Realizability and parametricity in pure type systems

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Parametric polymorphism

```
let rec f = function
   | [] -> 1
   | hd::tl -> 2 * (f tl)

val f : ∀α, α list → int
```

Parametricity polymorphism: parametric types behave uniformly over abstracted types.

If $\vdash_{\mathcal{F}} f : \forall\alpha, \alpha\texttt{list} \rightarrow \texttt{int}$ and $|l| = |l'|$ then $f\ l = f\ l'$.

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$$t_1 \sim_{\sigma\to\tau} t_2 \equiv \forall x_1\, x_2. x_1 \sim_\sigma x_2 \to (t_1\, x_1) \sim_\tau (t_2\, x_2)$$

$$t_1 \sim_\alpha t_2 \equiv R_\alpha\, t_1\, t_2$$

$$t_1 \sim_{\forall\alpha,\tau} t_2 \equiv \forall R_\alpha . t_1 \sim_\tau t_2$$

Two related functions map related inputs to related outputs.

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Abstraction theorem

If $\vdash t : \tau$ then we can prove that $t \sim_\tau t$.

Example

$\forall \alpha . \alpha \to \alpha$

$$f \sim_{\forall \alpha . \alpha \to \alpha} g$$

$$\equiv$$

$$\forall R . \forall xy . xRy \to (f\ x) R (g\ y)$$

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$\forall \alpha \beta . \alpha \to \beta \to \alpha$

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$$\equiv$$

$$\forall R_1 R_2 . \forall x_1 y_1 . x_1 R_1 y_1 \to \forall x_2 y_2 . x_2 R_2 y_2 \to (f\ x_1\ x_2) R_1 (g\ y_1\ y_2)$$

Parametricity – Abstraction theorem

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Application : Theorems for free!

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- By unfolding the definition of $\sim_{\forall \alpha . \alpha \to \alpha}$,

$$\forall R^{\alpha, \beta} \quad x : \alpha \quad y : \beta . x R y \to (t_\alpha\, x) R (t_\beta\, y)$$

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$$\forall g : \alpha \to \beta.\forall x : \alpha.g\,(t_\alpha\, x) = t_\beta\,(g\,x)$$

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- Which is equivalent to the fact that t is the identity function

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Specifying programs with formulas
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Adequacy theorem

If there exists a proof π of P, then there exists a program p_π and a proof π' of $p_\pi \Vdash P$.

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If $\forall x \exists y, \varphi(x, y)$ is a theorem, then there exists a program f such that $\forall x, \varphi(x, f(x))$.

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Representation theorem

Functions definable in system F are exactly those provably total in second-order arithmetic.

Pure type systems – Generalities

- A family of λ-calculi where types and terms are unified
- Provide a framework for studying dependent types
- Contains many famous type-systems:
  - simply typed λ-calculus,
  - Girard and Reynolds polymorphic λ-calculus (system F),
  - Huet-Coquand's Calculus Of Constructions ...
- It even contains inconsistent calculus ($\texttt{Type} : \texttt{Type}$)

- A PTS P is defined by a specification $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ where
  - $\mathcal{S}$ is a set of sorts,
  - $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$ a set of axioms,
  - $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$ a set of rules.
- Typing judgement $\Gamma \vdash_P A : B$ of the PTS $P = (\mathcal{S}, \mathcal{A}, \mathcal{R})$.

Pure type systems – Terms and typing rules

Terms

$$A, B \quad := \quad s \mid x \mid (A\,B) \mid \lambda x : A.B \mid \forall x : A.B \quad \text{(with } s \in \mathcal{S}\text{)}$$

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$$\textsc{Axiom}\ \frac{}{\vdash s_1 : s_2}\ (s_1, s_2) \in \mathcal{A}$$

$$\textsc{Abstraction}\ \frac{\Gamma, x : A \vdash C : B \qquad \Gamma \vdash (\forall x : A.B) : s}{\Gamma \vdash (\lambda x : A.C) : (\forall x : A.B)}$$

$$\textsc{Product}\ \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\forall x : A.B) : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

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+ Application + Start + Weakening

