gamma [e]^2 \, \mathsf{d}s$$

such that

- all constants should be given in upper bound;
- obtain local lower bounds;
- cost is practically negligible compared with cost of obtaining DG approximation.

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# *Decomposition of Error*

Error in flux may be split as

$$\boldsymbol{\sigma}_{\mathcal{P}}(e) = a\,\mathbf{grad}_{\mathcal{P}}\,e = \boldsymbol{\sigma}(\chi) + \operatorname{curl}\psi$$

where *conforming error* $\chi \in H^1_E(\Omega)$:

$$(a\,\operatorname{grad}\chi, \operatorname{grad}v) = (a\,\mathbf{grad}_{\mathcal{P}}\,e, \operatorname{grad}v) \quad \forall v \in H^1_E(\Omega)$$

and *non-conforming error* $\psi \in \mathcal{H}$:

$$(a^{-1}\operatorname{curl}\psi, \operatorname{curl}w) = (a^{-1}\boldsymbol{\sigma}_{\mathcal{P}}(e), \operatorname{curl}w) = (\mathbf{grad}_{\mathcal{P}}\,e, \operatorname{curl}w) \quad \forall w \in \mathcal{H}.$$

Orthogonal in broken energy norm

$$|||v|||^2 = \sum_{K\in\mathcal{P}} \|a^{1/2}\,\mathbf{grad}_{\mathcal{P}}\,v\|_K^2$$

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# *Estimation of Conforming Error*

# *Estimation of Conforming Error*

Want: Exploit local conservation property of DGFEM. i.e.

$$\int_{\partial K} g_K \, \mathsf{d}s + \int_K f \, \mathsf{d}\boldsymbol{x} = 0$$

where fluxes $g_K \in L_2(\partial K)$ given by

$$g_{K|\gamma} = \begin{cases} \mu_K \left( \langle \sigma_\nu(U_{\mathcal{P}}) \rangle - \kappa h_\gamma^{-1} \left[ U_{\mathcal{P}} \right] \right) & \text{on } \gamma \in \mathcal{E}_I(K) \\ \sigma_\nu(U_{\mathcal{P}}) - \kappa h_\gamma^{-1} (U_{\mathcal{P}} - g_D) & \text{on } \gamma \in \mathcal{E}_D(K) \\ g_N & \text{on } \gamma \in \mathcal{E}_N(K). \end{cases}$$

Same property holds for averaged flux.

# *Estimation of Conforming Error*

Let $v \in H^1_E(\Omega)$ be given. Then,

$$
\begin{aligned}
&(a \operatorname{grad} \chi, \operatorname{grad} v) \\
&\quad = \quad (a\, \mathbf{grad}_{\mathcal{P}}\, e, \operatorname{grad} v) \\
&\quad = \quad (f, v) + \int_{\Gamma_N} g_N v - (a\, \mathbf{grad}_{\mathcal{P}}\, U_{\mathcal{P}}, \operatorname{grad} v)
\end{aligned}
$$

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# Estimation of Conforming Error

Let $v \in H^1_E(\Omega)$ be given. Then,

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\end{aligned}$$

Now localise to elements using DG-flux $g_K$:

$$\begin{aligned}
&(a \operatorname{grad} \chi, \operatorname{grad} v) \\
&\quad = \quad \sum_{K \in \mathcal{P}} \left\{ (f, v)_K + \int_{\partial K} g_K v - (a\, \mathbf{grad}_{\mathcal{P}}\, U_{\mathcal{P}}, \operatorname{grad} v)_K \right\}
\end{aligned}$$

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# *Estimation of Conforming Error*

Suppose (see later) can find $\boldsymbol{\sigma}_K$ such that for all $v \in H^1_E(K)$

$$(\boldsymbol{\sigma}_K, \mathrm{grad}\, v)_K = (f, v)_K + \int_{\partial K} g_K v - (a\, \mathbf{grad}_{\mathcal{P}}\, U_{\mathcal{P}}, \mathrm{grad}\, v)_K.$$

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Then

$$(a \operatorname{grad} \chi, \operatorname{grad} v) = \sum_{K \in \mathcal{P}} (\boldsymbol{\sigma}_K, \operatorname{grad} v)_K$$

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Then

$$(a \operatorname{grad} \chi, \operatorname{grad} v) = \sum_{K \in \mathcal{P}} (\boldsymbol{\sigma}_K, \operatorname{grad} v)_K$$

Apply Cauchy-Schwarz and simplify to obtain

$$|||\chi|||^2 \leq \sum_{K \in \mathcal{P}} (A^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K).$$

Only the *value* of norm of $\boldsymbol{\sigma}_K$ matters, *not* $\boldsymbol{\sigma}_K$ *per se*.

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# *How to construct $\boldsymbol{\sigma}_K$?*

Want: $\boldsymbol{\sigma}_K$ such that for all $v \in H^1_E(K)$

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- local conservation property of DG means that data satisfies compatibility condition
- assume (remove later) that data $f$ and $g$ are piecewise polynomial

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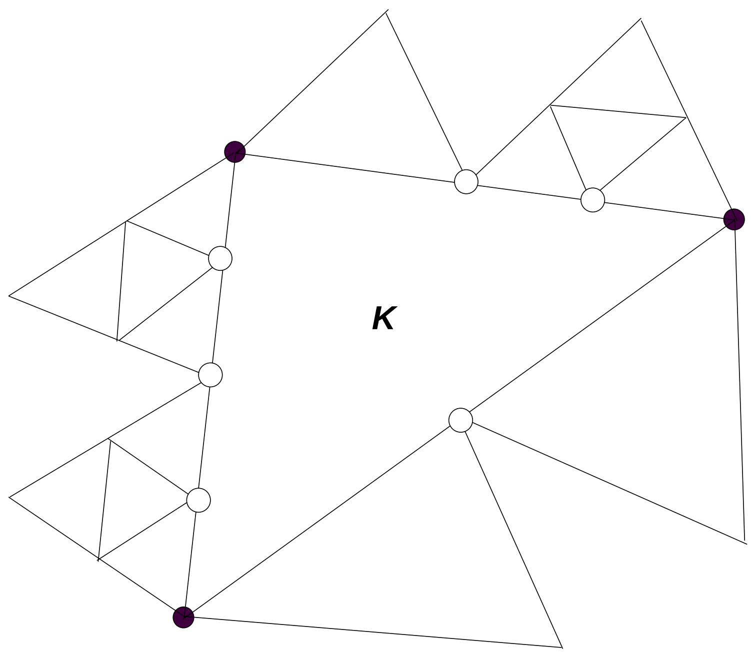
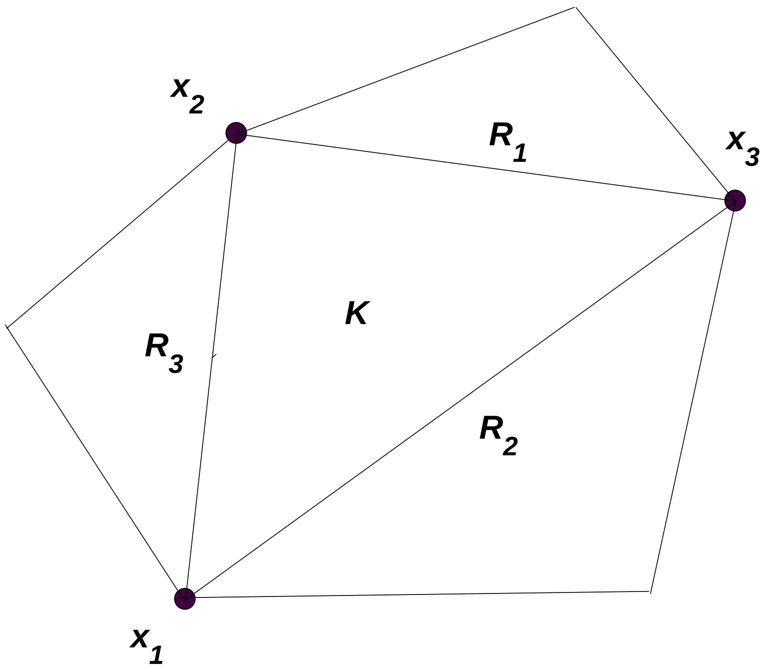



DG-flux $g_K$ is *discontinuous* piecewise polynomial on boundary of element.

# *Special Case: No hanging nodes*

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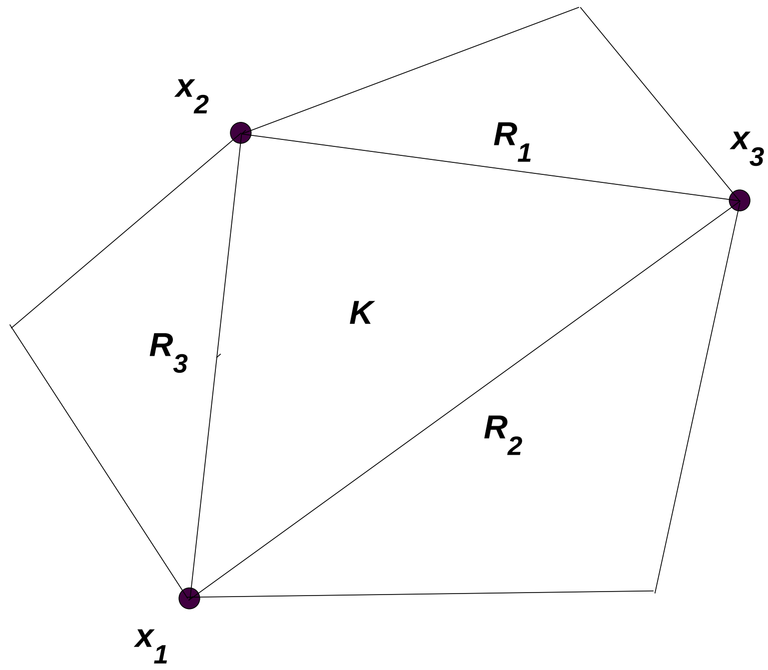


- DG-flux $g_K$ is *continuous* piecewise polynomial on each edge;
- Source term $f$ *continuous* polynomial on element;
- Neumann data $g$ *continuous* polynomial on each edge.

Means we can find a simple formula for norm of $\boldsymbol{\sigma}_K \in RT_0(K)$.

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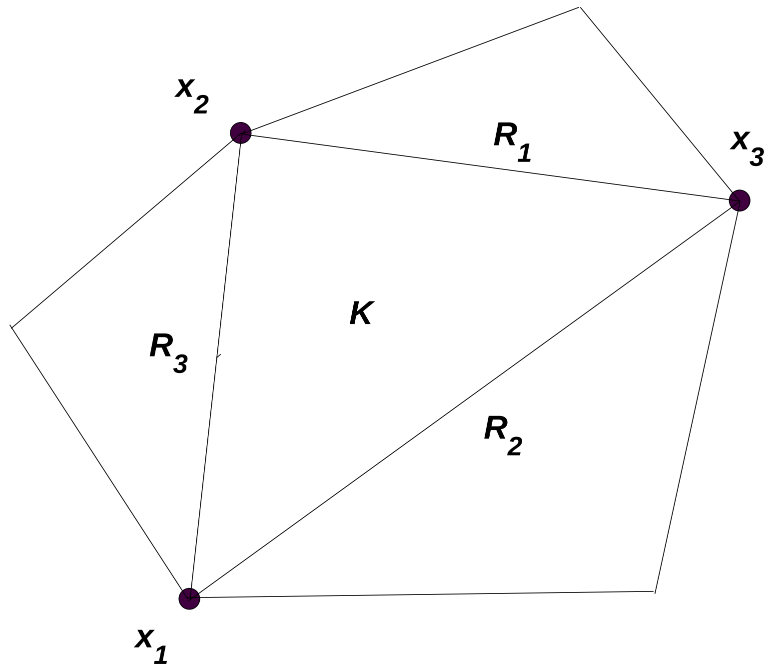
# Special Case: No hanging nodes




$$\boldsymbol{M}_K = \frac{|\gamma_j||\gamma_k|}{4|K|^2} \int_K (\boldsymbol{x} - \boldsymbol{x}_j)^\top \boldsymbol{A}^{-1} (\boldsymbol{x} - \boldsymbol{x}_k) \, \mathsf{d}\boldsymbol{x}$$

and $\vec{R} = (R_1, R_2, R_3)$, then (c.f. MA, SINUM 2006)

$$(A^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K) = \vec{R}^\top \boldsymbol{M}_K \vec{R}$$

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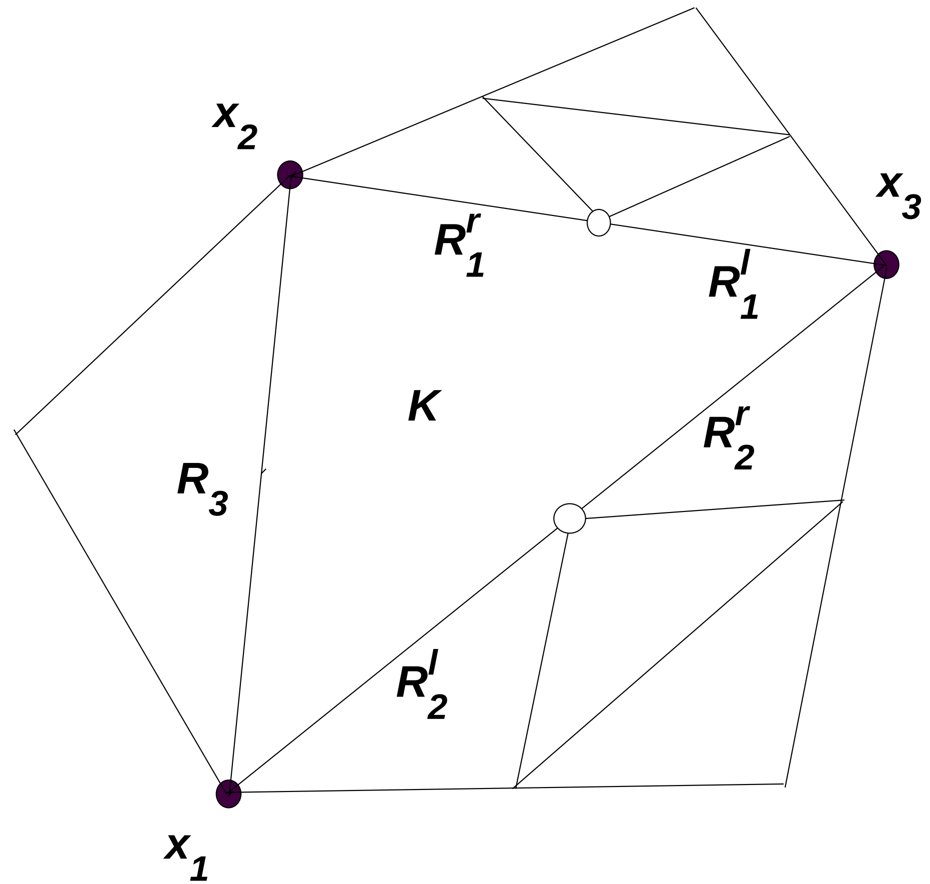
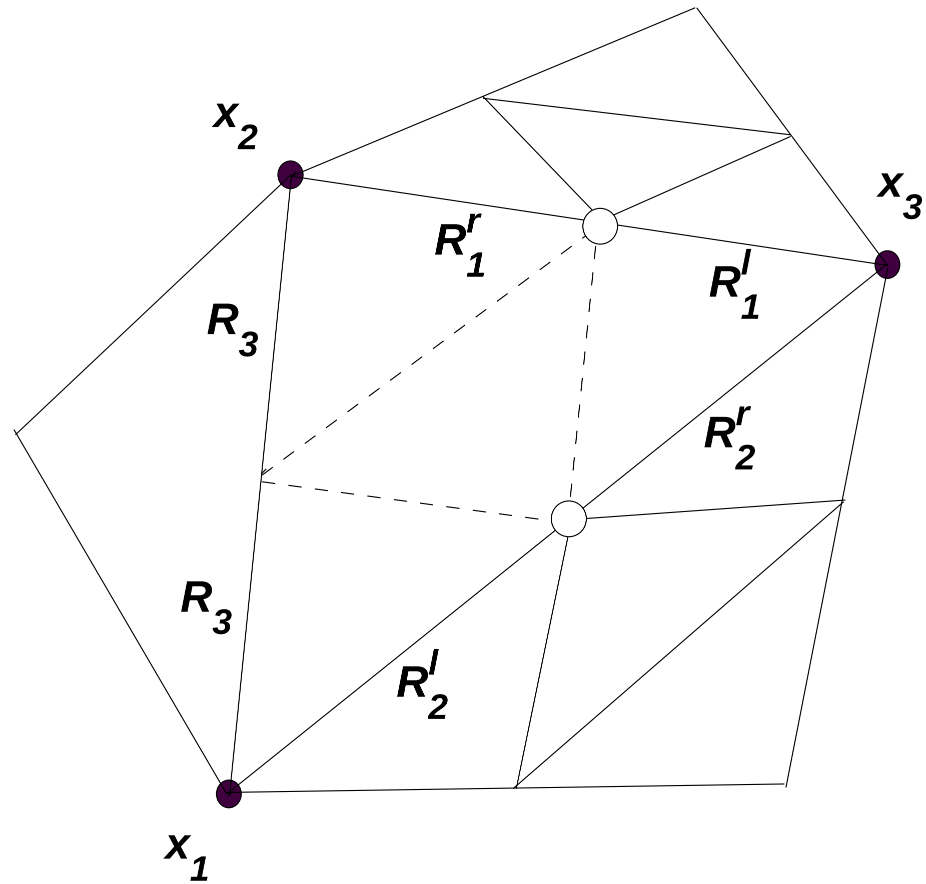
# *Special Case: One hanging node per edge.*




Boundary terms *discontinuous* on element edges.

