Constructing More Quadratic APN Functions with the QAM Method

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Outline

Plan of this Section

Why Generate Quadratic APN Functions?

Definition (APN function)

A function $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ is Almost Perfect Non-linear (APN) if

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has at most two solutions for all $a \neq 0, b$.

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Does there exist an APN permutation on $\mathbb{F}_2^n$ for n even?

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- $n \geq 8$ **???**

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The Big APN Problem

Does there exist an APN permutation on $\mathbb{F}_2^n$ for n even?

- $n = 4$ **No.**
- $n = 6$ **Yes! [Dillon et al. 09]**
  Find permutation in the *CCZ-class* of a known APN function (the "Kim mapping")
- $n \geq 8$ **???**

Equivalence Relations

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F and G are *E(xtented) A(ffine) equivalent* if $G(x) = (B \circ F \circ A)(x) + C(x)$, where A, B, C are affine and A, B are permutations.

Definition (CCZ-Equivalence)

$F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ and $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ are *C(arlet)-C(harpin)-Z(inoviev) equivalent* if

$$\Gamma_G = \left\{(x, G(x)), \forall x \in \mathbb{F}_2^n\right\} = L\left(\left\{(x, F(x)), \forall x \in \mathbb{F}_2^n\right\}\right) = L(\Gamma_F),$$

where $L : \mathbb{F}_2^{n+m} \rightarrow \mathbb{F}_2^{n+m}$ is an affine permutation.

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Total (without redundancy) **21112**

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2. What's the overlap between the classes of known funcitons?

   not much!

Plan of this Section

Definition of the QAM

Definition (Quadratic Homogeneous Functions)

Quadratic functions without linear or constant terms are called **quadratic homogeneous functions**:

$$F(x) = \sum_{1 \leq j < i \leq n} c_{i,j} x^{2^{i-1}+2^{j-1}} \in \mathbb{F}_{2^n}[x].$$

Definition (QAM)

Let $H = (h_{i,j})_{n \times n}$ be an $n \times n$ matrix of $\mathbb{F}_{2^n}$. It is a **Quadratic APN Matrix (QAM)** if

1. it is symmetric and the elements in its main diagonal are all zeros; and
2. every nonzero linear combination of its rows has rank $n - 1$.

Properties

$$H = \begin{pmatrix} 0 & g^{34} & g^{81} & g^{83} & g^{170} & g^{106} & \mathbf{x}_{13} & \mathbf{x}_{7} \\ g^{34} & 0 & g^{68} & g^{162} & g^{166} & g^{85} & \mathbf{x}_{12} & \mathbf{x}_{6} \\ g^{81} & g^{68} & 0 & g^{136} & g^{69} & g^{77} & \mathbf{x}_{11} & \mathbf{x}_{5} \\ g^{83} & g^{162} & g^{136} & 0 & g^{17} & g^{138} & \mathbf{x}_{10} & \mathbf{x}_{4} \\ g^{170} & g^{166} & g^{69} & g^{17} & 0 & g^{34} & \mathbf{x}_{9} & \mathbf{x}_{3} \\ g^{106} & g^{85} & g^{77} & g^{138} & g^{34} & 0 & \mathbf{x}_{8} & \mathbf{x}_{2} \\ \mathbf{x}_{13} & \mathbf{x}_{12} & \mathbf{x}_{11} & \mathbf{x}_{10} & \mathbf{x}_{9} & \mathbf{x}_{8} & 0 & \mathbf{x}_{1} \\ \mathbf{x}_{7} & \mathbf{x}_{6} & \mathbf{x}_{5} & \mathbf{x}_{4} & \mathbf{x}_{3} & \mathbf{x}_{2} & \mathbf{x}_{1} & 0 \end{pmatrix}$$

Theorem (Yu et al.[1])

There exists a one to one correspondence between quadratic homogeneous APN functions and QAMs.

[1] Y. Yu, M. Wang, Y. Li, A matrix approach for constructing quadratic APN functions. Designs Codes and Cryptography 73, p.587-600 (2014).

Generating New Functions

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4. Let $\{x_1, \dots\}$ take different values and check if we have a QAM.

Sorting the Result

This approach works (see later)!

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A better statement

How to partition the functions obtained into

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How to ensure that we do not generate the same function multiple times?

A better statement

How to partition the functions obtained into CCZ-equivalence classes?

Invariant-based Approach

Theorem ([Yos12][2])

Quadratic APN functions are CCZ-equivalent if and only if they are EA-equivalent.

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Thickness spectrum: A property of the *Walsh zeroes* of F.