System F

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The PTS F has the following specification

$$\mathcal{S}_F = \{\star, \Box\} \qquad \mathcal{A}_F = \{(\star, \Box)\} \qquad \mathcal{R}_F = \{(\star, \star, \star), (\Box, \star, \star)\}$$

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Only two kinds of product :

- Arrow type $(\sigma \to \tau)$: $(\star, \star, \star)$
- Type quantification $(\forall \alpha, \tau)$: $(\Box, \star, \star)$

$$\Gamma \vdash t : \tau : \star$$

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- We can prove that $\Gamma \vdash \tau : \star$ and $\Gamma \vdash x : \sigma : \star$ then $x \notin \tau$. Therefore $\forall x : \sigma.\tau$ can always be written $\sigma \to \tau$.

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- We can also prove that inhabitants of $\star$ are either :
  $$\alpha,\ \sigma \to \tau \text{ or } \forall \alpha : \star.\tau.$$

System F – Examples

Example

- $\mathsf{Nat} \equiv \forall\alpha : \star.(\alpha \to \alpha) \to (\alpha \to \alpha)$
- $0 \equiv \lambda(\alpha : \star)(f : \alpha \to \alpha)(x : \alpha).x$
- $\mathsf{Succ} \equiv \lambda(n : \mathsf{Nat})(\alpha : \star)(f : \alpha \to \alpha)(x : \alpha).f\,(n\,\alpha\,f\,x)$

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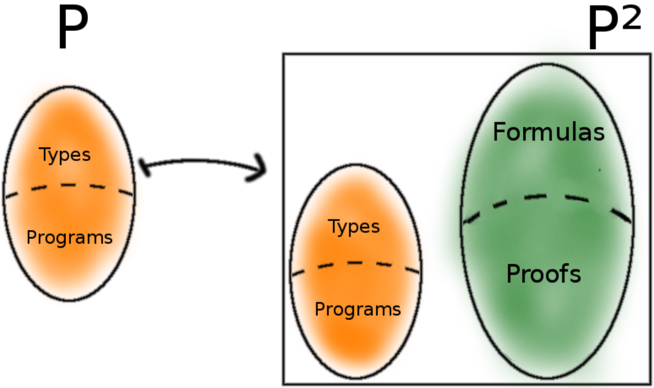
From P to P^2 – From realizers to logic



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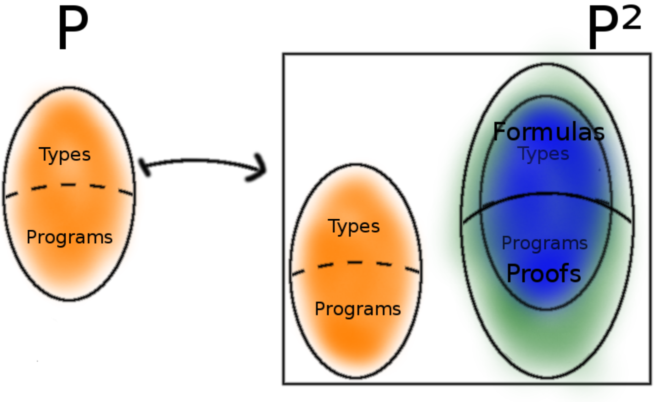

From P to P^2 – Definitions

Given a PTS $P = (\mathcal{S}, \mathcal{A}, \mathcal{R})$, we define $P^2 = (\mathcal{S}^2, \mathcal{A}^2, \mathcal{R}^2)$ by

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\mathcal{S}^2 & = & \mathcal{S} \cup \{\, \lceil \mathsf{s} \rceil \;\mid s \in \mathcal{S}\} \\
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- For each axiom (s_1, s_2) we add the axiom $(\lceil s_1 \rceil, \lceil s_2 \rceil)$.
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  2. We allow quantification over programs,
  3. We allow the formation of predicates.