# *Special Case: One hanging node per edge.*




Introduce virtual refinement into four congruent sub-triangles.

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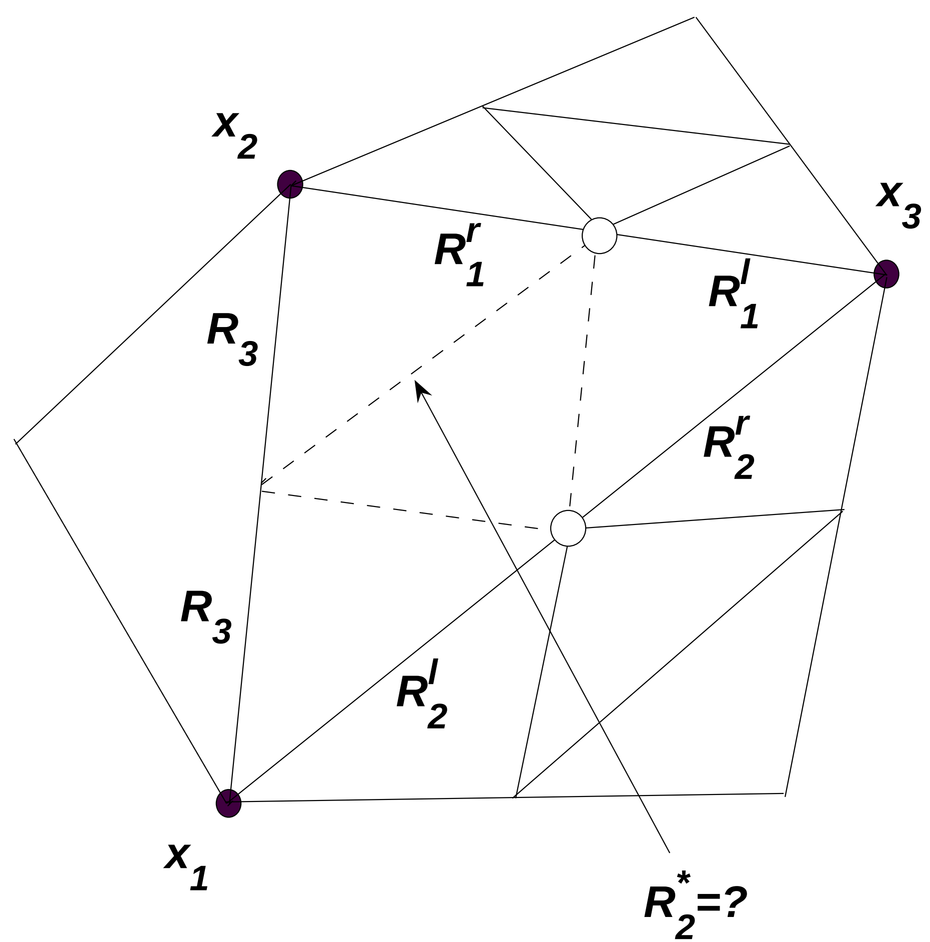
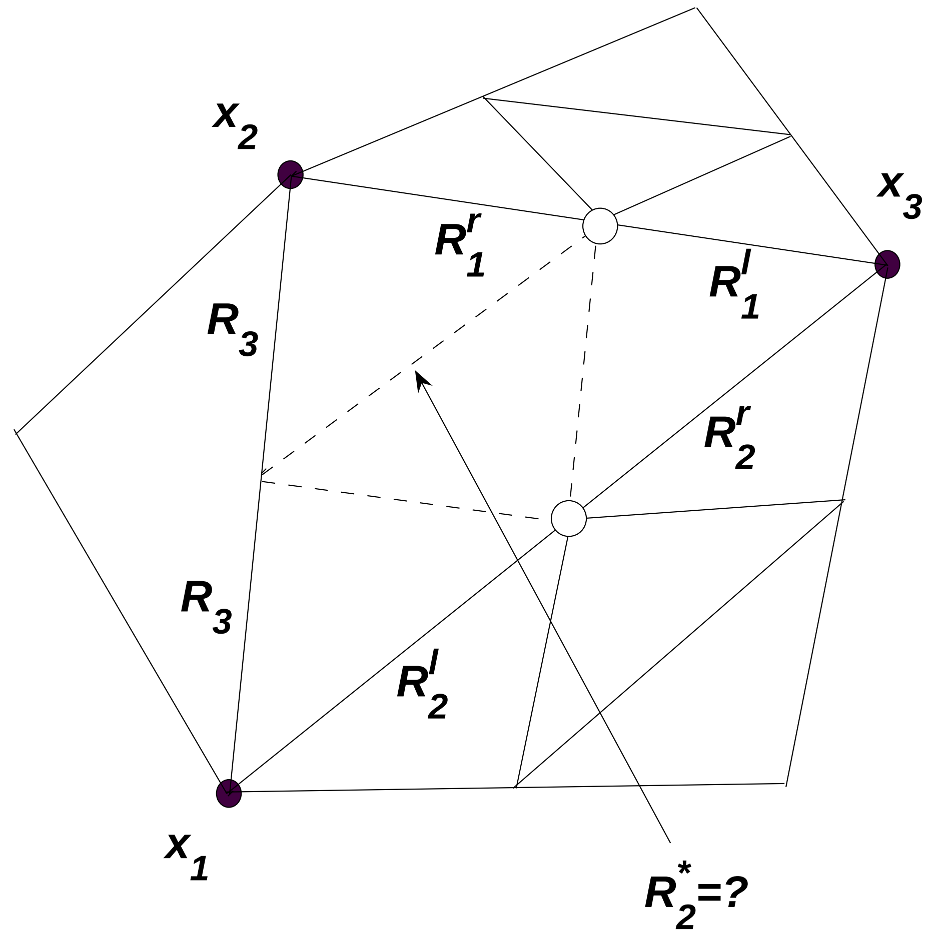



How to choose 'residuals' on new internal edges?

# Special Case: One hanging node per edge.




... choose $R_2^*$ so that sub-element has compatible data (i.e. preserves *local* conservation property).

# *Special Case: One hanging node per edge.*

Let $\boldsymbol{M}_K$ denote *same* matrix as before, viz.

$$\boldsymbol{M}_K = \frac{|\gamma_j||\gamma_k|}{4|K|^2} \int_K (\boldsymbol{x} - \boldsymbol{x}_j)^\top \boldsymbol{A}^{-1} (\boldsymbol{x} - \boldsymbol{x}_k) \, \mathsf{d}\boldsymbol{x}$$

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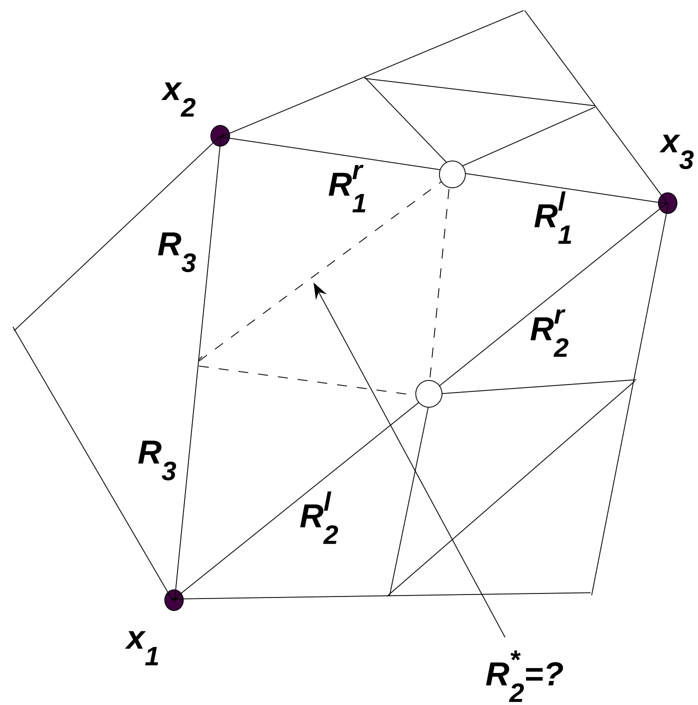


Define: $\vec{R}_2 = (R_1^r, R_2^*, R_3)$

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Let $\vec{R}_1, \ldots, \vec{R}_4$ denote vectors in $\mathbb{R}^3$ formed from residuals on boundaries of virtual elements, then

$$(A^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K) = \frac{1}{4} \sum_{k=1}^{4} \vec{R}_k^\top \boldsymbol{M}_K \vec{R}_k.$$

... essentially for free.

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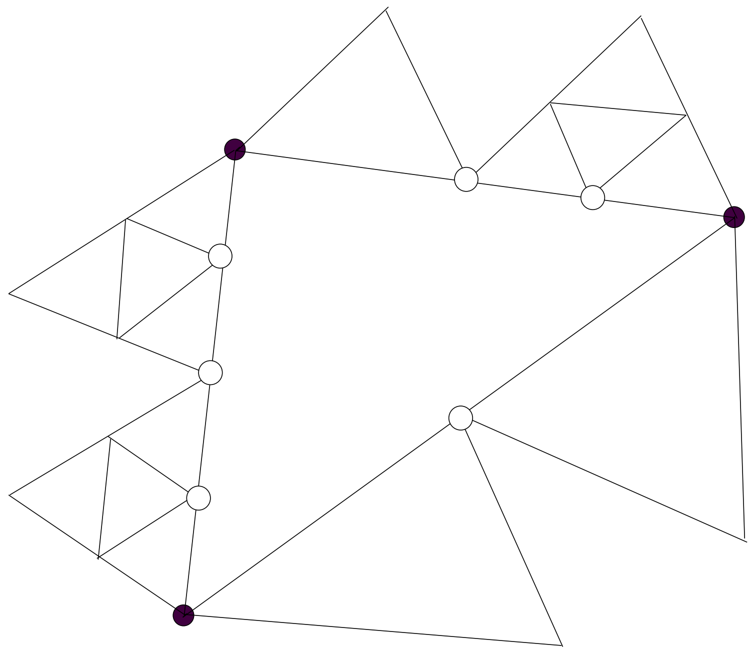
# *General Case: Arbitrary number of hanging nodes*



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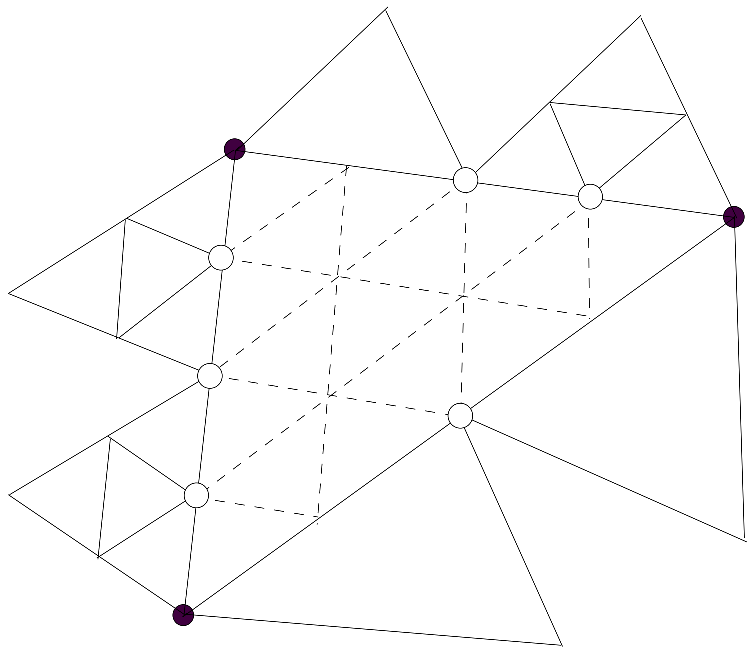

Introduce *virtual refinements* as before.

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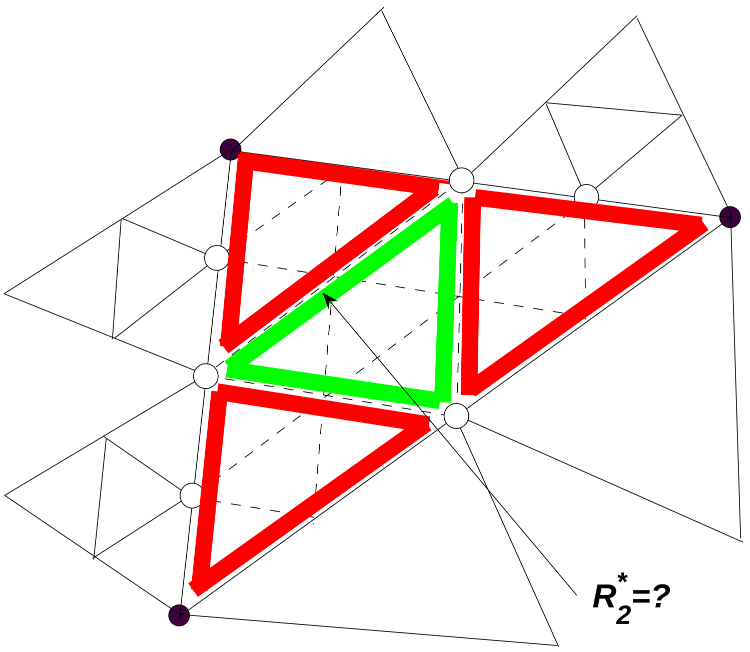


Decompose into four sub-domains with *one fewer edge node*.

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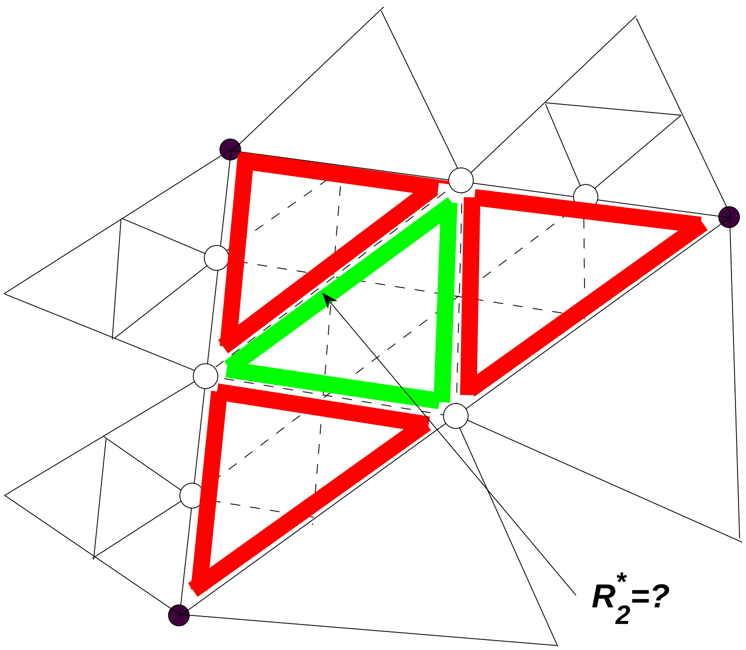


Decompose into four sub-domains with *one fewer edge node*.

Proceed *recursively* on each sub-domain to reduce to situation of no edge nodes.

