CHAPITRE 2: OTHER OPTIMIZATION METHODS, EXAMPLE OF MONOTONIC ALGORITHM

1. Introduction

In this chapter, we present other approaches to solve the optimization problem associated to an optimal control problem, i.e., its optimality system. We briefly describe the shooting method and detail a specific method, namely, the monotonic algorithms, for which we prove the convergence.

2. Shooting method

In this section, we briefly present another approach to solve optimality systems. Recall that in the case of the optimal control considered in Chapter 1, this ones reads

$$\begin{cases} \dot{y}(t) & = A(y(t), c(t)) \\ y(t=0) & = y_{init} \\ \dot{p}(t) + \partial_y A(y(t), c(t))^T p(t) & = 0 \\ p(t=T) & = 2(y(T) - y_{cible}) \\ 2\alpha c(t) + \partial_c A(y(t), c(t))^T p(t) & = 0. \end{cases}$$

In the gradient methods considered in Chapter 1, the first two equations are solved exactly forward and backward, respectively (in a discretized version), whereas a fixed point iteration is applied on the last one. In shooting methods, one proceeds in a very different way. One consider as an unknown $p(t=0)$, solve forward the equations on y and p using $c(t) = -\frac{1}{2\alpha}\partial_c A(y(t), c(t))^T p(t)$ and iterate to solve $p(t=T) = 2(y(T) - y_{cible})$. The Newton method is generally used to design the iteration.

3. Monotonic Algorithms

3.1. **Idea of the Monotonic algorithms.** Let us briefly present the monotonic schemes in the simple case of ordinary differential equations (ODE). Let M, K be three square matrices in $\mathcal{M}_m(\mathbb{R})$, $\alpha > 0$ and $T > 0$. Consider the optimal control problem corresponding to the minimization of the functional J defined by:

$$J(c) = -y(T) \cdot y_{target} + \frac{\alpha}{2} \int_0^T c^2(t) dt,$$

where " $\cdot$ " denotes the usual scalar product of $\mathbb{R}^m$. Here, the state $y : [0, T] \to \mathbb{R}^m$ and the control $c : [0, T] \to \mathbb{R}$ are linked by the ODE:

$$\begin{cases} \dot{y}(t) = \big(M + c(t)K\big) y(t), \ \forall t \in (0, T) \\ y(0) = y_{init}, \end{cases}$$

where the initial condition y_{init} being fixed.

Given two controls c and c' and the corresponding states y and y', we first note that:

$$\begin{aligned} J(c') - J(c) &= -\big(y'(T) - y(T)\big) \cdot y_{target} \\ &\quad + \frac{\alpha}{2} \int_0^T \big(c'(t) - c(t)\big)\big(c(t) + c(t)\big) dt. \end{aligned}$$

We then introduce an auxiliary function $p : [0, T] \to \mathbb{R}^m$ associated to y and c by

$$\left\{ \begin{array}{lcl} p'(t) &=& -\big(M^* + c(t)K^*\big)p(t), \\ p(T) &=& -y_{target}, \end{array} \right.$$

where M^* and K^* are the transposed matrices of M and K.
Focusing on the first term of the right hand side of this equation, we get:

$$-\big(y'(T) - y(T)\big) \cdot y_{target} = \int_0^T \big(c'(t) - c(t)\big) K y'(t) \cdot p(t) dt.$$

Thus, we finally obtain:

$$J(c') - J(c) = \frac{\alpha}{2} \int_0^T \big(c'(t) - c(t)\big) \left(\frac{2}{\alpha} K y'(t) \cdot p(t) + c'(t) + c(t) \right) dt.$$

A simple way to guarantee that c' gives a functional value lower than c, consists in imposing that:

$$(1) \qquad \big(c'(t) - c(t)\big) \left(\frac{2}{\alpha} K y'(t) \cdot p(t) + c'(t) + c(t) \right) \leq 0.$$

Following this approach, the sequence $(c^k)_{k \in \mathbb{N}}$ defined iteratively by the implicit equation

$$c^{k+1}(t) - c^k(t) = -\lambda \left(\frac{2}{\alpha} K y^{k+1}(t) \cdot p^k(t) + (c^{k+1}(t) + c^k(t)) \right)$$

where y^{k+1} and p^k correspond to c^{k+1} and c^k respectively, optimizes J monotonically since

$$J(c^{k+1}) - J(c^k) = -\frac{\alpha}{\lambda} \int_0^T \big(c^{k+1}(t) - c^k(t)\big)^2 dt \leq 0.$$

3.2. **Discretized monotonic schemes.** We now focus on a finite dimensional Schrödinger equation which reads:

$$(2) \qquad \left\{ \begin{array}{l} \dot{y}(t) = i\big(A + c(t)B\big)y(t), \ \forall t \in (0, T) \\ y(0) = y_{init} \end{array} \right.$$

the initial condition y_{init} being fixed. It can easily been shown that if the solution exists, it satisfies $\|y(t)\|_2 = \|y_{init}\|_2$.