$\Sigma_F^k(0)$: How many tuples $(x_1, \ldots, x_k)$ such that:

$$x_1 + \ldots + x_k = 0; \text{ and } F(x_1) + \ldots + F(x_k) = 0 .$$

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Ortho-derivative: $\pi_F : \mathbb{F}_2^n \to \mathbb{F}_2^n$ is the unique function such that $\pi_F(0) = 0$ and, for all x, a:

$$\pi_F(a) \cdot \big(F(x+a) + F(x) + F(a) + F(0)\big) = 0.$$

Its affine equivalence-class is an EA-class invariant.

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Implementation aspects

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Γ-ranks	$O(2^{2\omega n})$	`gamma_rank(F)`
Thickness spectrum	?	`thickness_spectrum(F)`
Σ_F^k	$O(n2^{2n})$	`sigma_multiplicities(F, k)`
π_F	$O(2^{2n})$	`ortho_derivative_label(F)`

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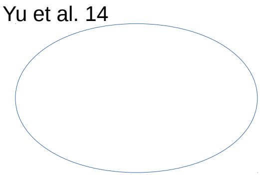
Are my APN functions new?

```
from collections import defaultdict
from sboxU import *

ea_counters = defaultdict

known_apn_functions = eightBitAPN.all_quadratics()
for f in known_apn_functions:
    ea_counters[ortho_derivative_label(f)] += 1

new_QAMs = [[0, ..., 255], ... ]
updated_apn_functions = known_apn_functions[:]
for f in new_QAMs:
    l = ortho_derivative_label(f)
    ea_counters[l] += 1
    if ea_counters[l] == 1:
        updated_apn_functions.append(f)
```

Plan of this Section

8-bit Quadratic APN Generation



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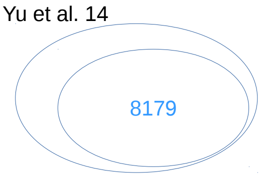

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Yu et al. 14

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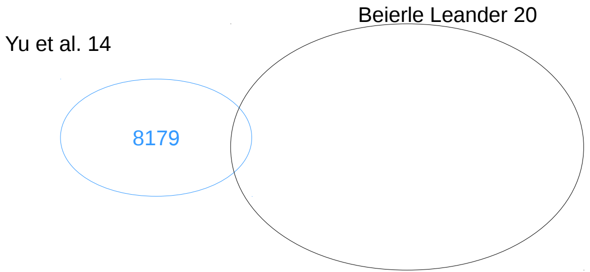

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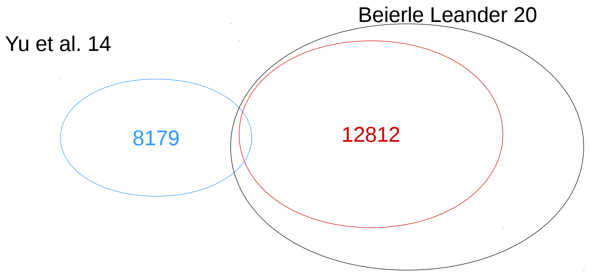

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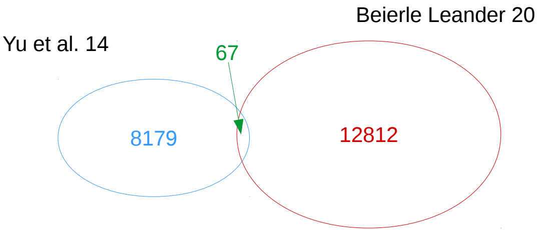

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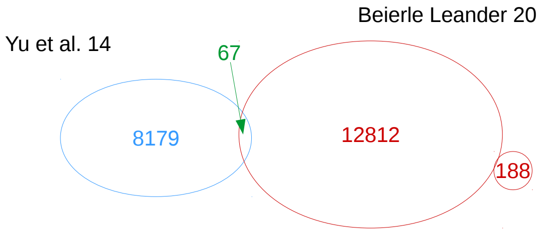

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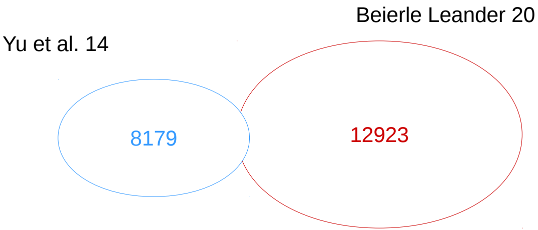