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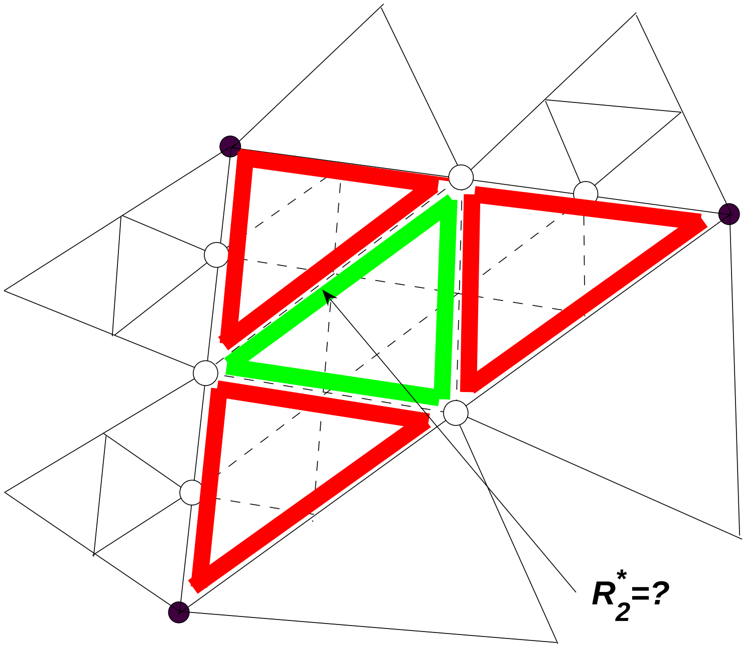


Decompose into four sub-domains with *one fewer edge node*.

Proceed *recursively* on each sub-domain to reduce to situation of no edge nodes.

Accumulate *norms* of $\boldsymbol{\sigma}_K$ over sub-domains to obtain value over original element ... again, practically for free.

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# *Computable Upper Bound on Conforming Error*

**Theorem 4**

$$\|\!|\chi|\!\|^2 \leq \sum_{K \in \mathcal{P}} \eta^2_{\mathsf{CF},K}$$

*where*

$$\eta^2_{\mathsf{CF},K} = (\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K),$$

*is computed using recursive procedure.*

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# *Computable Upper Bound on Conforming Error*

**Theorem 6** *(MA & Rankin, 2008)*

$$\|\!|\chi|\!\|^2 \leq \sum_{K \in \mathcal{P}} \eta_{\mathsf{CF},K}^2$$

*where*

$$\eta_{\mathsf{CF},K}^2 = (\boldsymbol{A}^{-1}\boldsymbol{\sigma}_K, \boldsymbol{\sigma}_K),$$

*is computed using recursive procedure.*

- *Also local lower bound up to generic constant (depends on number of levels of hanging nodes).*
- *Case of non-polynomial data $f$ and $g$ introduces usual* oscillation *terms (we give* all *multiplicative constants explicitly).*

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# *Estimation of Non-Conforming Error*

# *Estimation of the Non-conforming Error*

Pick $u^* \in H^1_E(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$.

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Recall, $\psi \in \mathcal{H} = \{w \in H^1(\Omega)/\mathbb{R} : w \text{ constant on } \Gamma_N\}$ satisfies

$$
\begin{aligned}
&(A^{-1} \operatorname{curl} \psi, \operatorname{curl} w) \\
&\quad = \quad (\mathbf{grad}_{\mathcal{P}}\, e, \operatorname{curl} w) \quad \forall w \in \mathcal{H}
\end{aligned}
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$$\begin{aligned} &(A^{-1}\,\mathrm{curl}\ \psi, \mathrm{curl}\, w) \\ &= (\mathbf{grad}_{\mathcal{P}}\, e, \mathrm{curl}\, w) \quad \forall w \in \mathcal{H} \\ &= (\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathrm{curl}\, w) + (\mathrm{grad}(u - u^*), \mathrm{curl}\, w) \end{aligned}$$

Observe that

$$(\mathrm{grad}(u - u^*), \mathrm{curl}\, w) = \int_{\Gamma_N} (u - u^*) \frac{\partial w}{\partial s} = 0.$$

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&\quad = \quad (\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathrm{curl}\, w) + 0
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$$

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&\quad = (\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathrm{curl}\, w) + 0
\end{aligned}$$

Choose $w = \psi$ and apply Cauchy-Schwarz to get

$$(A^{-1}\,\mathrm{curl}\ \psi, \mathrm{curl}\ \psi) \le (A\,\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}})).$$

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(A^{-1}\,\mathrm{curl}\ \psi, \mathrm{curl}\ \psi) \leq (A\,\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}})).
$$

Equality holds when $u^* = u - \phi$, so

$$
(A^{-1}\,\mathrm{curl}\ \psi, \mathrm{curl}\ \psi) = \min_{u^*}(A\,\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}})).
$$

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# *Estimation of the Non-conforming Error*

Let $u^* \approx u$ be any smooth ($H^1$) approximation.
Obtain computable upper bound for non-conforming error

$$(A^{-1}\,\mathrm{curl}\;\psi, \mathrm{curl}\;\psi) \leq (A\,\mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}}), \mathbf{grad}_{\mathcal{P}}(u^* - U_{\mathcal{P}})).$$

Question: How to choose $u^*$ to obtain good bound?

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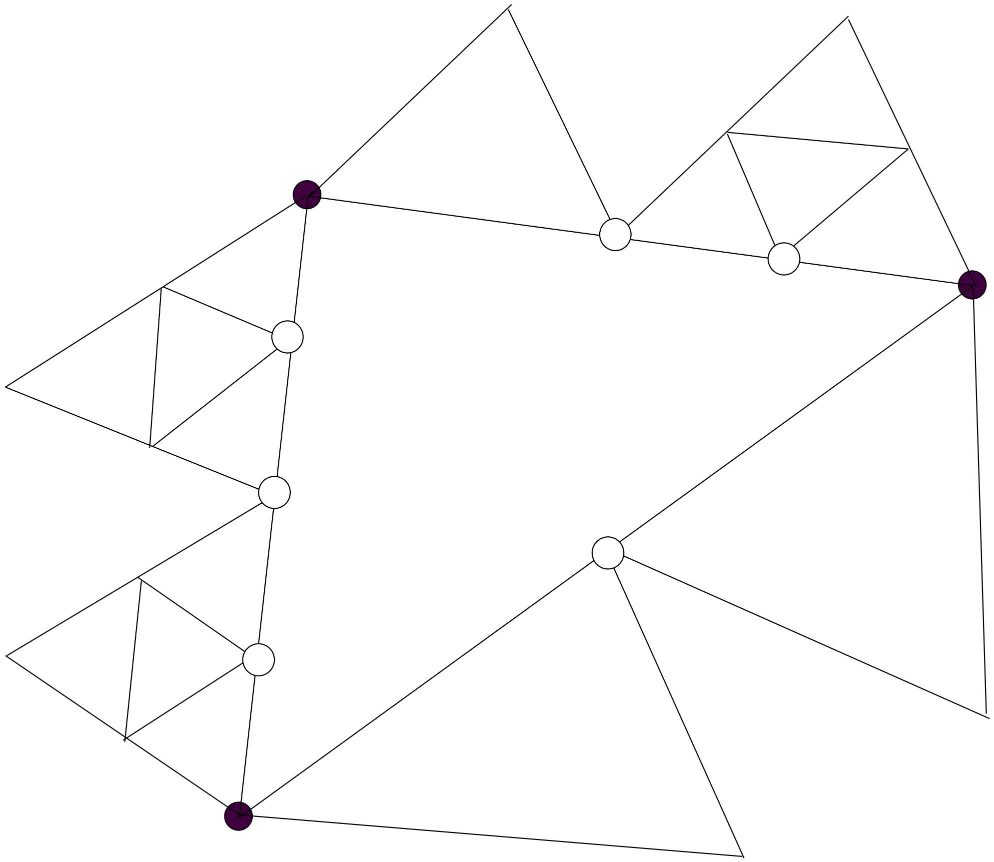
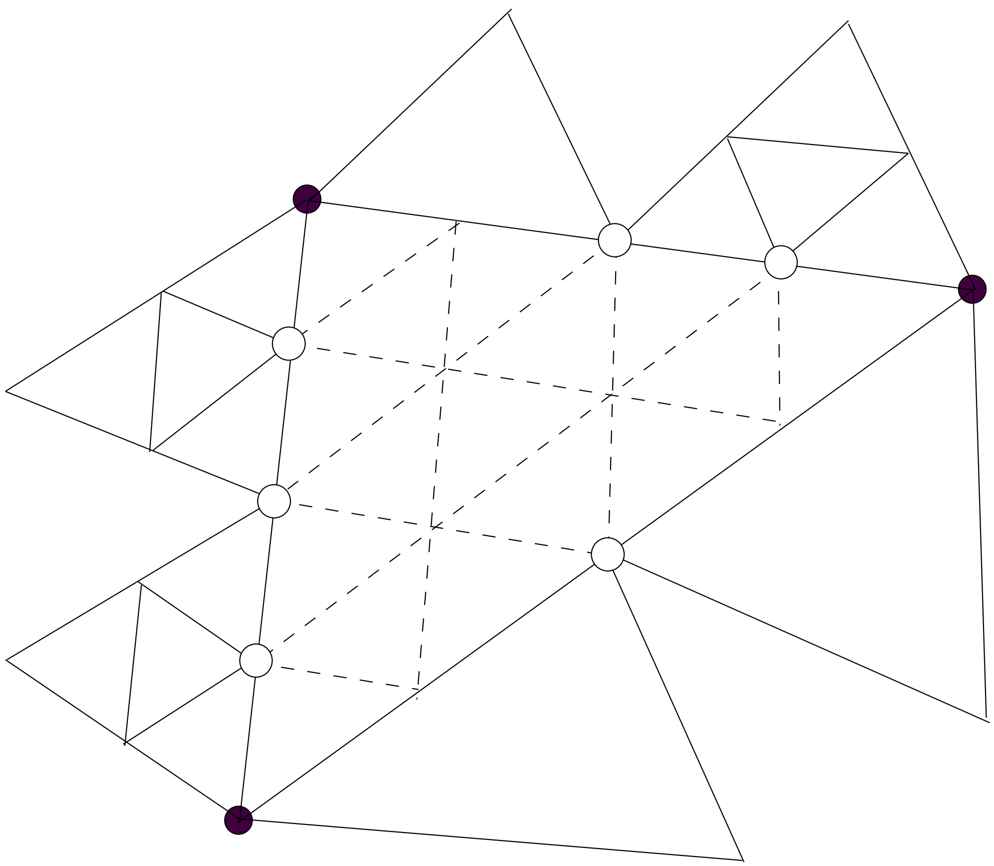
# *Construction of* $u^*$



DG FEM approximation $U_{\mathcal{P}}$ known but *discontinuous*.

# *Construction of* $u^*$



Introduce *virtual* refinement again, and choose $u^*$ to be continuous piecewise linear on *virtual* mesh.

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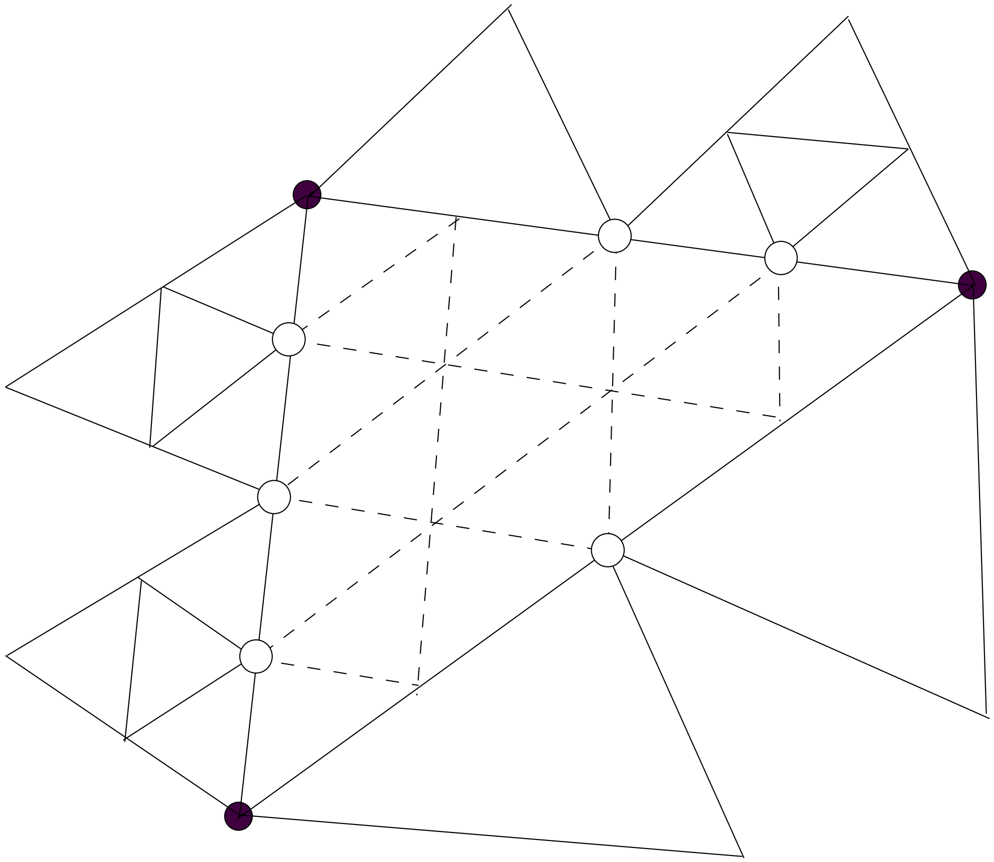
# Construction of $u^*$



Values at regular nodes $\bullet$ and at hanging nodes $\circ$ obtained by averaging values of $U_{\mathcal{P}}$ at node.

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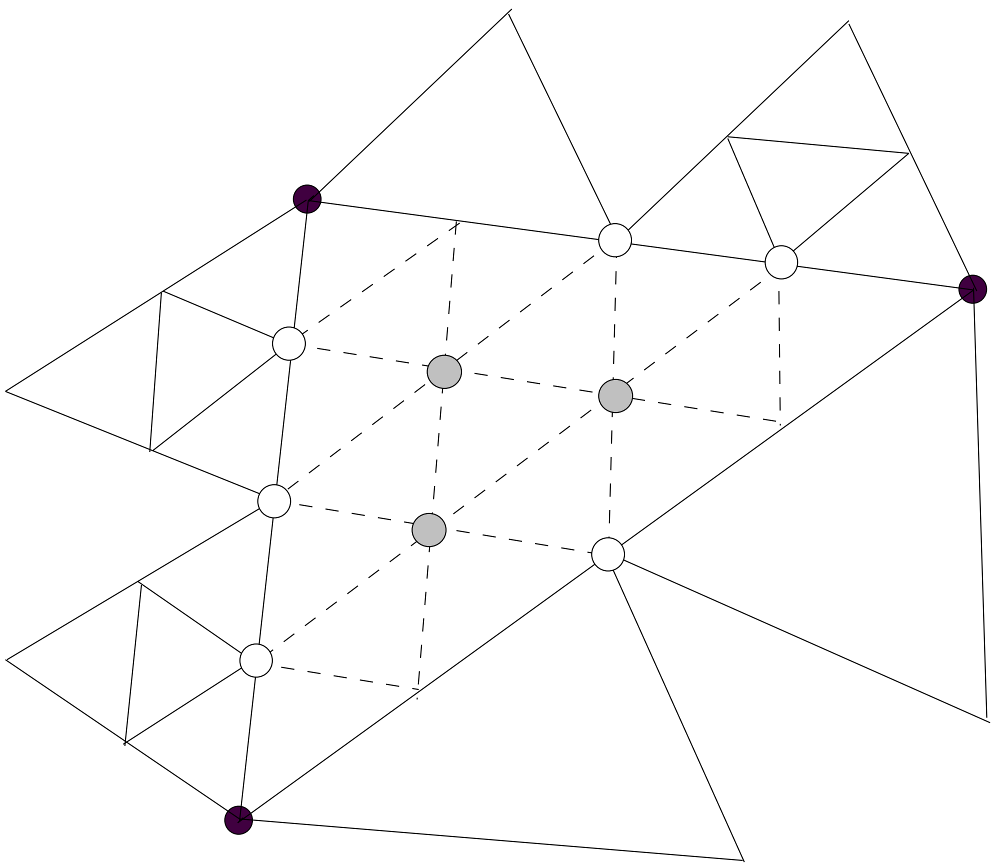
# *Construction of* $u^*$



Values at virtual interior nodes chosen equal to $U_{\mathcal{P}}$ at node.