Remarque 1. *In infinite dimensional setting, the Schrödinger equation is given by*

$$i\partial_t y(x, t) - [H - \mu(x)c(t)]y(x, t) = 0,$$

for which we also have $\|y(t)\|_2 = \|y_{init}\|_2$. This equation governs the evolution of a quantum system, described by its wave function y, that interacts with a laser pulse of amplitude c, the control variable. The factor μ is the dipole moment operator of the system. In what follows, $H = -\Delta + V$ where Δ is the Laplacian operator and $V = V(x)$ the electrostatic potential in which the system evolves.

In this part, for any prescribed sequences c, c', the approximations of the state y and the adjoint state p are thus defined by the semi-discretized propagation equations:

$$\text{(3)} \qquad \left\{ \begin{array}{rcl} y_{n+1} & = & e^{iA\Delta T} e^{ic_n B \Delta T} y_n \\ & & \\ y_0 & = & y_{init}, \end{array} \right.$$

Recall that this discretization of (2) preserves the L^2-norm of the propagated vectors, which means:

$$\text{(4)} \qquad \forall n = 0, \ldots, N, \ \|y_n\|_2 = \|y_{init}\|_2 = 1.$$

We consider the following time discretization of the discrete cost functional:

$$J_{\Delta T}(c) = -Re < y_N, y_{target} > + \alpha \Delta T \sum_{n=0}^{N-1} |c_n|^2.$$

Let us also introduce norms on $\mathbb{R}^N$ corresponding to the time discretization:

$$\|c\|_1 = \Delta T \sum_{n=0}^{N-1} |c_n|, \ \|c\|_2 = \left(\Delta T \sum_{n=0}^{N-1} |c_n|^2 \right)^{\frac{1}{2}}.$$

Note that the inner product associated with the norm $\|.\|_2$ is defined by :

$$c.c' = \Delta T \sum_{n=0}^{N-1} c_n \ c'_n.$$

3.3. **Definition of the schemes.** A computation of the variation in the cost functional values similar to the one done in Section 3.1 leads to:

$$J_{\Delta T}(c^{k+1}) - J_{\Delta T}(c^k) = \frac{\alpha}{2} \Delta T \sum_{n=0}^{N-1} (c_n^{k+1} - c_n^k) \left(c_n^{k+1} + c_n^k + \frac{2}{\alpha} Re(< i\widetilde{B}(c_n^{k+1}, c_n^k) y_n^{k+1}, p_n^k >) \right),$$

where

$$\widetilde{B}(c_n^{k+1}, c_n^k) = \frac{\exp(i(c_n^{k+1} - c_n^k) B \Delta T) - Id}{i(c_n^{k+1} - c_n^k)\Delta T}.$$

Remarque 2. *At this step, we can already use this equation to debug our code: one can indeed compute independently both sides of the identity and check that they gives the same result.*

This function is an approximation of μ inasmuch as:

$$\text{(5)} \qquad \|\widetilde{B}(h) - B\|_* \leq \Delta T \|B\|_*^2 |h|,$$

which can be obtained by the mean value inequality.Given initial control fields c^0 and the associated state y^0 and adjoint state p^0, suppose that for some $k \geq 1$, y^{k-1}, p^{k-1}, c^{k-1} are known. The computation of y^k, p^k, c^k is achieved as follows:

$$(6)\qquad \begin{cases} y_{n+1}^k &= e^{iA\Delta T} e^{ic_n^k B\Delta T} y_n^k \\ c_n^k &= (1-\delta)c_n^{k-1} + \delta\left(-\frac{1}{\alpha} Re < i\widetilde{B}(c_n^k, c_n^{k-1}) y_n^k, p_n^{k-1} >\right) \\ y_0^k &= y_{init}, \end{cases}$$

$$(7)\qquad \begin{cases} p_n^k &= e^{-ic_n^k B\Delta T} e^{-iA\Delta T} p_{n+1}^k \\ p_N^k &= -y_{target}. \end{cases}$$

