Algebraic cryptanalysis: how Gröbner bases techniques can be used in cryptanalysis

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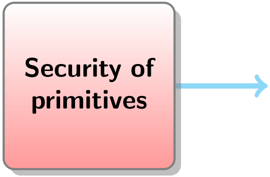
Algebraic Cryptanalysis



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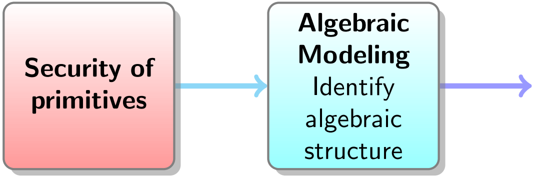

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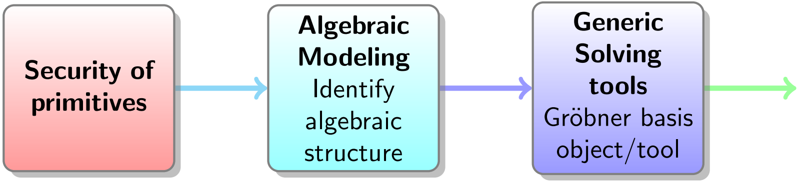

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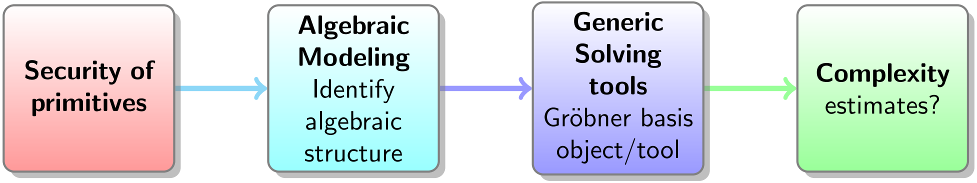

Algebraic Cryptanalysis can be devastating

Famous practical cryptanalyses in $\simeq$ **2 days**:

- attacking first HFE Challenge (80 bits) (J.-C. Faugère and Joux 2003)
- attacking finalist Rainbow (128 bits) (Beullens 2022)

Many other examples in the literature.

Algebraic Modeling

Principle: write a Polynomial System

$$\begin{cases} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{cases}, \quad \deg(f_i)=d_i, f_i \in \mathbb{K}[x_1,\dots,x_n].$$

such that finding the set of solutions

$$V(f_1,\dots,f_m) = \{(x_1,\dots,x_n) \in \overline{\mathbb{K}}^n : f_i(x_1,\dots,x_n)=0, \forall i \in \{1..m\}\}$$

gives (part of) the secret.

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Ideally: *any* solution is related to the secret!

- Otherwise, we have to deal with spurious solutions.
- Solutions in $\mathbb{F}_q$: algebraic constraint! add the field equations $x_i^q - x_i$.

Solving the algebraic system using Gröbner bases (object)

- A particular basis of the ideal

$$I(f_1, \ldots, f_m) = \langle f_1, \ldots, f_m \rangle$$

  that solves the ideal-membership problem.
- Depends on the choice of a monomial ordering.

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A hard problem

- Ideal Membership testing is EXPSPACE-complete,
- Existence of solutions to a system of polynomial equations over a finite field is NP-complete (Fraenkel and Yesha 1979),

Gröbner basis algorithms

General algorithms, for any input system:

- Buchberger (Buchberger 1965),
- F4 (J.-C. Faugère 1999),
- F5 (J.-C. Faugère 2002).

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Specific algorithms, for a particular class of systems:

The algorithms will terminate in a predictable time.
The result is not always a Gröbner basis of the system.
For random instances in the specific class, the result is a Gröbner basis.

Properties of monomial orderings

Different monomial orderings have different properties

- the *lex* order (Lexicographical): in Shape Position, for a zero-dimension ideal, the lex basis is

$$\begin{cases} x_1 - g_1(x_n), \\ \vdots \\ x_{n-1} - g_{n-1}(x_n), \\ g_n(x_n), \end{cases}$$

  with $\deg(g_n) = D$ the number of solutions to the system.
- the *grevlex* order (Graded Reverse Lexicographical): usually the best one w.r.t. the complexity.
- the *elim* order (Elimination): two blocks of variables $\mathbf{x} > \mathbf{y}$.