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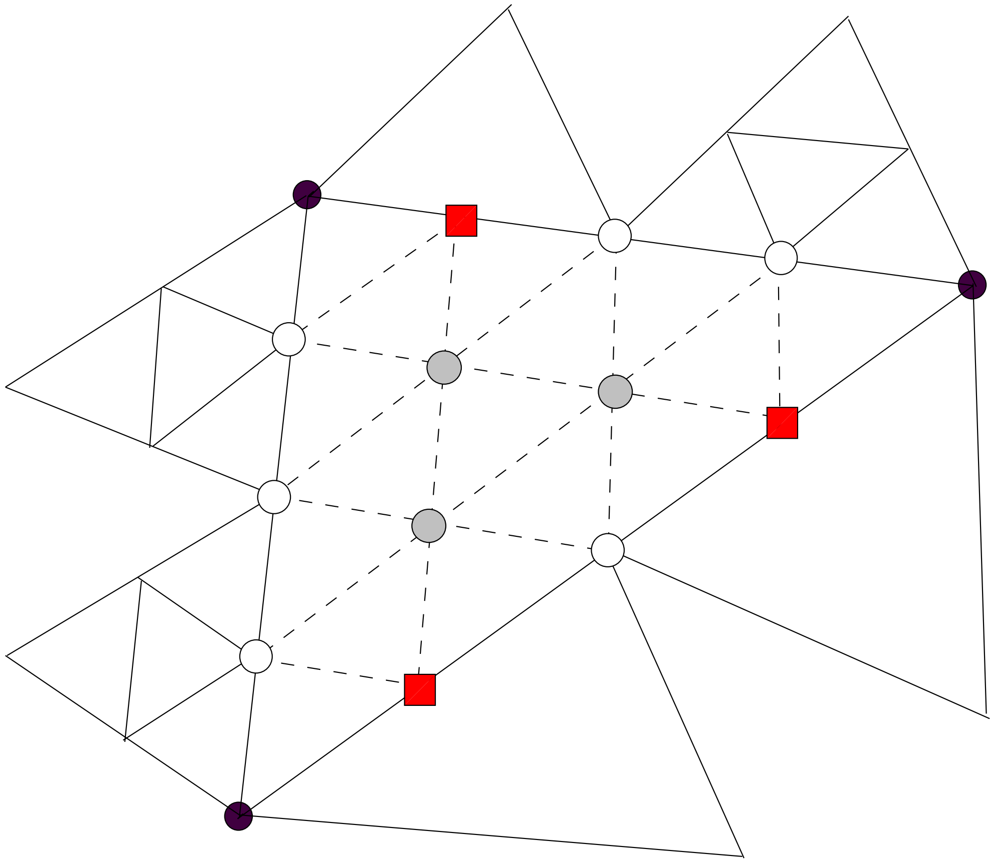
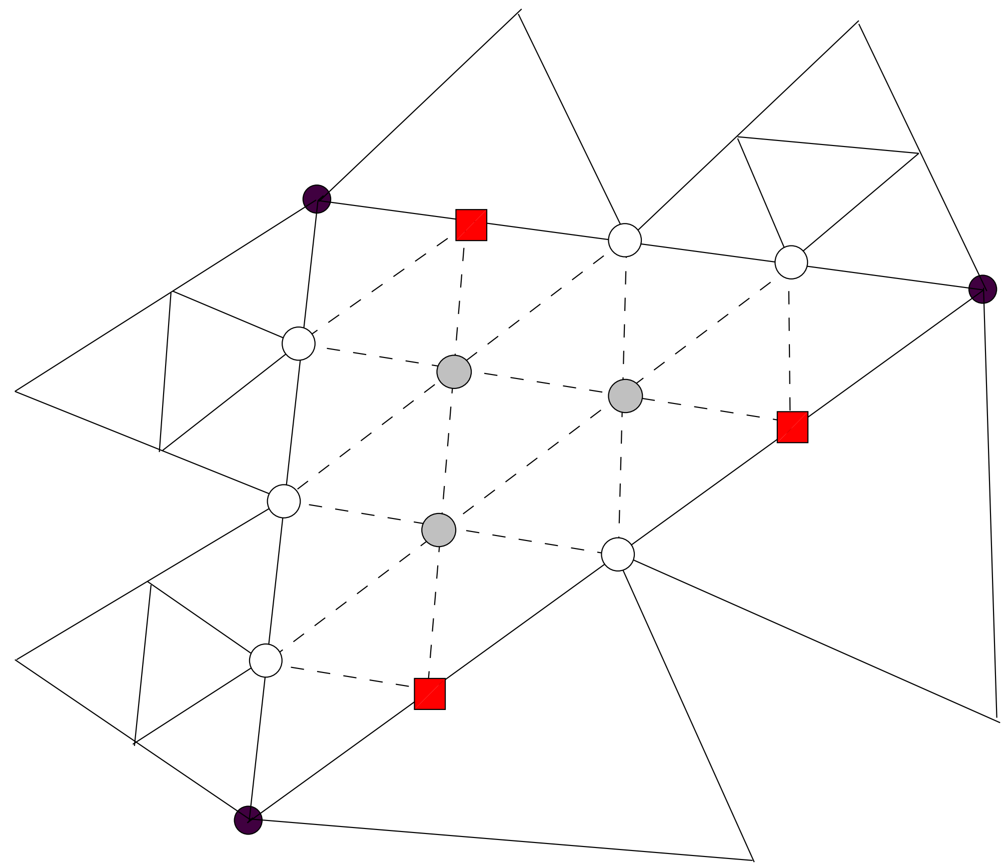
# *Construction of* $u^*$



Values at virtual edge nodes obtained by interpolating values of $u^*$ at two nearest $\bullet$ or $\circ$ nodes.

# *Construction of* $u^*$



Values at virtual edge nodes obtained by interpolating values of $u^*$ at two nearest $\bullet$ or $\circ$ nodes.

On Dirichlet boundary $\Gamma_D$, take $u^*$ equal to Dirichlet data.

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# *Estimation of the Non-conforming Error*

**Theorem 7** *Explicit upper bound for non-conforming error*

$$\|\psi\|^2_{A^{-1}} \leq \sum_{K \in \mathcal{P}} \eta^2_{\mathsf{nc},K}$$

*where*

$$\eta_{\mathsf{nc},K} = \|U_{\mathcal{P}} - u^*\|_K \,.$$

*is computed using recursive procedure based on $\boldsymbol{S}_K$ (local stiffness matrix).*

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# Estimation of the Non-conforming Error

**Theorem 8** *Explicit upper bound for non-conforming error*

$$\|\psi\|^2_{A^{-1}} \leq \sum_{K\in\mathcal{P}} \eta^2_{\mathsf{nc},K}$$

*where*

$$\eta_{\mathsf{nc},K} = \|U_{\mathcal{P}} - u^*\|_K \,.$$

*is computed using recursive procedure based on* $\boldsymbol{S}_K$ *(local stiffness matrix).*

Also

- lower bounds;
- non-homogeneous Dirichlet conditions.

(Full details in MA & Rankin, 2008)

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# *Estimation of Total Error in Energy Norm*

Upper bound on total error

$$|||e|||^2 \leq \sum_{K \in \mathcal{P}} (\eta^2_{\mathsf{CF},K} + \eta^2_{\mathsf{NC},K})$$

# *Estimation of Total Error in Energy Norm*

Upper bound on total error

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Question: Does this really control error?

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# *Estimation of Total Error in Energy Norm*

Upper bound on total error

$$|||e|||^2 \leq \sum_{K \in \mathcal{P}} (\eta^2_{\mathsf{CF},K} + \eta^2_{\mathsf{NC},K})$$

Question: Does this really control error?

$$|||e|||^2 = \sum_{K \in \mathcal{P}} \|a^{1/2}\, \mathbf{grad}_{\mathcal{P}}\, e\|^2_K.$$

i.e. ... broken energy norm does not "see" jumps between elements.
Two possibilities ...

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# Estimation in DG-Energy Norm

First Possibility: Estimate error in DG-Energy Norm

$$|\!|\!|e|\!|\!|_{DG}^2 = |\!|\!|e|\!|\!|^2 + \sum_{\gamma\in\partial\mathcal{P}} \frac{\kappa}{h_\gamma} \|[e]\|_\gamma^2$$

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# *Estimation in DG-Energy Norm*

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$$|||e|||_{DG}^2 = |||e|||^2 + \sum_{\gamma\in\partial\mathcal{P}} \frac{\kappa}{h_\gamma} \|[e]\|_\gamma^2$$

Can use the fact that $[e] = [u] - [U_{\mathcal{P}}] = -[U_{\mathcal{P}}]$ is computable.
Then, we obtain

$$|||e|||_{DG}^2 \leq \sum_{K\in\mathcal{P}} (\eta_{\mathsf{CF},K}^2 + \eta_{\mathsf{NC},K}^2) + \sum_{\gamma\in\partial\mathcal{P}} \frac{\kappa}{h_\gamma} \|[U_{\mathcal{P}}]\|_\gamma^2 \leq C\,|||e|||_{DG}^2 \,.$$

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# *Estimation in DG-Energy Norm*

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$$\|e\|_{DG}^2 = \|e\|^2 + \sum_{\gamma \in \partial \mathcal{P}} \frac{\kappa}{h_\gamma} \|[e]\|_\gamma^2$$

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... but *who cares about DG-norm*?

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# *Estimation of Conforming Error*

Second Possibility: ... observe that estimator for broken energy norm *ALREADY* bounds the jump terms.

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For $\kappa$ *sufficiently large*, there holds

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How large? *SAME* bound needed for DGFEM to be well-posed.

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# *Estimation of Conforming Error*

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How large? *SAME* bound needed for DGFEM to be well-posed.
Hence, obtain two-sided estimator in more natural norm

$$\|a^{1/2}\,\mathbf{grad}_{\mathcal{P}}\, e\|^2 = |\!|\!|e|\!|\!|^2 \le \sum_{K\in\mathcal{P}} (\eta_{\mathsf{CF},K}^2 + \eta_{\mathsf{NC},K}^2) \le C\, |\!|\!|e|\!|\!|^2 .$$

Generalises result from (MA, SINUM 2007) to case where there are hanging nodes (see MA & Rankin, 2008).

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# Numerical Example—Poisson Problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1) \times (0,1) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Data $f$ chosen so that true solution is

$$u(x,y) = \begin{cases} (1-x-y)^2(1-x)(1-y) & \text{if } x+y > 1 \\ 0 & \text{if } x+y \le 1 \end{cases}$$

on $\Omega = (0,1) \times (0,1)$.

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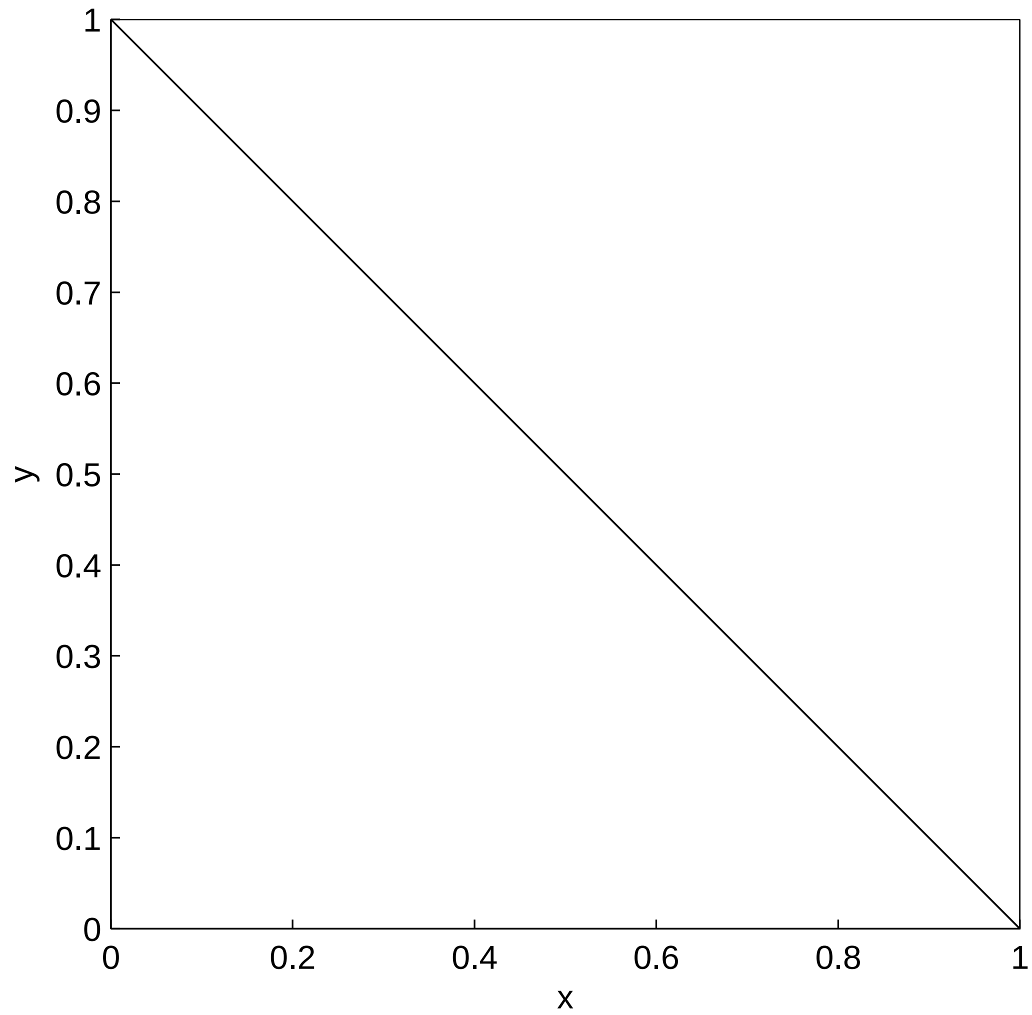
# SIPDG ($\tau = 1$; 2-levels of hanging nodes; $\kappa = 10$)

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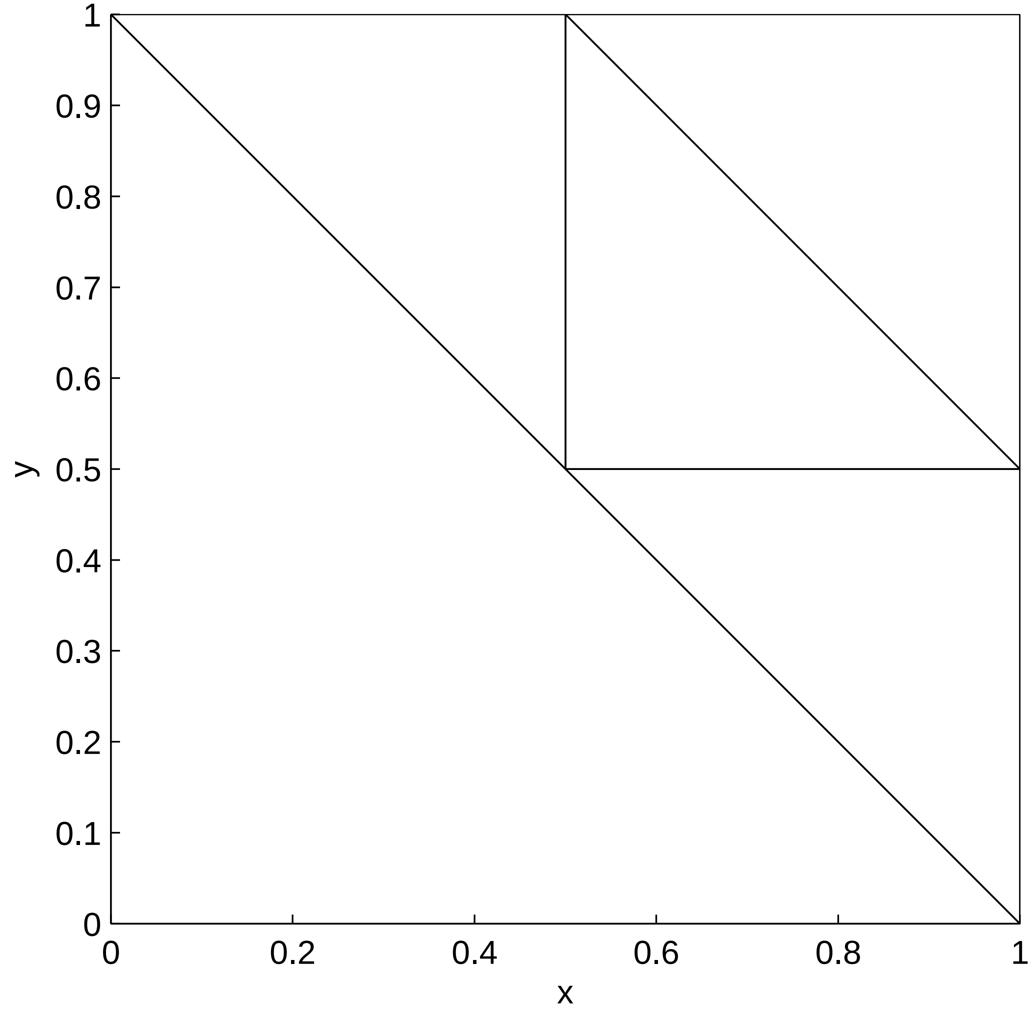
# *SIPDG* ($\tau = 1$; *2-levels of hanging nodes;* $\kappa = 10$)