A bit of vocabulary

- a type inhabits an original sort s

$$\Gamma \vdash A : s$$

- a formula inhabits a lifted sort $\lceil s \rceil$

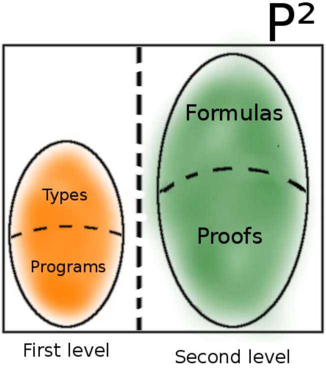
$$\Gamma \vdash A : \lceil s \rceil$$

- a program inhabits a type

$$\Gamma \vdash A : B : s$$

- a proof inhabits a formula

$$\Gamma \vdash A : B : \lceil s \rceil$$



- types & programs are first-level terms
- formulas & proofs are second-level terms

Second-order logic F^2

The PTS F^2 has the following specification:

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- $(\lceil\star\rceil, \lceil\star\rceil, \lceil\star\rceil)$ allows to build implication $P \to Q$.

Second-order logic F^2

The PTS F^2 has the following specification:

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\mathcal{S}_F^2 & = & \{ & \star, \Box, \lceil\star\rceil, \lceil\Box\rceil \quad \} \\
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Second-order logic F^2 – A stratified presentation

- We can prove that F^2 is equivalent to this presentation:

programs:

$$t, t_1, t_2 \quad := \quad x \quad | \ \lambda x : \tau . t \ | \ \Lambda \alpha . \tau \ | \quad (t_1 \, t_2) \quad | \quad (t \, \tau)$$

types:

$$\tau, \sigma \quad := \quad \alpha \quad | \ \sigma \rightarrow \tau \ | \ \forall \alpha . \tau$$

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- + a proof system
- In the PTS presentation, proofs are represented by terms

Second-order logic: F^2 – Examples

Here are some examples in F^2.

- Truth: $\top \equiv \forall X : \lceil\star\rceil . X \to X$
  and is proved by $\lambda X : \lceil\star\rceil (h : X) . h$

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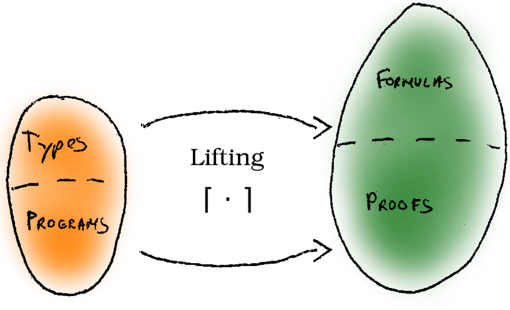
  $\lambda(\alpha : \star)(x : \alpha)(X : \alpha \to \lceil \star \rceil)(h : X\, x).h$
- The induction principle over Nat:

$$N \equiv \lambda x : \mathrm{Nat}\,.\forall X : \mathrm{Nat} \to \lceil \star \rceil.(\forall y : \mathrm{Nat}\,.X\, y \to X\,(\mathrm{Succ}\ y)) \to X\, 0 \to X\, x$$

Lifting and projection

Lifting

$\lceil\cdot\rceil$ embeds the first level toward its copy.

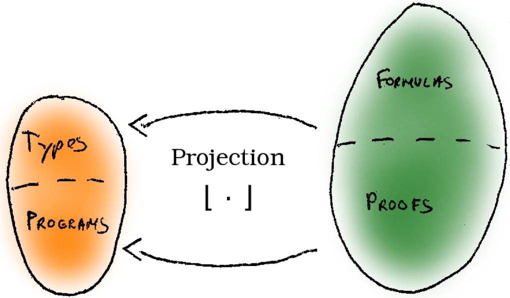


$$\lceil\forall\alpha:\star.\alpha\to\alpha\rceil \equiv \forall X:\lceil\star\rceil.X\to X$$
$$\lceil\mathsf{Nat}\rceil \equiv \forall X:\lceil\star\rceil.(X\to X)\to X\to X$$

Lifting and projection

Projection

$\lfloor \cdot \rfloor$ collapses the second level toward the first level.