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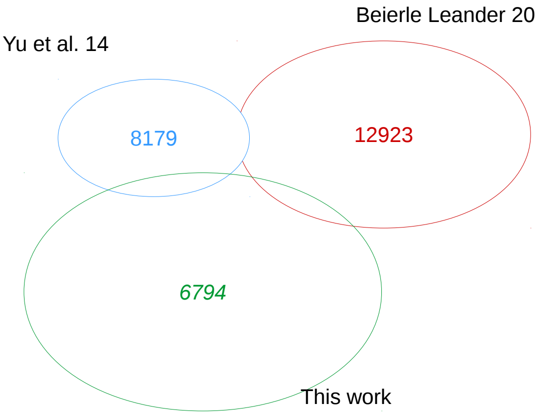

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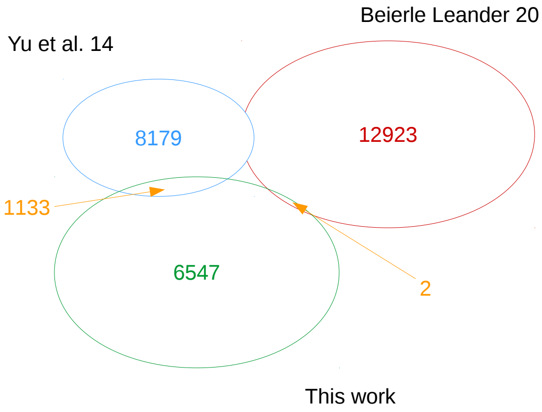

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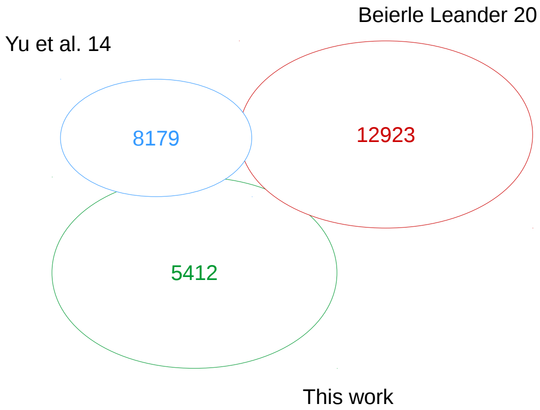

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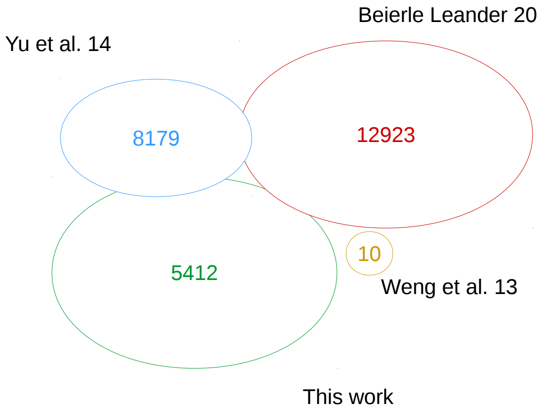

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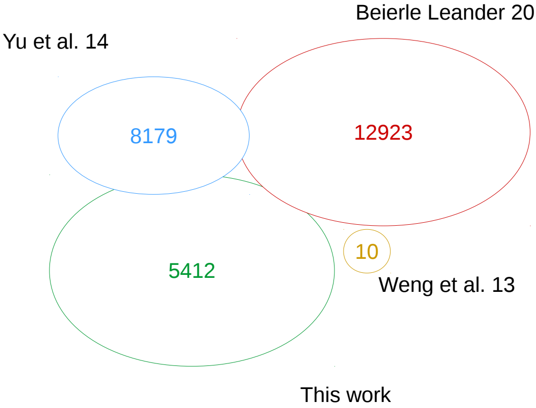


```
len(sboxU.eightBitAPN.all_quadratics()) = 26524
```

Total Number of APN Functions

A simple test

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- $n = 7$ For $k = 230$ (out of 488), $P_{230} \approx 79\%$

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- $n = 8$ For $k = 25624$, $P_{25624} \approx 79\%$

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Conjecture

There are at least $50,000$ quadratic APN functions on 8 bits.

How to get them?

Using the QAM method

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- $n = 7$ We need $3000 \approx 8 \times \ell_7$ QAMs to obtain all $\ell_n = 488$
- $n = 8$?

Conjecture

For $n = 8$, we would need to generate $4 \times \ell_8 \approx 200,000$ QAMs to generate all of them, i.e. about 50 CPU·year.

Plan of this Section

Conclusion

There are many 8-bit quadratic APN functions!