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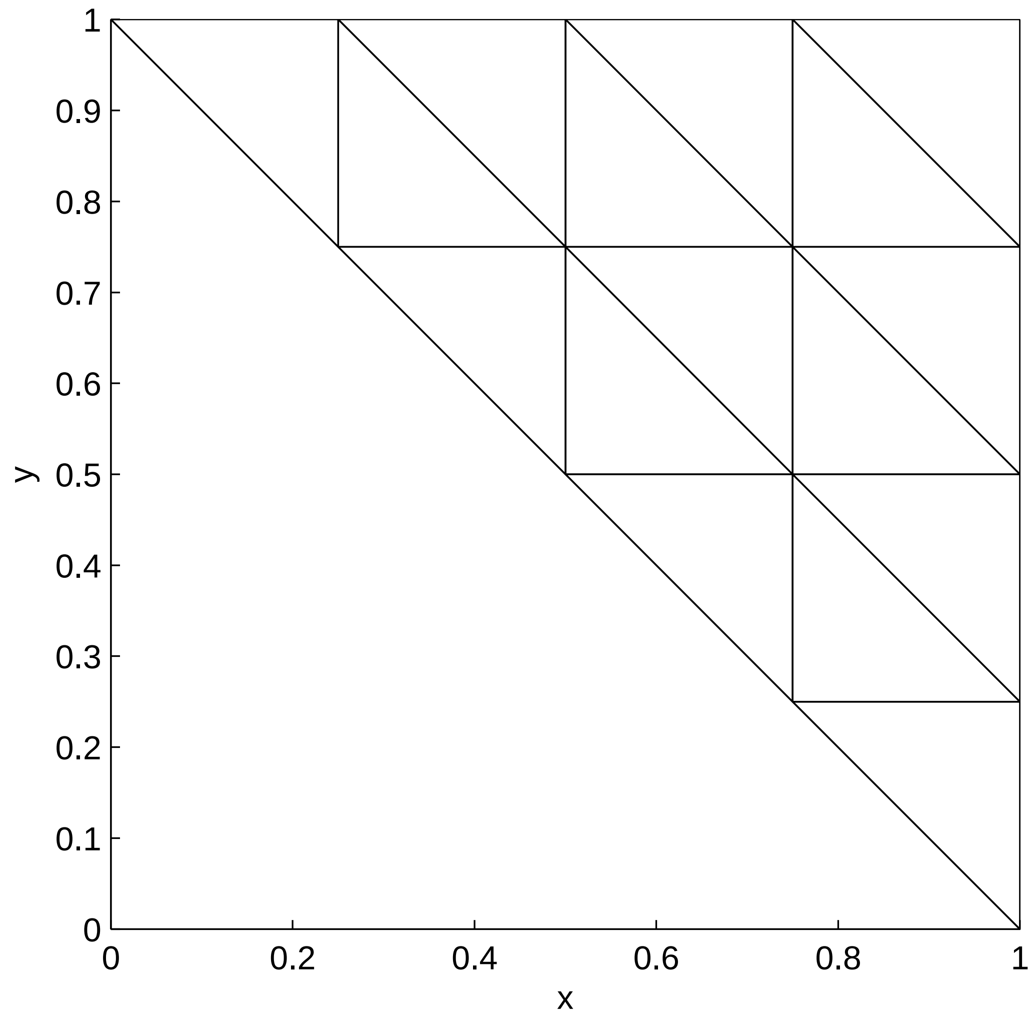


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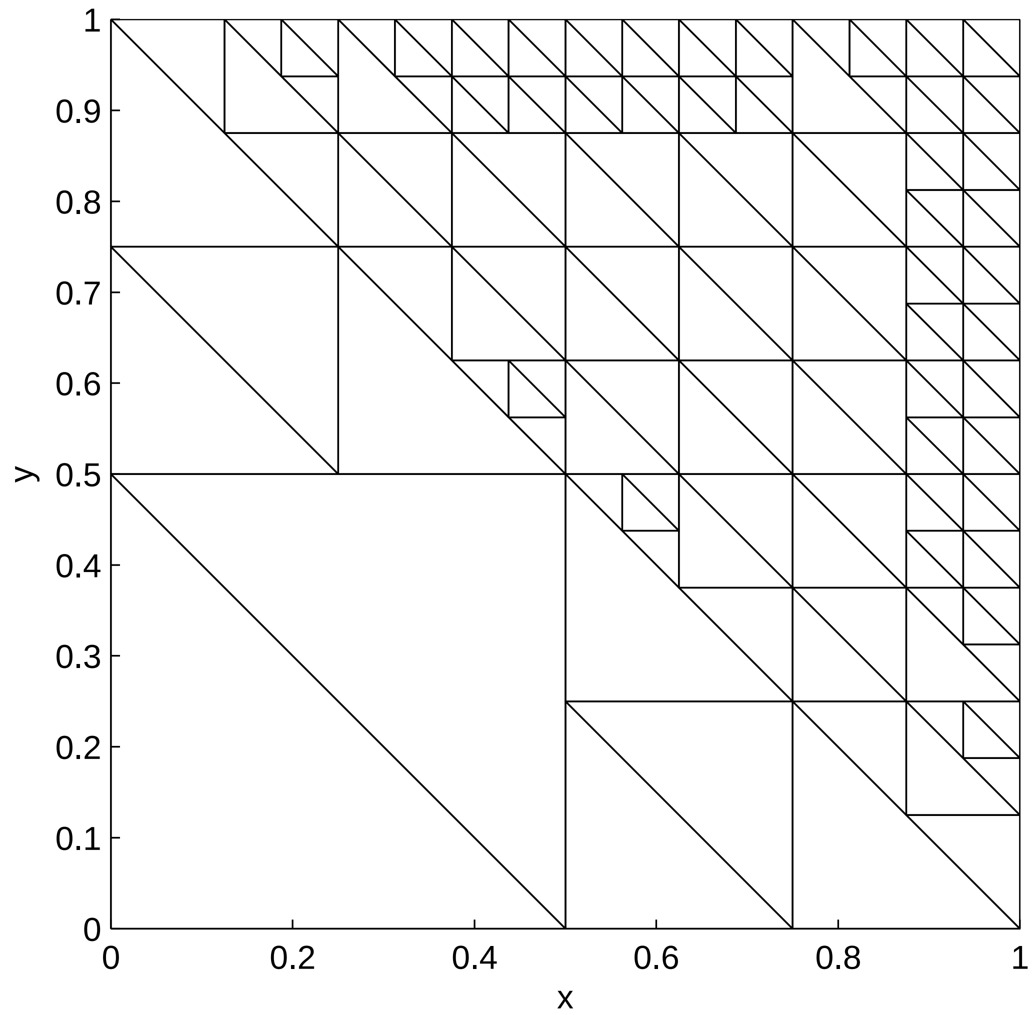
# *SIPDG ($\tau = 1$; 2-levels of hanging nodes; $\kappa = 10$)*

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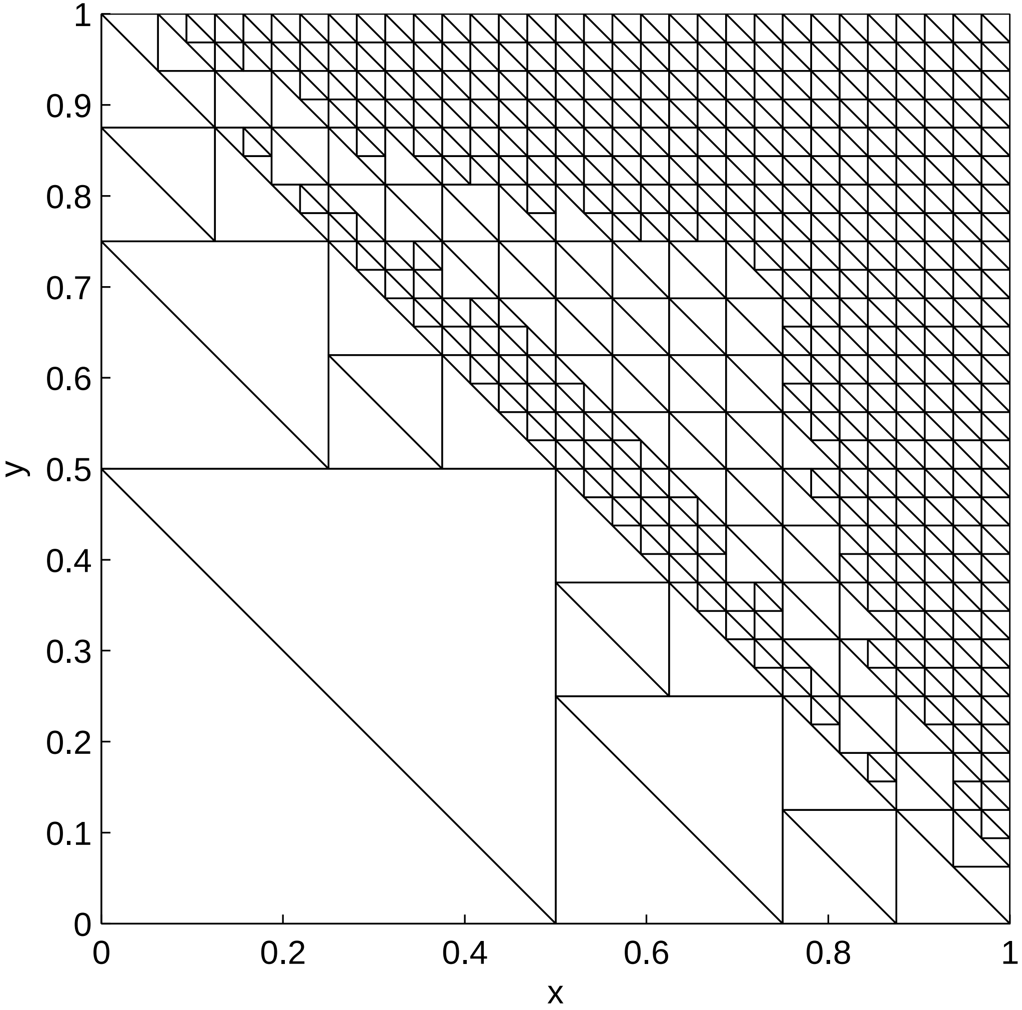
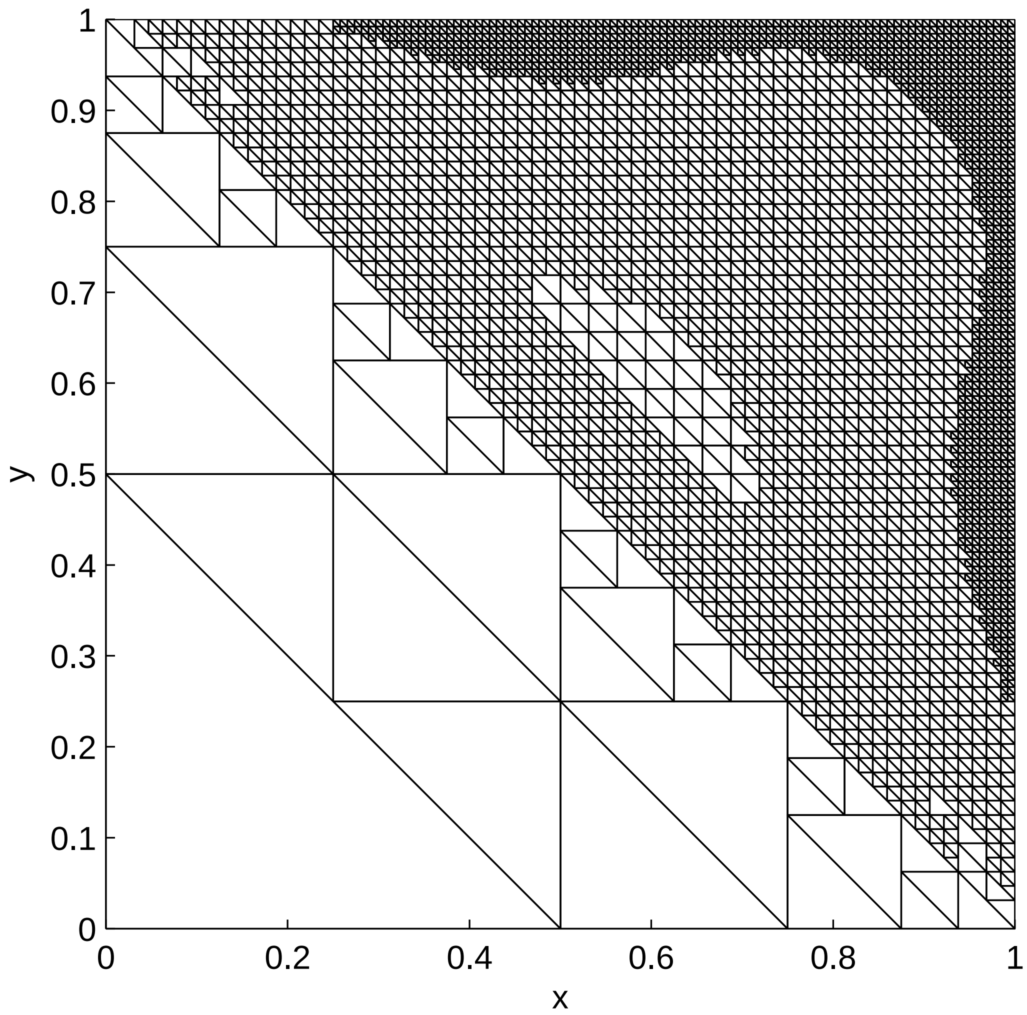


# SIPDG ($\tau = 1$; 2-levels of hanging nodes; $\kappa = 10$)

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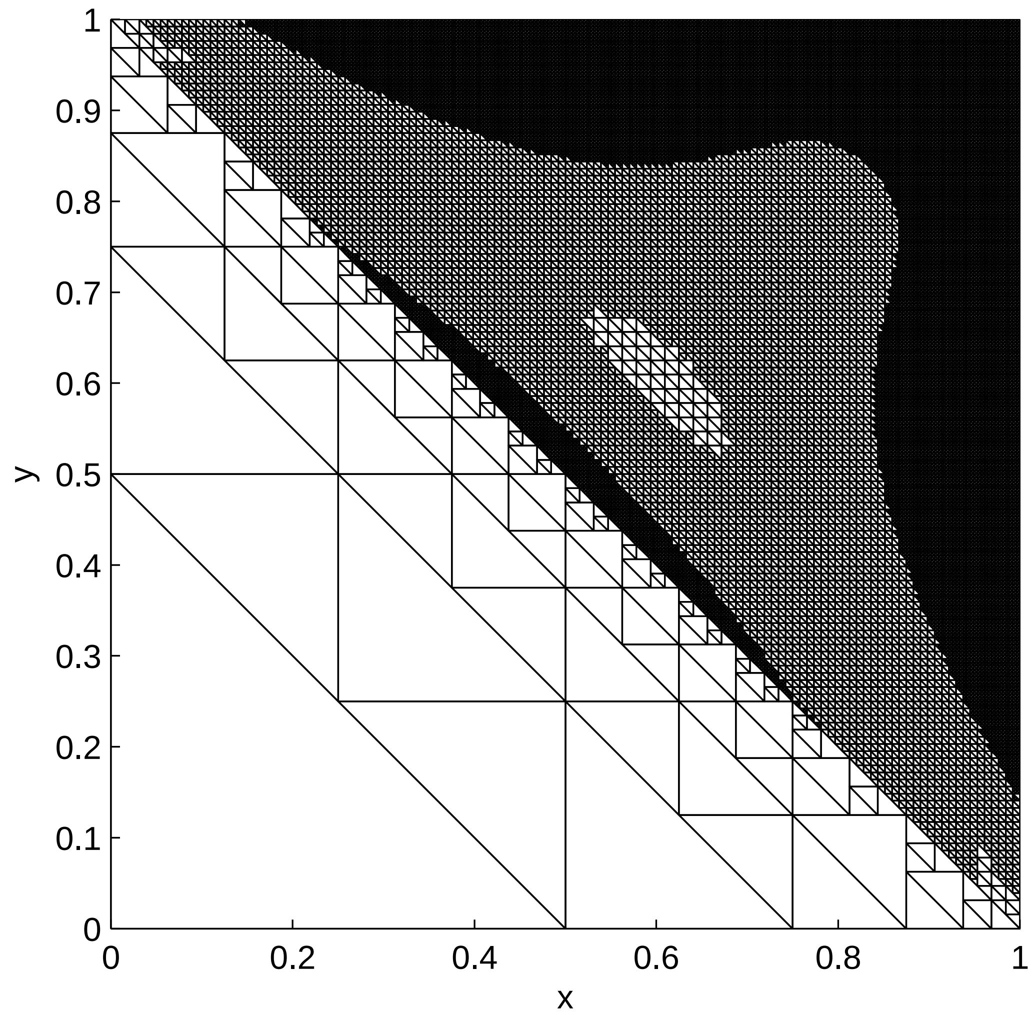