$$\lfloor t_1 =_\tau t_2 \rfloor \equiv \lfloor \forall X : \tau \to \lceil \star \rceil . X\, t_1 \to X\, t_2 \rfloor \equiv \forall \alpha : \star . \alpha \to \alpha$$

$$\begin{aligned} \lfloor N\, t \rfloor &\equiv \lfloor \forall X : \mathsf{Nat} \to \lceil \star \rceil . (\forall y : \mathsf{Nat} . X\, y \to X\, (\mathsf{Succ}\, y)) \to X\, 0 \to X\, t \rfloor \\ &\equiv \forall \alpha . (\alpha \to \alpha) \to \alpha \to \alpha \equiv \mathsf{Nat} \end{aligned}$$

Lifting and projection – Lemmas

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If $A \longrightarrow_\beta B$, then either $\lfloor A \rfloor \longrightarrow_\beta \lfloor B \rfloor$ or $\lfloor A \rfloor = \lfloor B \rfloor$.

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Projection is the left inverse of lifting

$$\lfloor \lceil A \rceil \rfloor = A$$

Strong normalization

Theorem (Normalization)

If P is strongly normalizing, so is P^2.

Strong normalization

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If P is strongly normalizing, so is P^2.

Proof sketch.

If a term A is typable in P^2 and not normalizable, then :

- one of the first-level subterms of A is not normalizable, or
- the first-level term $\lfloor A \rfloor$ is not normalizable.

□

Parametricity and realizability in PTS's

In the following sections,

- We are going to define a parametricity relation :
  $(A, B) \in [\![C]\!]$ (we no longer use the notation $A \sim_C B$)
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$$\overbrace{(\ \underset{\substack{\uparrow \\ \textit{Program}}}{\cdot}\ ,\ \underset{\substack{\uparrow \\ \textit{Program}}}{\cdot}\ ) \in [\![\ \underset{\substack{\uparrow \\ \textit{Type}}}{\cdot}\ ]\!]}^{\textit{Formula}} \qquad\qquad \overbrace{\underset{\substack{\uparrow \\ \textit{Program}}}{\cdot}\ \Vdash\ \underset{\substack{\uparrow \\ \textit{Formula}}}{\cdot}}^{\textit{Formula}}$$

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Program / Program / Type
Type / Type / Sort

Program / Formula
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Parametricity in PTS's

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  - a unary notation $[\![\cdot]\!]$
- We want to satisfy the abstraction theorem:

Theorem (abstraction)

If $\Gamma \vdash A : B : s$, *then*

$$[\![\Gamma]\!] \;\vdash\; [\![A]\!] \;:\; (A, A) \in [\![B]\!] \;:\; \lceil s \rceil$$

Parametricity in PTS's – Products, sorts and variables

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\begin{aligned}
&(A_1, A_2) \in [\![\forall x : B.C]\!] \equiv \\
&\qquad \forall (x_1 : B)(x_2 : B).(x_1, x_2) \in [\![B]\!] \to (A_1\, x_1, A_2\, x_2) \in [\![C]\!]
\end{aligned}
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Parametricity in PTS's – Products, sorts and variables

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Parametricity in PTS's – Example

$$(t_1, t_2) \in [\![\forall\alpha : \star.\alpha \to \alpha]\!] \quad \equiv \quad \forall(\alpha_1 : \star)(\alpha_2 : \star)(\alpha_R : \ (\alpha_1, \alpha_2) \in [\![\star]\!]\ ). \\ (t_1\,\alpha_1, t_2\,\alpha_2) \in [\![\alpha \to \alpha]\!]$$

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Parametricity in PTS's

- Here is the transformation for the product:

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- If we have $\vdash (\lambda x : B.A) : (\forall x : B.C)$, since we want to satisfy the abstraction theorem, we must take

$$[\![\lambda x : B.A]\!] \equiv \lambda (x_1 : B)(x_2 : B)(x_R : (x_1, x_2) \in [\![B]\!]).[\![A]\!]$$

Parametricity in PTS's

- Here is the transformation for the product:

$$(A_1, A_2) \in [\![\forall x : B.C]\!] \equiv \\ \forall(x_1 : B)(x_2 : B)(x_R : (x_1, x_2) \in [\![B]\!]).(A_1\, x_1, A_2\, x_2) \in [\![C]\!]$$

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- Symmetrically, we need to take $[\![(A\, B)]\!] \equiv ([\![A]\!]\; B\; B\; [\![B]\!])$.