Monomial ordering examples

Lexicographical ordering $x_1 > \cdots > x_n$

$$x_1^{\alpha_1} \ldots x_n^{\alpha_n} > x_1^{\beta_1} \ldots x_n^{\beta_n} \text{ iff } \begin{cases} \alpha_j = \beta_j & \forall j < i, \\ \alpha_i > \beta_i. \end{cases}$$

Graded Reverse Lexicographical ordering $x_1 > \cdots > x_n$

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Elimination Ordering $\mathbf{x} > \mathbf{y}$

$$\mathbf{x}^{\alpha}\mathbf{y}^{\beta} > \mathbf{x}^{\alpha'}\mathbf{y}^{\beta'} \text{ iff } \begin{cases} \alpha >_1 \alpha' \\ \text{or } \alpha = \alpha' \text{ and } \beta >_2 \beta'. \end{cases}$$

Family of random zero-dimensional systems

Hypotheses for cryptanalysis

- the variety is zero-dimensional (otherwise, change the modeling!).
- the instances are “random” (not the system).

Change of ordering FGLM for zero-dimensional systems

- The FGLM (J.-C. Faugère, Gianni, Lazard, and Mora 1993) Algorithm performs a change of ordering in complexity

$$O(nD^3),$$

n number of variables, $n \to \infty$, D degree of the ideal (number of solutions).

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- Sparse versions for generic systems grevlex to lex (J.-C. Faugère and Mou 2017) in

$$O\left(\sqrt{\frac{6}{n\pi}}D^{2+\frac{n-1}{n}}\right).$$

Systems with 0 or 1 solution

The grevlex and lex bases are the same:

- If the system has 1 solution:

$$\begin{cases} x_1 - a_1, \\ \vdots \\ x_n - a_n, \end{cases}$$

  where $(a_1, \ldots, a_n) \in \mathbb{F}_q^n$ is the solution.
- If the system has no solution:

$$\langle 1 \rangle.$$

Adding the field equations

Should I add the field equations to the system?

- Does the ideal have solutions in the algebraic closure of $\mathbb{F}_q$? How many?
- Is the maximal degree D reached during the computation smaller than q?
- Are there solutions in $\overline{\mathbb{F}_q}$ that I'm not interested in?

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When should I add the field equations?

- from the beginning,
- to the lex basis (gcd).

Complexity of computing a Gröbner basis

- worst case: doubly exponential! polynomials of degree d^{2^n} in the basis, any monomial ordering (Mayr and Meyer 1982).
- zero-dimensional, grevlex: simply exponential (Lazard 1983; Giusti 1984).
- relation to linear algebra for the computation: Macaulay matrices.

Tools from computer algebra toward complexity analysis

$$\text{System} \begin{cases} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_m(x_1, \ldots, x_n) \end{cases}, \quad \deg(f_i) = d_i, f_i \in \mathbb{K}[x_1, \ldots, x_n].$$

- Macaulay Matrices (Macaulay 1902):

$$\mathcal{M}_d(\{f_1, \ldots, f_m\}) = \begin{matrix} \vdots \\ (t, i) \\ \vdots \end{matrix} \overset{t'}{\begin{pmatrix} & & \\ & \text{coeff}(tf_i, t') & \\ & & \end{pmatrix}}$$

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- Describes the vector space $\langle tf_i : \deg(tf_i) = d \rangle_{\mathbb{K}}$.
- Linear algebra on the Macaulay matrices up to degree D computes a Gröbner basis (Lazard 1983,Giusti 1984).

Complexity bounds

Linear algebra on the Macaulay matrix of degree D

A Gröbner basis of a system $(f_1, \ldots, f_m) \in \mathbb{K}[x_1, \ldots, x_n]$ up to degree D for a graded monomial ordering can be computed in, at most,

$$O\left(mD\binom{n+D-1}{D}^{\omega}\right) \qquad n, m \to \infty.$$

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Main challenges

- Estimate D.
- Identify unnecessary computations to reduce the complexity, e.g. to $O\left(\binom{n+D}{D}^{\omega}\right)$.
- If there are fall degree at degree $< D$, construct a better strategy (algorithm) to take that into account, and estimate its complexity.

Generic Complexity analysis

Known classes of particular systems (not exhaustive)

- **regular** systems (Macaulay 1916), $\#$ eq $\leqslant \#$ vars,
- **determinantal** systems (Conca and Herzog 1994),
- **semi-regular** systems (Bardet, J.-C. Faugère, and Salvy 2004), $\#$ eq $\geqslant \#$ vars,
- solutions in $\mathbb{F}_2$: **boolean semi-regular** systems (Bardet, J.-C. Faugère, Salvy, and Yang 2005),
- **bi-regular bilinear** systems (J.-C. Faugère, Safey El Din, and P.-J. Spaenlehauer 2011).