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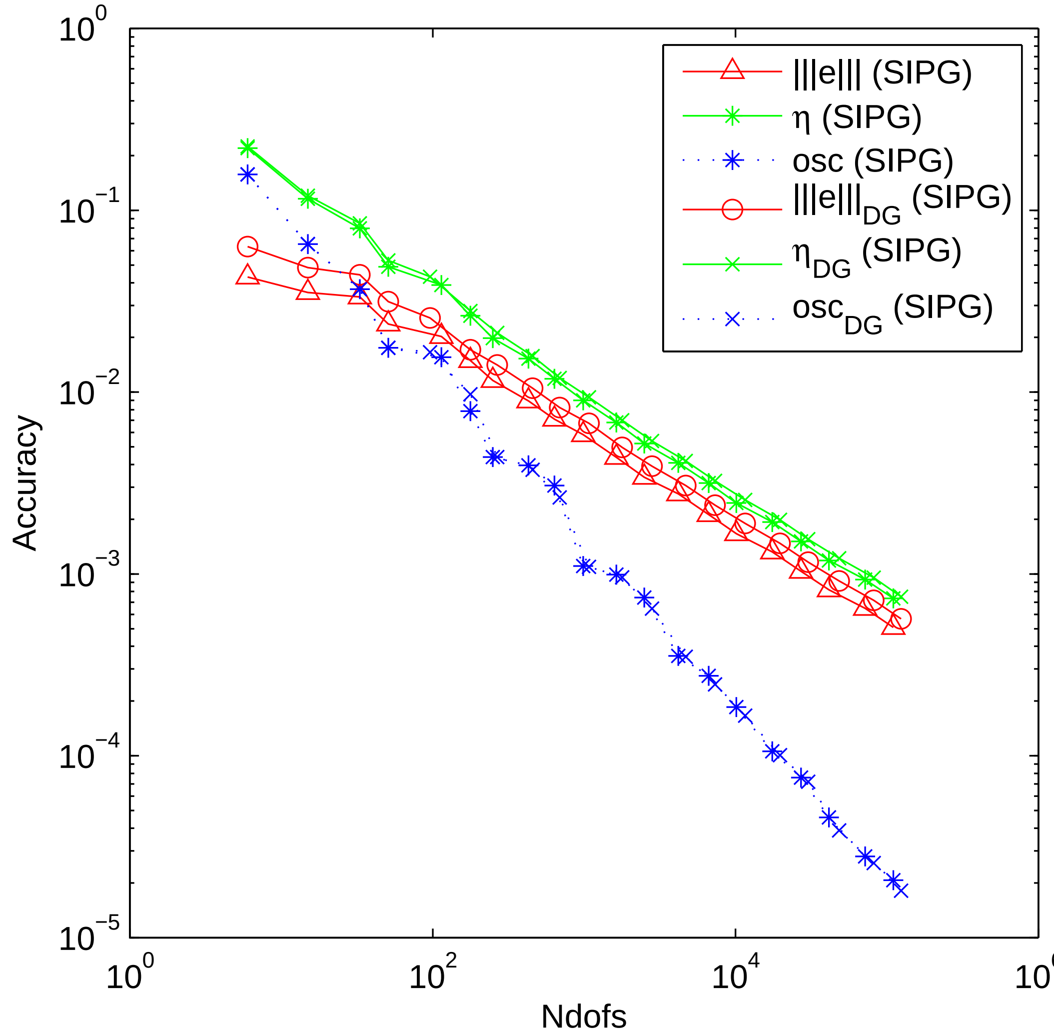
# *Performance of Estimators*

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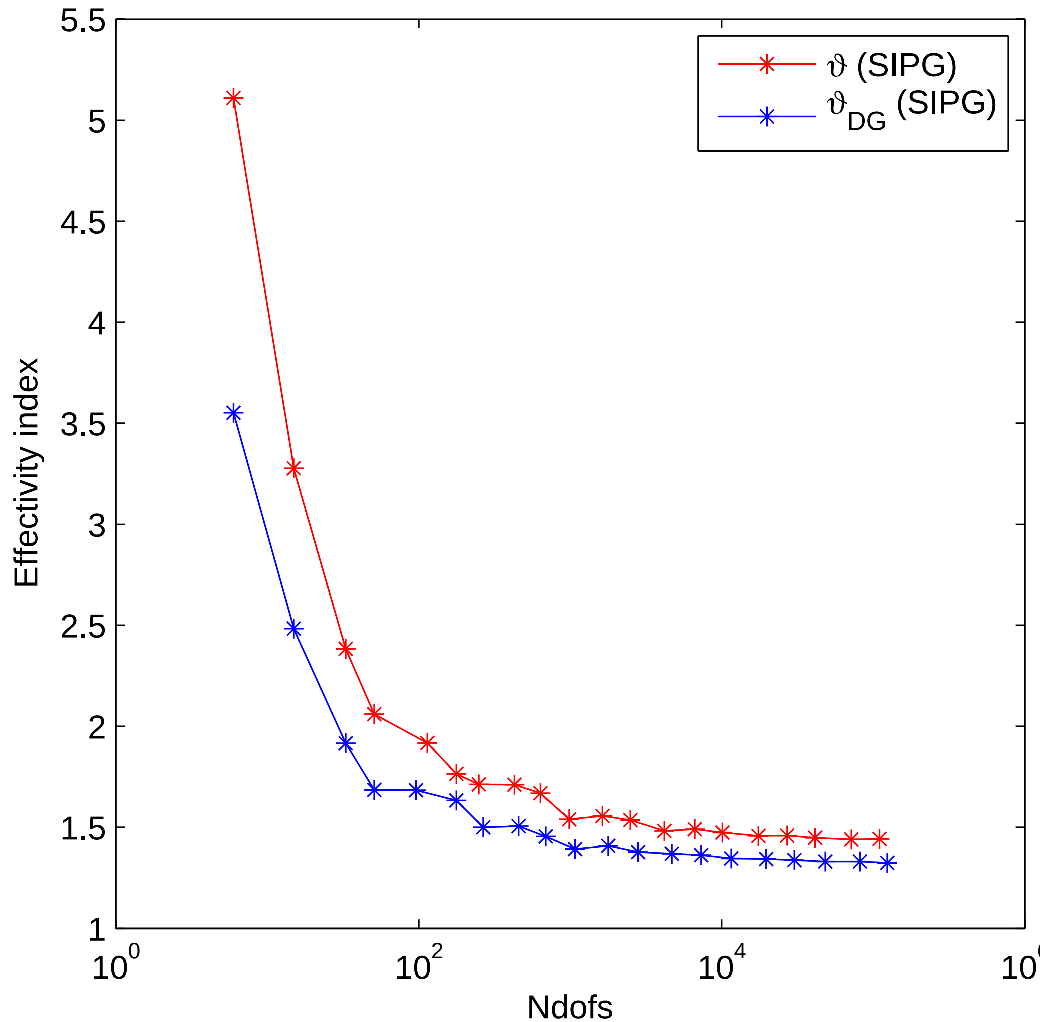
# *Effectivity Index of Estimators*

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# *Numerical Example—L-shaped Domain $hp$-DGFEM*

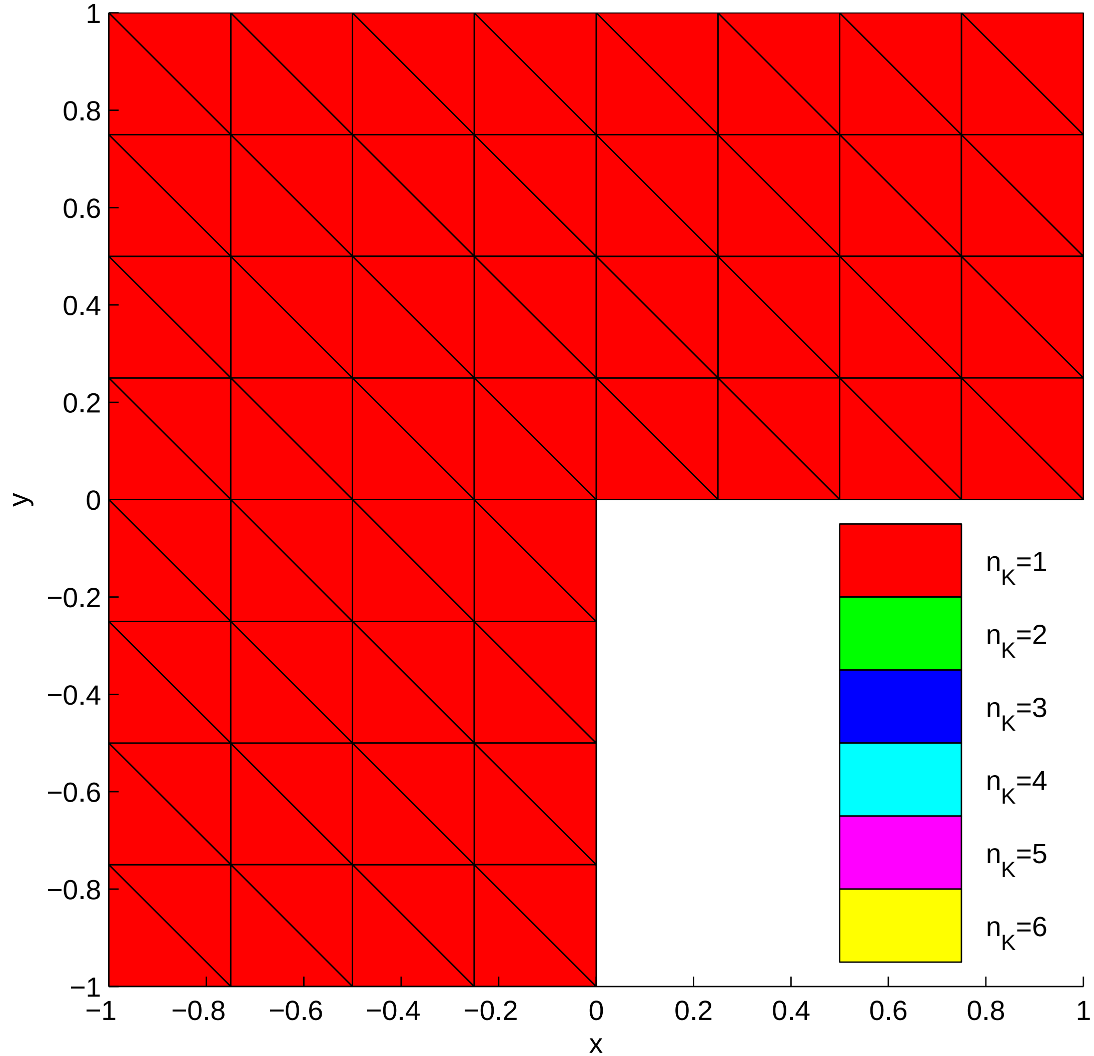
$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1) \times (0,1) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Data $f$ chosen so that true solution is

$$u(x,y) = (1 - x^2)(1 - y^2) r^{2/3} \sin \frac{2}{3}\theta$$

on usual L-shaped domain.

# *SIPDG ($\tau = 1$; 1-levels of hanging nodes;)*

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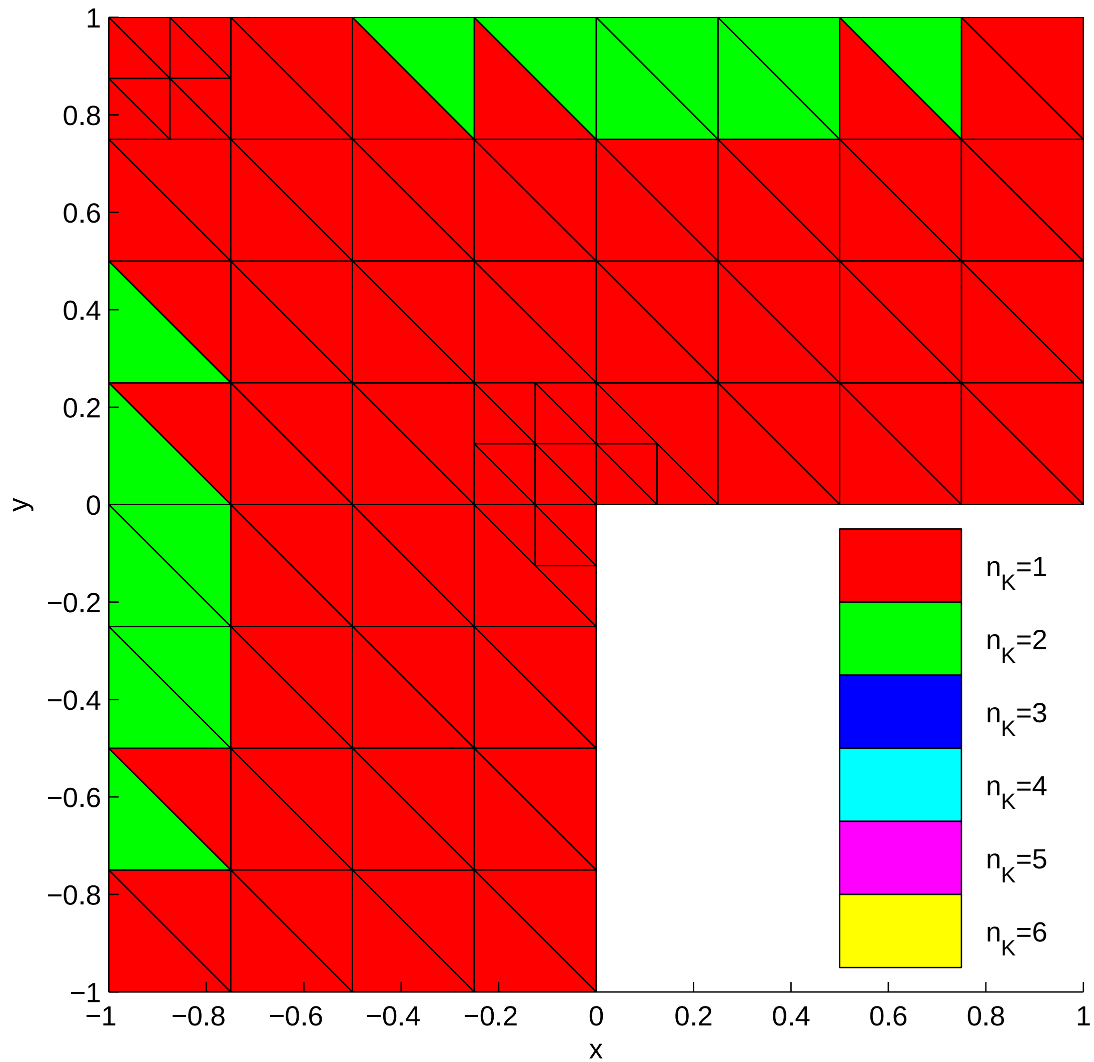