Parametricity in PTS's – The whole definition

Definition (parametricity)

$(C_1, C_2) \in [\![s]\!]$
$(C_1, C_2) \in [\![\forall x : A.\, B]\!]$

$(C_1, C_2) \in [\![T]\!]$
$[\![x]\!]$
$[\![\lambda x : A.B]\!]$
$[\![A\, B]\!]$
$[\![T]\!]$

Parametricity in PTS's – The whole definition

Definition (parametricity)

$$
\begin{array}{lcl}
(C_1, C_2) \in [\![s]\!] & = & C_1 \to C_2 \to \lceil s \rceil \\
(C_1, C_2) \in [\![\forall x : A.\, B]\!] & = & \forall (x_1 : A)(x_2 : A)(x_R : (x_1, x_2) \in [\![A]\!]). \\
 & & \quad (C_1\, x_1, C_2\, x_2) \in [\![B]\!] \\
(C_1, C_2) \in [\![T]\!] & = & ([\![T]\!]\; C_1\; C_2)\ \underline{\text{otherwise}} \\
[\![x]\!] & & \\
[\![\lambda x : A.B]\!] & & \\
[\![A\, B]\!] & & \\
[\![T]\!] & &
\end{array}
$$

Theorem (abstraction)

If $\Gamma \vdash A : B : s$, *then* $[\![\Gamma]\!] \vdash [\![A]\!] : (A, A) \in [\![B]\!] : \lceil s \rceil$

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[\![A\, B]\!] & = & [\![A]\!]\, B\, B\, [\![B]\!] \\
[\![T]\!] & = & \lambda (x_1\, x_2 : T).(x_1, x_2) \in [\![T]\!]\ \underline{\text{otherwise}}
\end{array}
$$

Theorem (abstraction)

If $\Gamma \vdash A : B : s$, *then* $[\![\Gamma]\!] \vdash [\![A]\!] : (A, A) \in [\![B]\!] : \lceil s \rceil$

Parametricity in PTS's – The *n*-ary version

Definition (parametricity)

$$
\begin{array}{lcl}
\overline{C} \in [\![s]\!]_n & = & \overline{C} \to \lceil s \rceil \\
\overline{C} \in [\![\forall x : A.\, B]\!]_n & = & \forall \overline{x : A}.\, \forall x_R : \overline{x} \in [\![A]\!]_n.\, \overline{z\, x} \in [\![B]\!]_n \\
\overline{C} \in [\![T]\!]_n & = & [\![T]\!]_n\, \overline{C} \ \underline{\text{otherwise}} \\
[\![x]\!]_n & = & x_R \\
[\![\lambda x : A.\, B]\!]_n & = & \lambda \overline{x : A}.\, \lambda x_R : \overline{x} \in [\![A]\!]_n.\, [\![B]\!]_n \\
[\![A\, B]\!]_n & = & [\![A]\!]_n\, \overline{B}\, [\![B]\!]_n \\
[\![T]\!]_n & = & \lambda \overline{z : T}.\, \overline{C} \in [\![T]\!]_n \ \underline{\text{otherwise}}
\end{array}
$$

Theorem (abstraction)

If $\Gamma \vdash A : B : s$, *then* $[\![\Gamma]\!]_n \vdash [\![A]\!]_n : \overline{A} \in [\![B]\!]_n : \lceil s \rceil$

Realizability in second-order logic

In traditional presentation of realizability:

- $t \Vdash P \to Q \equiv \forall x, x \Vdash P \to (t\,x) \Vdash Q$

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There are two kinds of quantification:

- First-level quantification
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- First-level quantification : uniform,
- Second-level quantification : things happen.