Difference between classes

$$O\left(mD\binom{n+D-1}{D}^{\omega}\right) \qquad\qquad n,m\rightarrow\infty.$$

Examples of quadratic equations:

- $m=n$ regular system: : $D\leqslant n+1$,
- $m=n+1$ semi-regular system: $D\leqslant\lceil\frac{n+2}{2}\rceil$,
- $m=n$ regular bilinear system with $\lfloor\frac{n}{2}\rfloor$ variables x and $\lceil\frac{n}{2}\rceil$ variables y: $D\leqslant\lceil\frac{n}{2}\rceil$.
- $m=n$ regular over $\mathbb{F}_2$: $D\simeq\frac{n}{11}$, $O(\binom{n}{D}^{\omega})$

Algebraic attack

For each class we know

- relations between rows in the Macaulay matrices,
- the rank of the Macaulay matrices for generic systems,
- the maximal degree $D \rightarrow$ complexity estimates,
- a specific Gb algorithm that is more efficient.

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If the system is not in a known class

- Identify a generic behavior,
- Identify a specific algorithm to compute the Gb,
- Create a new class!

Homogeneous vs Affine

- All bounds are given for homogeneous polynomials.

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- In this case, D is the *first fall degree*, also the *solving degree* and the *degree of regularity* and is related to the complexity.
- Otherwise, first fall degree is not related to complexity estimates!

The MinRank Problem

- `Input`: integers $r, m, n \in \mathbb{N}$, and K matrices $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_K \in \mathbb{F}_q^{m \times n}$
- `Output`: $(x_1, \ldots, x_K) \in \mathbb{F}_q$, not all zero, such that

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- No need to add the field equations: already in the ideal!
- For very small q (e.g. $q = 2$): adding small degree equations can speed up the computation.

MinRank problem $\text{Rank}\left(\sum_{i=1}^{K} x_i \boldsymbol{M}_i\right) \leqslant r$

- Kipnis-Shamir modeling (Kipnis and Shamir 1999)

$$\left(\sum_{i=1}^{K} x_i \boldsymbol{M}_i\right) \begin{pmatrix} \boldsymbol{I}_{n-r} \\ -\boldsymbol{R} \end{pmatrix} = 0_{m\times(n-r)}, \quad \boldsymbol{R} \in \mathbb{F}_q^{r\times(n-r)}, x_i \in \mathbb{F}_q \qquad \text{(KS)}$$

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- Same ideal ! (Bardet and Bertin 2022; Guo and Ding 2022)

Specific systems

GeMSS signature scheme (Casanova, J. Faugère, Macario-Rat, Patarin, et al. 2019)

- alternate candidate (3rd Round of the NIST process) that suffered a MinRank attack (Tao, Petzoldt, and Ding 2021),
- the system has m solutions in an extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$,
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$\mathbb{F}_{q^m}$-linear codes

- the equations are not linearly independent! + a lot of linear equations.
- Bardet, Briaud, Bros, Gaborit, and Tillich 2022: specific analysis of the system.

Gröbner bases in Magma

Computer algebra system `magma`

- default strategy: compute the grevlex basis, then change to the lex basis using FGLM.
- `lex` by default, you can specify `"grevlex"` in the polynomial ring.
- grevlex basis computed using F4, with several heuristics (`SetVerbose("Faugere",2)`)
- an input parameter for HFE-like systems, to save memory and time.

Conclusion

- A powerful tool to solve problems that have an algebraic modeling,
- A lot of parameters to choose,
- Design specific algorithms for specific class of systems to be efficient,
- Already a lot of applications on arithmetization-oriented symmetric-key primitives.

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A PhD position is available in Rouen, starting in fall, algebraic cryptanalysis.

- Banea, J., P. Briaud, D. Cabarcas, R. Perlner, et al. (Aug. 2022). "Improving Support-Minors Rank Attacks: Applications to GeMSS and Rainbow". In: *Advances in Cryptology – CRYPTO 2022*. Ed. by Y. Dodis and T. Shrimpton. Vol. Part III. LNCS. Cham: Springer Nature Switzerland, pp. 376–405.
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