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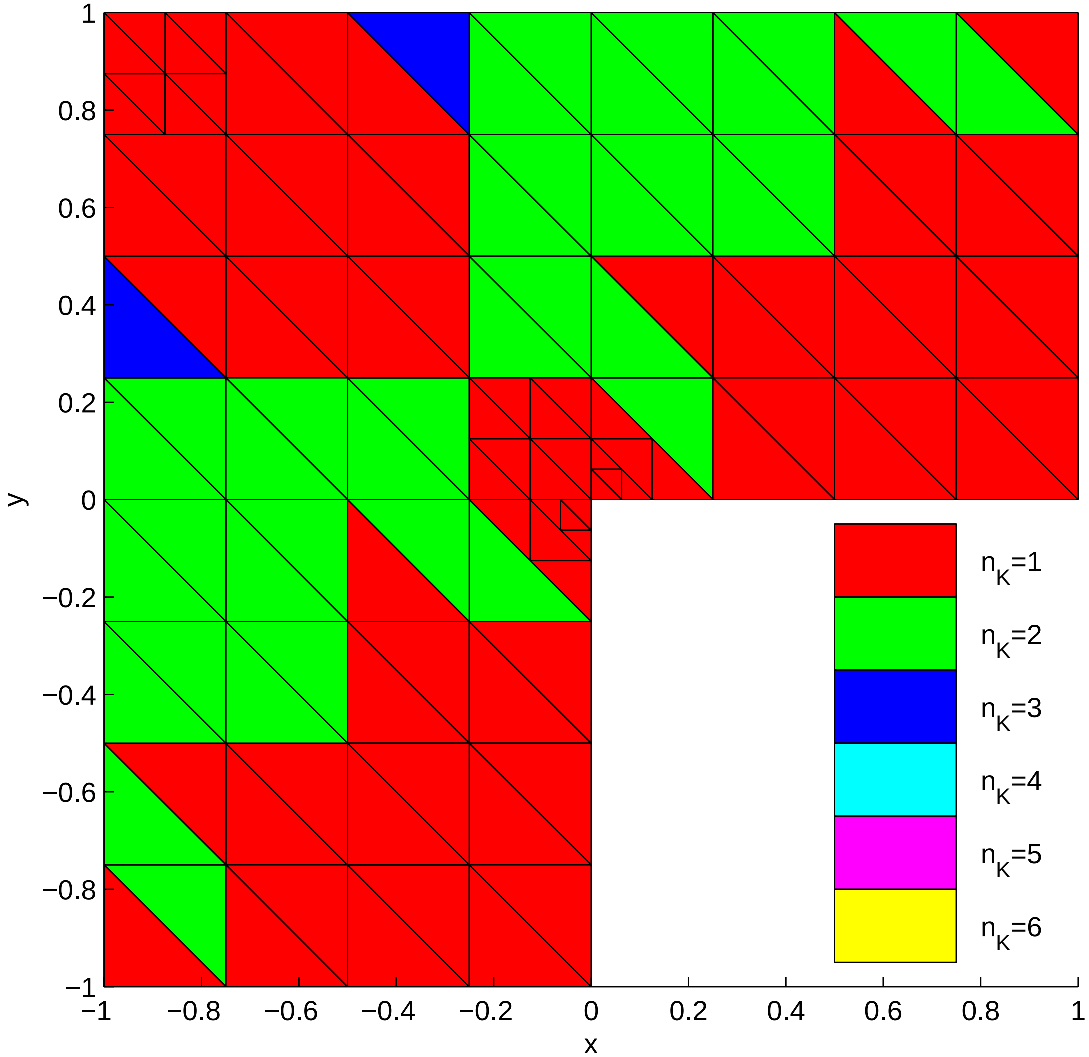


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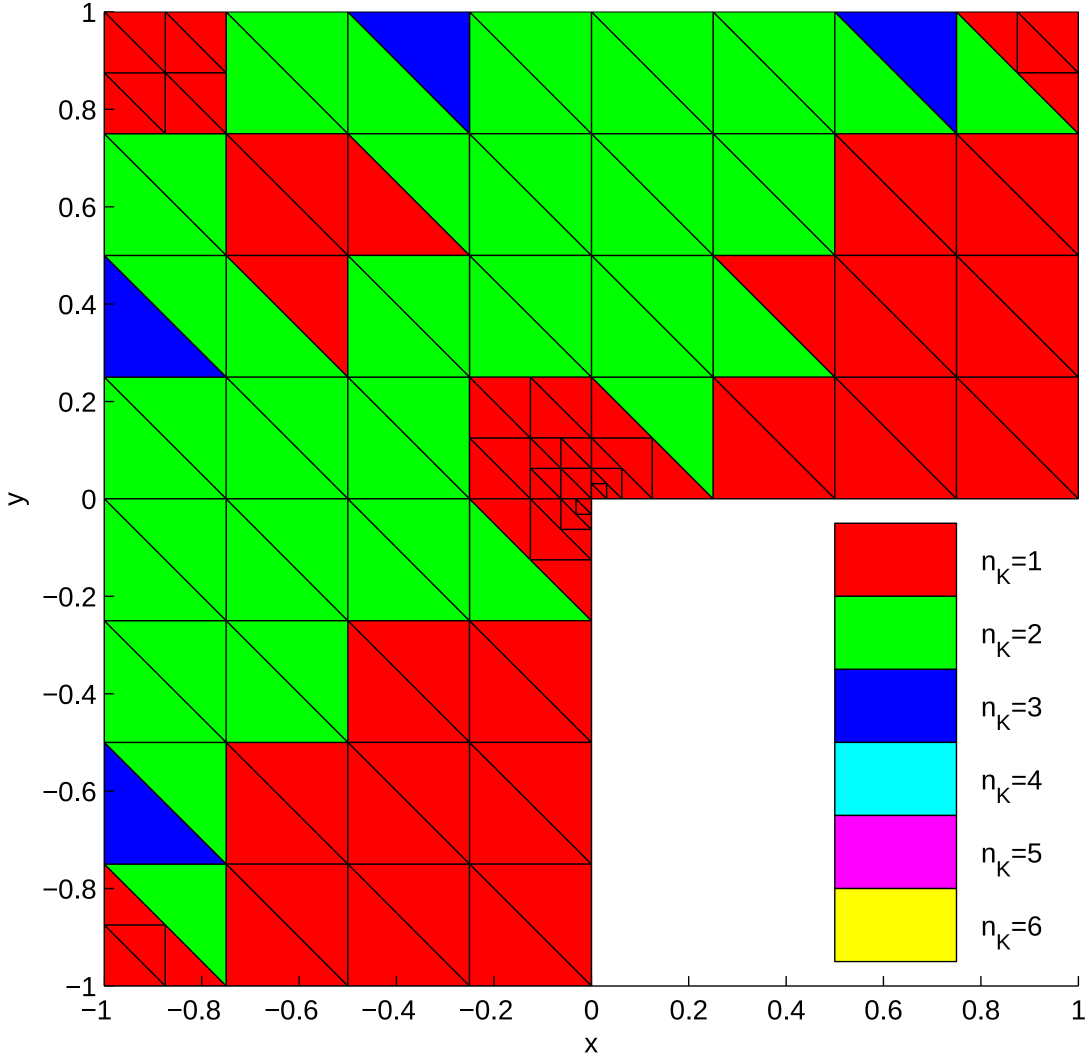


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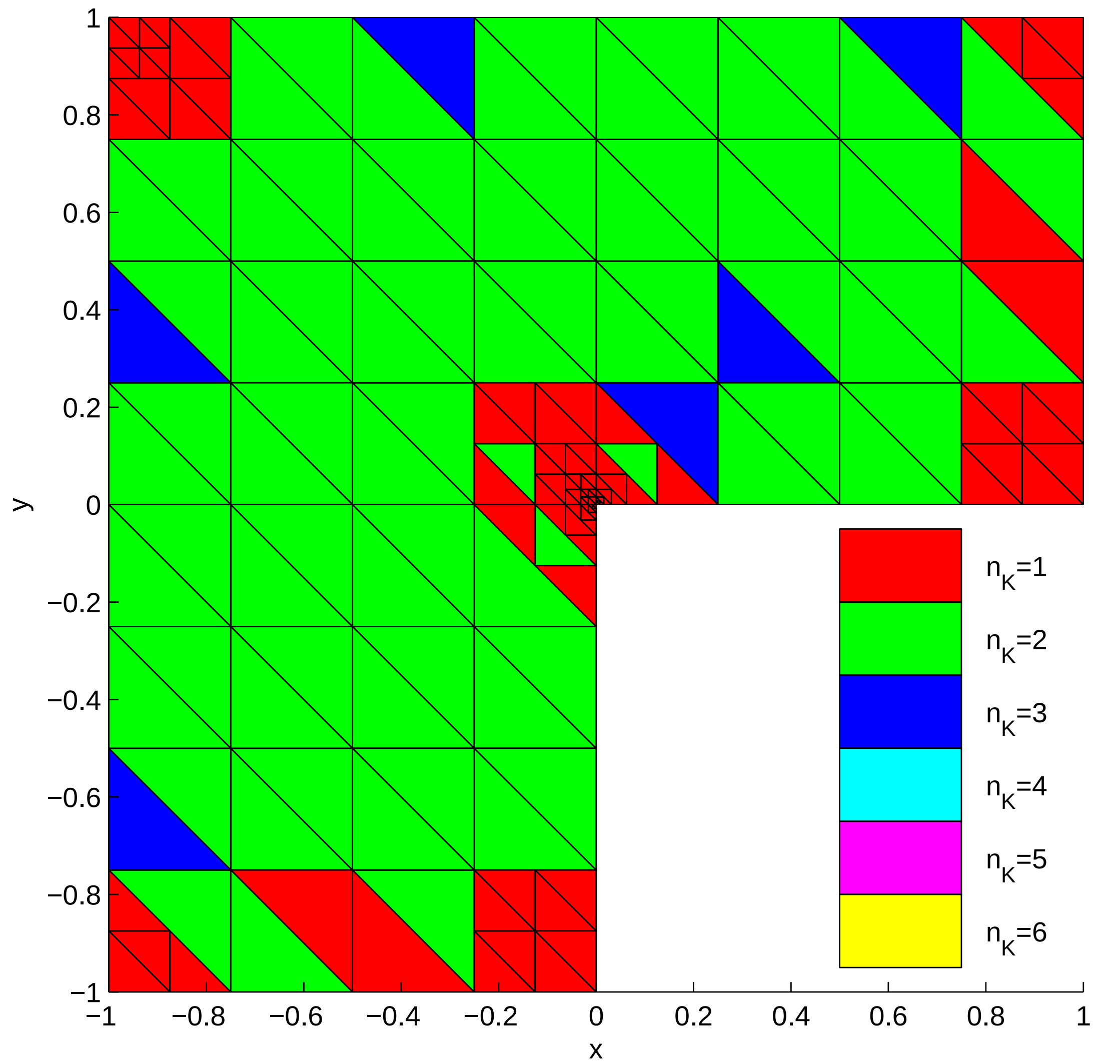


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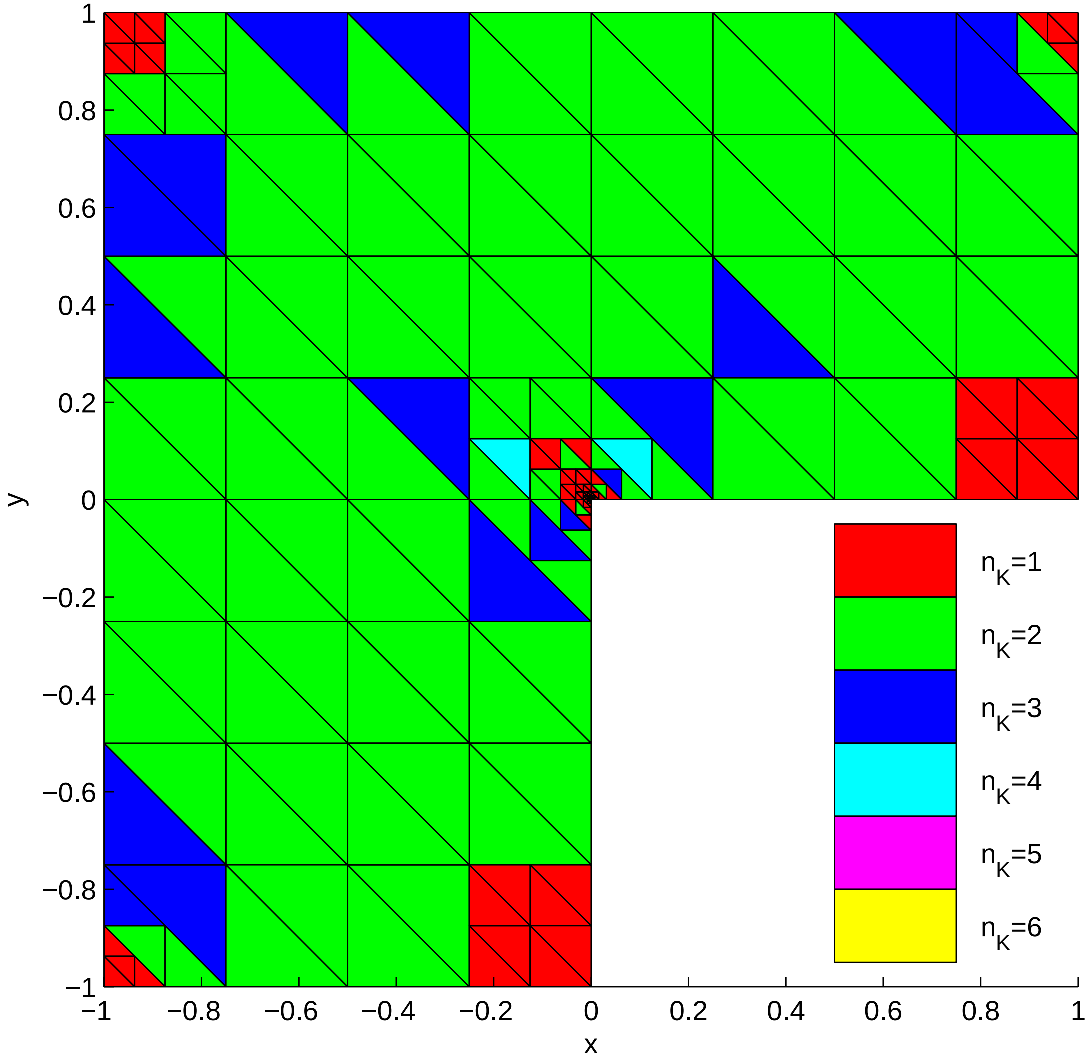


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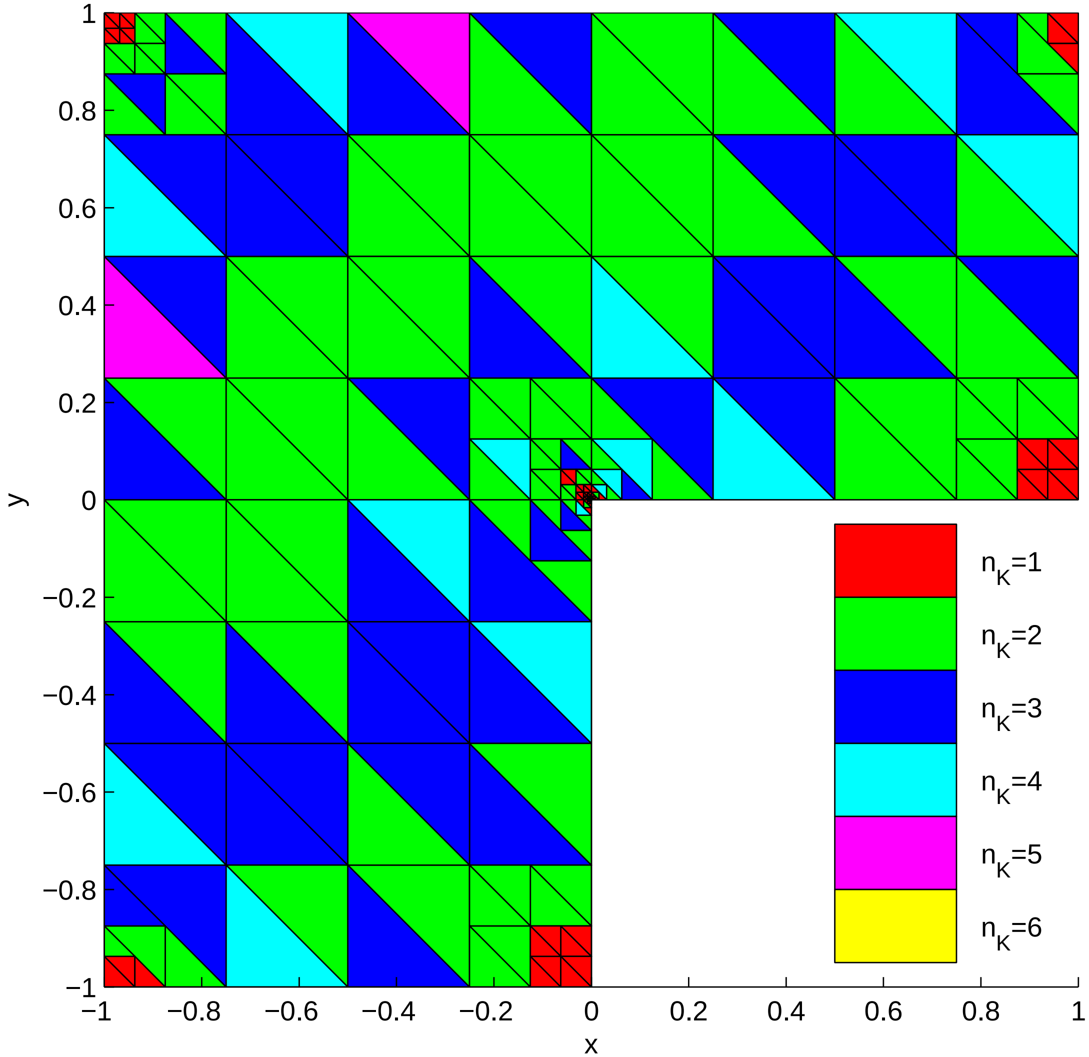