Pure Type Systems – A technical detail: sort annotations

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$$\textsc{Product}\ \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x \quad : A \ \vdash B : s_2}{\Gamma \vdash (\forall x \quad : A.B)\ : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

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$$\textsc{Product}\ \frac{\Gamma \vdash A : s_1 \qquad \Gamma, x^{s_1} : A \vdash B : s_2}{\Gamma \vdash (\forall x^{s_1} : A.B) : s_3}\ (s_1, s_2, s_3) \in \mathcal{R}$$

- We can distinguish the two kinds of quantification:
  - First-level quantification of the form $\forall x^{s} : A.B$,
  - Second-level quantification of the form $\forall x^{\lceil s \rceil} : A.B$.

Realizability in PTS's

- We define at the same time :
  - a binary notation $\cdot \Vdash \cdot$
  - a unary notation $\langle \cdot \rangle$
- We want to satisfy the adequacy theorem:

Theorem (adequacy)

If $\Gamma \vdash A : B : \lceil s \rceil$, *then*

$$\langle \Gamma \rangle \ \vdash \ \langle A \rangle \ : \ \lfloor A \rfloor \Vdash B \ : \ \lceil s \rceil$$

Realizability in PTS's – The products

- First level quantification :

$$C \Vdash \forall x^s : A.B = \forall x^s : A.C \Vdash B$$

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$$C \Vdash \lceil s \rceil = C \to \lceil s \rceil$$

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$$t \Vdash \forall x : \tau.P \equiv \forall x : \tau.t \Vdash Q$$

$$t \Vdash P \to Q \equiv \forall x : \lfloor P \rfloor.x \Vdash P \to (t\, x) \Vdash Q$$

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$$t \Vdash \forall X : \tau_1 \to \cdots \to \tau_n \to \lceil \star \rceil.P \equiv \forall \alpha : \star.\forall X : \tau_1 \to \cdots \to \tau_n \to \alpha \to \lceil \star \rceil.(t\,\alpha) \Vdash P$$

Realizability in PTS's – The whole definition

Definition (realizability)

$$
\begin{array}{lcl}
C \Vdash \lceil s \rceil & = & C \to \lceil s \rceil \\
C \Vdash \forall x^s : A.B & = & \forall x^s : A. C \Vdash B \\
C \Vdash \forall x^{\lceil s \rceil} : A.B & = & \forall (\lfloor x \rfloor^s : \lfloor A \rfloor)(x^{\lceil s \rceil} : \lfloor x \rfloor \Vdash A).(C\,\lfloor x \rfloor) \Vdash B \\
C \Vdash F & = & \langle F \rangle\, C \ \underline{\text{otherwise}}
\end{array}
$$

Theorem (adequacy)

If $\Gamma \vdash A : B : \lceil s \rceil$, *then*

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C \Vdash F & = & \langle F \rangle\, C \ \underline{\text{otherwise}} \\
\hline
\langle x^{\lceil s \rceil} \rangle & = & x^{\lceil s \rceil} \\
\langle \lambda x^s : A.B \rangle & = & \lambda x^s : A.\langle B \rangle \\
\langle \lambda x^{\lceil s \rceil} : A.B \rangle & = & \lambda (\lfloor x \rfloor^s : \lfloor A \rfloor)(x^{\lceil s \rceil} : \lfloor x \rfloor \Vdash A).\langle B \rangle \\
\langle (A\, B)_s \rangle & = & (\langle A \rangle\, B)_s \\
\langle (A\, B)_{\lceil s \rceil} \rangle & = & ((\langle A \rangle\, \lfloor B \rfloor)_s\, \langle B \rangle)_{\lceil s \rceil} \\
\langle T \rangle & = & \lambda z^s : \lfloor T \rfloor .\, z \Vdash T \ \underline{\text{otherwise}}
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$$

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If $\Gamma \vdash A : B : \lceil s \rceil$, *then*

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From realizability to parametricity

Theorem (realizability increases arity of parametricity)

$$\left(B, \overline{C}\right) \in [\![A]\!]_{n+1} = B \Vdash \left(\overline{C} \in [\![A]\!]_n\right)$$

and

$$[\![A]\!]_{n+1} = \langle [\![A]\!]_n \rangle$$

Lemma (0-parametricity is lifting)

$$[\![A]\!]_0 \equiv \lceil A \rceil$$

We can define parametricity with lifting+realizability:

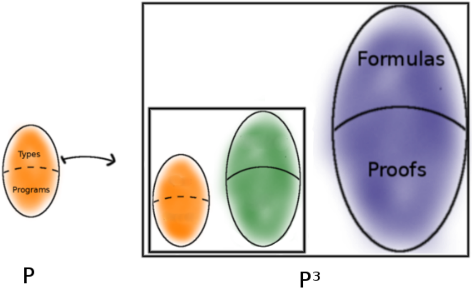
Corollary (From realizability to parametricity)

$$\overline{z} \in [\![A]\!]_n = z_1 \Vdash z_2 \Vdash \cdots \Vdash z_n \Vdash \lceil A \rceil$$

and

$$[\![A]\!]_n = \langle \cdots \langle \lceil A \rceil \rangle \cdots \rangle$$

A third level – From parametricity to realizability



Theorem (From parametricity to realizability)

If A is a second-level term, then

$$z \Vdash A = \lfloor \lceil z \rceil \in [\![A]\!]_1 \rfloor \qquad \textit{and} \qquad \langle A \rangle = \lfloor [\![A]\!]_1 \rfloor$$

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$$\forall x_1 ... x_n : \mathrm{Nat}, N\,x_1 \to \cdots \to N\,x_n \to N\,(f\,x_1\,...\,x_n)$$

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- Conversely : if $\vdash_P p : \mathrm{Nat} \to \mathrm{Nat}$ we can find π_p such that $\vdash_{P^2} \pi_p : \forall x : \mathrm{Nat}, N\,x \to N\,(p\,x)$.

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Theorem

Arithmetic functions representable in P are those provably total in P^2.

Inductive types

- Encoding of conjunction:

$$\textbf{data}\ \_ \wedge \_ : \lceil s \rceil \to \lceil s \rceil \to \lceil s \rceil\ \textbf{where}$$
$$\quad \mathit{conj} : \Pi\, P\, Q : \lceil s \rceil . P \to Q \to P \wedge Q$$

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$$\begin{array}{l}\textbf{data}\ \_ \wedge \_ : \lceil s \rceil \to \lceil s \rceil \to \lceil s \rceil\ \textbf{where} \\ \quad conj : \Pi\, P\, Q : \lceil s \rceil . P \to Q \to P \wedge Q\end{array}$$

- Projection $\lfloor \wedge \rfloor = \times$:

$$\begin{array}{l}\textbf{data}\ \_ \times \_ : s \to s \to s\ \textbf{where} \\ \quad (\_, \_) : \Pi\, \alpha\, \beta : s.\alpha \to \beta \to \alpha \times \beta\end{array}$$

$$\begin{aligned}
\textbf{data } \langle\wedge\rangle : \; & \Pi(\alpha : s).(\alpha \to \lceil s \rceil) \to \\
& \Pi(\beta : s).(\beta \to \lceil s \rceil) \to \\
& \alpha \times \beta \to s \textbf{ where} \\
\langle conj \rangle : \; & \Pi(\alpha : s)(P : \alpha \to \lceil s \rceil) \\
& \quad (\beta : s)(Q : \beta \to \lceil s \rceil)(x : \alpha)(y : \beta). \\
& \quad P\,x \to Q\,y \to \langle\wedge\rangle\,\alpha\,P\,\beta\,Q\,(x, y)
\end{aligned}$$

By definition, $t \Vdash P \wedge Q$ means $\langle\wedge\rangle\,\lfloor P \rfloor\,\langle P \rangle\,\lfloor Q \rfloor\,\langle Q \rangle\,t$. We have

$$t \Vdash P \wedge Q \Leftrightarrow (\pi_1\,t) \Vdash P \wedge (\pi_2\,t) \Vdash Q$$

where π_1 and π_2 are projections upon cartesian product.

Conclusion

- We gave a systematic way to formalize the meta-theory to study a programming language
- An account of parametricity and realizability in PTSs
- We exposed links between the two
- Extension: works with inductive types