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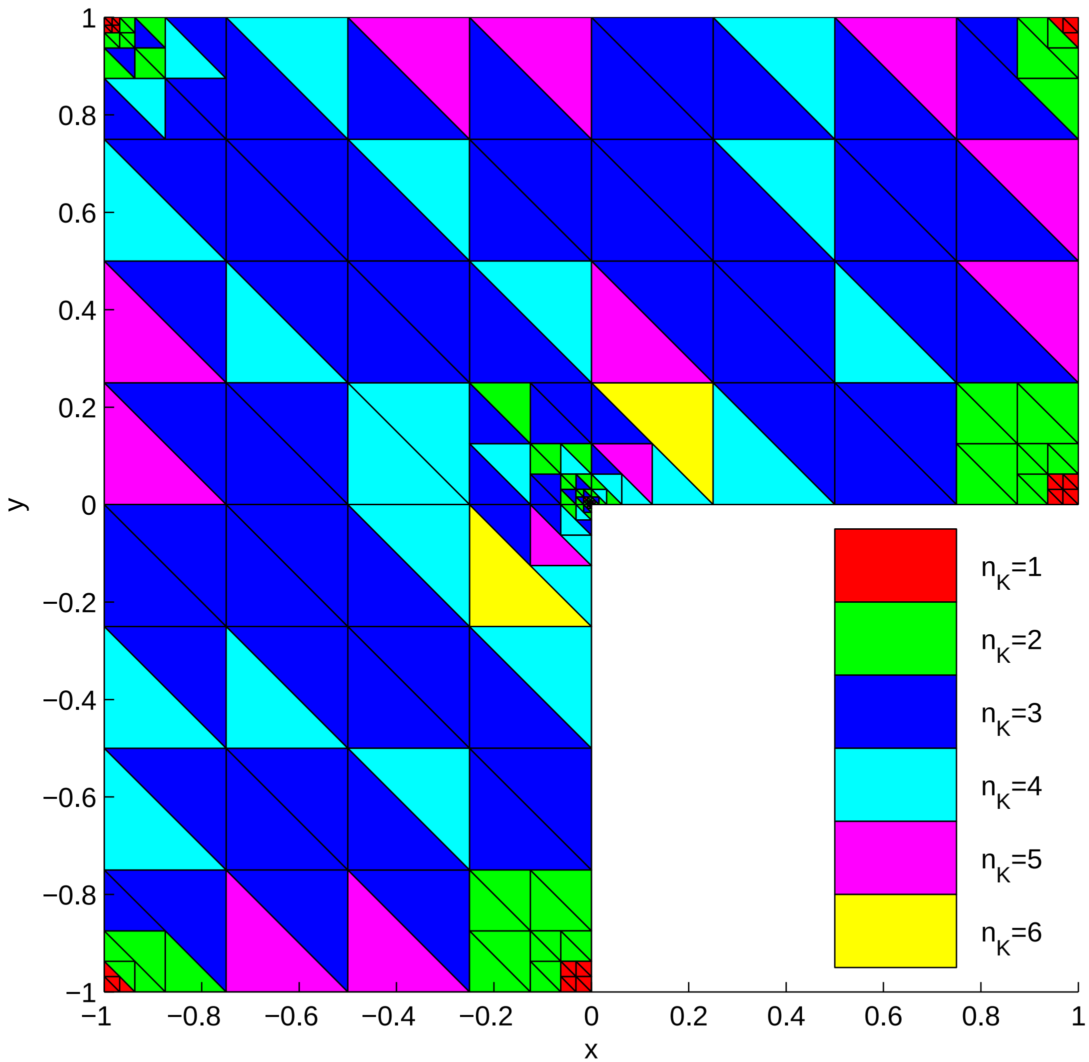


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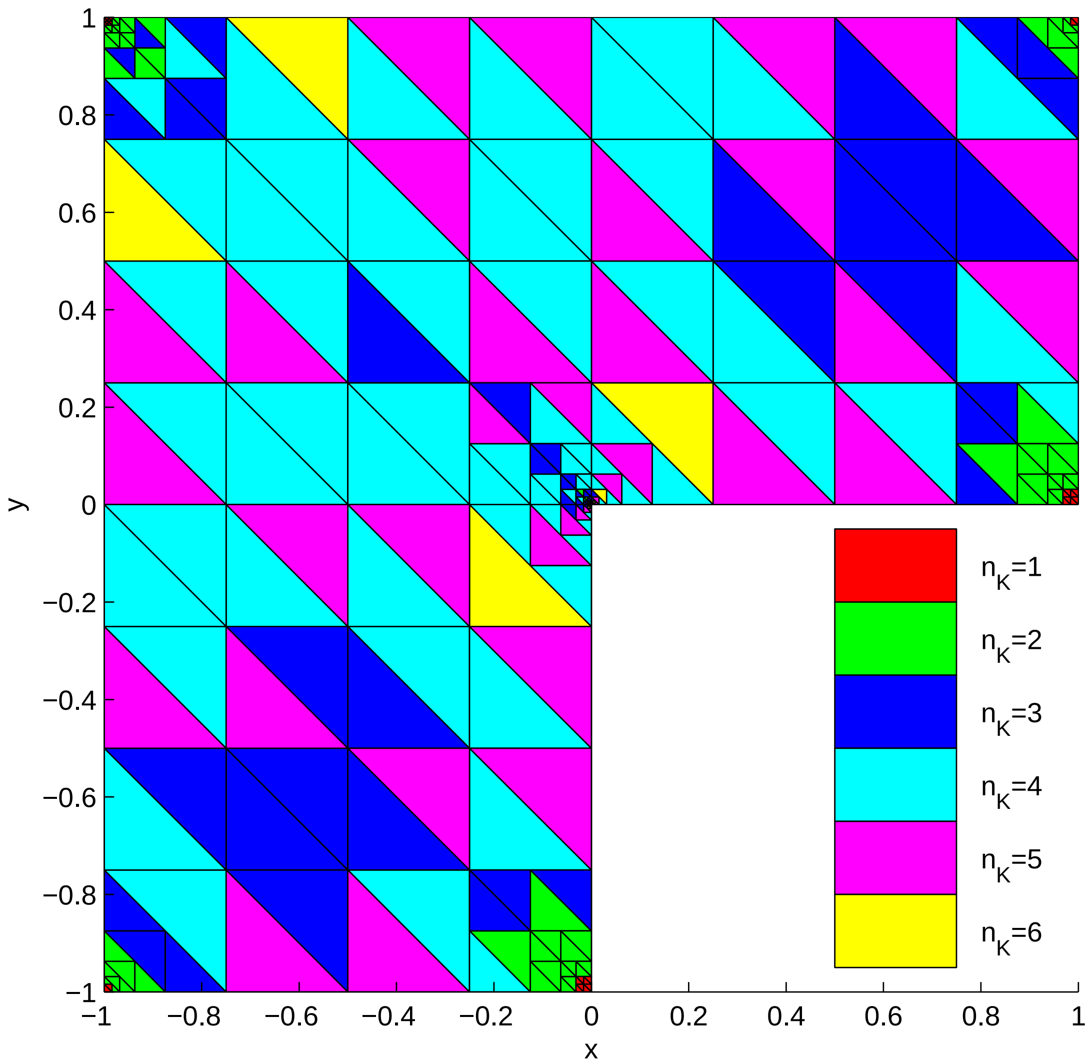


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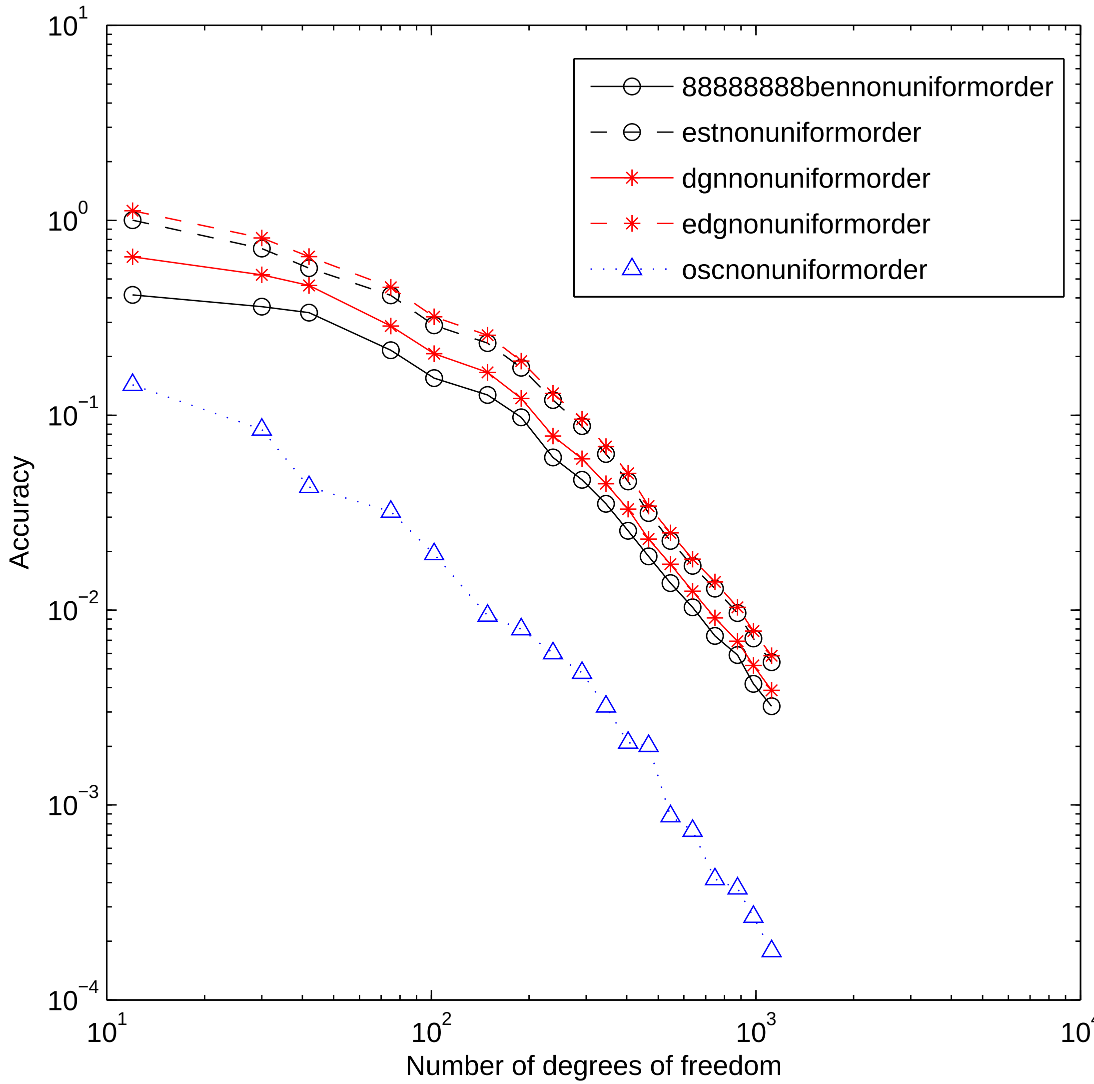


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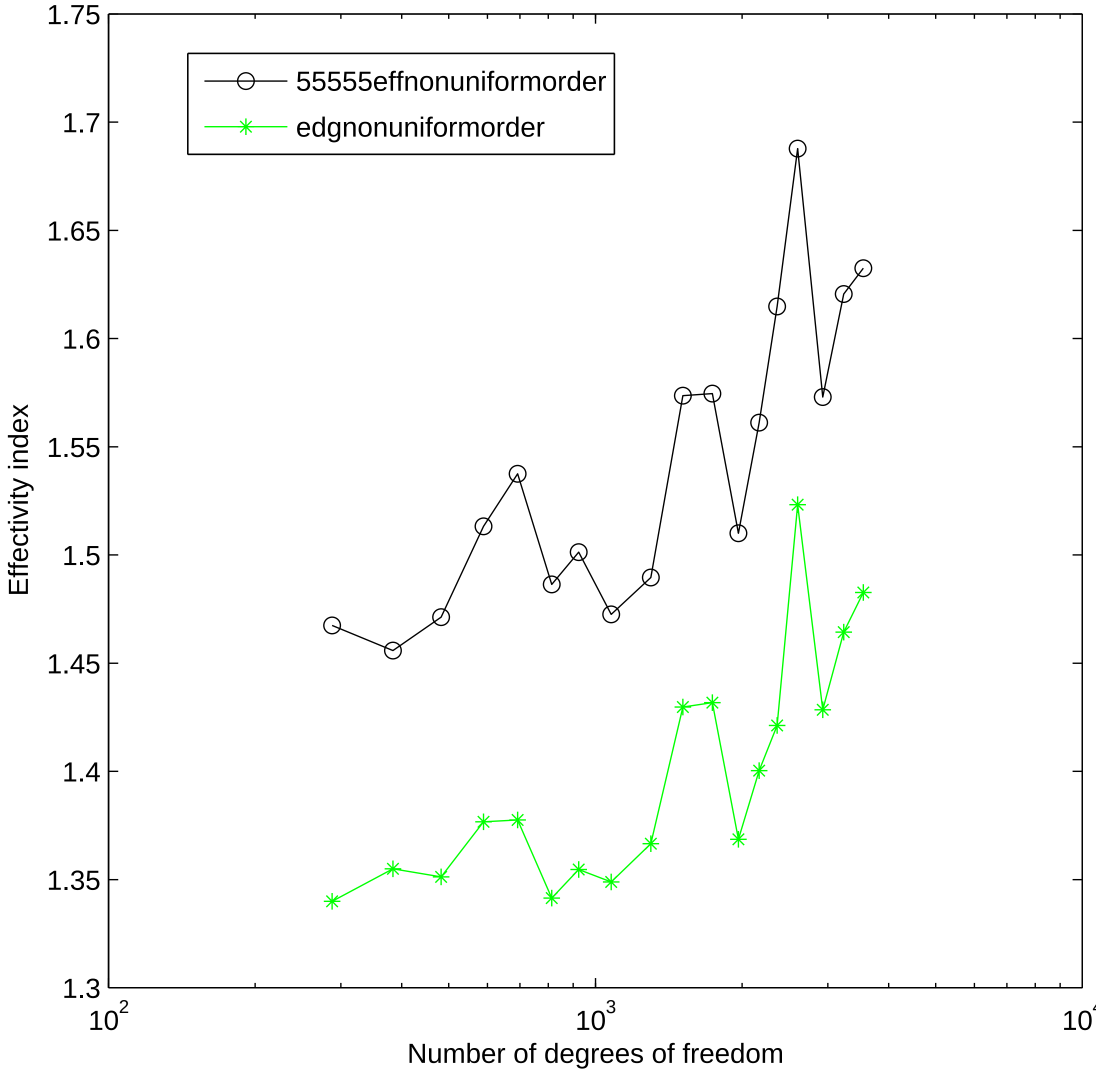


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