This scheme is design in such a way that

$$c_n^{k+1} - c_n^k = -\lambda\left(c_n^{k+1} + c_n^k + \frac{2}{\alpha} Re(< i\widetilde{B}(c_n^{k+1}, c_n^k) y_n^{k+1}, p_n^k >)\right)$$

meaning that

$$(8)\qquad J_{\Delta T}(c^{k+1}) - J_{\Delta T}(c^k) = -\frac{\alpha}{2\lambda}\Delta T \sum_{n=0}^{N-1} (c_n^{k+1} - c_n^k)^2,$$

with $\lambda = \frac{\delta}{2-\delta}$, i.e., $\delta = \frac{2\lambda}{1+\lambda}$. Subsequently, the initial value c^0 of the monotonic schemes is considered fixed. A first property of $(c^k)_{k\in\mathbb{N}}$ defined in (6) is that this sequences is bounded. Indeed, the following result can be proved by induction.

Lemme 1. *Assume that $\|B\| < +\infty$. Given an initial field c^0, let us defined M by:*

$$(9)\qquad M = \max(\|c^0\|_\infty, \max(1, \frac{\delta}{2-\delta})\frac{\|B\|_*}{\alpha}).$$

The sequence $(c^k)_{k\in\mathbb{N}}$ are well defined and match the following conditions:

$$(10)\qquad \forall k \in \mathbb{N},\ \forall n = 0, \dots, N-1,\ |c_n^k| \leq M.$$

Proof. Suppose that (6) and (7) admit solution c_n^{k+1}; let us prove (10). First, the mean value inequality yields:

$$(11)\qquad \forall x, y \in \mathbb{R}, \frac{\|e^{B(x-y)i\Delta T} - Id\|_*}{|x-y|\Delta T} \leq \|B\|,$$

from which we deduce that $\|B^\star\|_* \leq \|B\|$ from the definition (13). Secondly, because of (6) and (7), we have

$$\|y_{n+1}^k\|_2 = \|y_{init}\|_2 = 1,\ \|p_n^{k-1}\|_2 = \|y_{target}\|_2 = 1.$$

Thanks to the Cauchy-Schwartz inequality, Inequality (11) and the definition of $\widetilde{B}$ give:

$$\forall y, p \in \mathbb{R}^m, \|y\|_2 = \|p\|_2 = 1,\ c, c' \in \mathbb{R} | < i\widetilde{B}(c, c')y, p > | \leq \frac{\|B\|}{\alpha}.$$

Assume $|c_n^k| \leq M$ has been obtained, then:

$$(12)\qquad |c_n^{k+1}| \leq |1-\delta| M + \delta\frac{\|B\|}{\alpha}.$$

If $\delta \leq 1$, then $M = \frac{\|B\|}{\alpha}$ and $|c_n^k| \leq |1-\delta|M + \delta M = M$, otherwise $M = \frac{\delta}{2-\delta}\frac{\|B\|}{\alpha}$ and in this case $|c_n^k| \leq |1-\delta|M + \delta\frac{2-\delta}{\delta}M = (\delta-1)M + (2-\delta)M = M$, which ends the proof of (10) by induction.

Given a field c^k, (10)implies that $f : x \mapsto (1-\delta)c_n^k + \delta\frac{2}{\alpha}Re(- < i\widetilde{B}(x, c_n^k)y_n^{k+1}, p_n^k >)$ maps the interval $[-M, M]$. The intermediate value theorem states that there exists a fixed-point for f in $[-M, M]$. □

Identity (8), combined with (10) leads to the following result.

Theorem 1. *The implicit schemes (6)–(7) ensure the monotonic convergence of the cost functional $J_{\Delta T}$ insofar as there exists l_{c^0} such that:*

$$\lim_{k \to +\infty} J_{\Delta T}(c^k) = \ell_{c^0}.$$

Proof. Because of (8), the sequence $(J_{\Delta T}(\varepsilon^k))_{k\in\mathbb{N}}$ is monotonically decreasing. By means of (10), we know that:

$$\forall k \in \mathbb{N}, \quad J_{\Delta T}(c^k) \geq -1,$$

hence the existence of ℓ_{c^0}. □

We keep the notation ℓ_{c^0} in the sequel.

3.4. **Cauchy property of the monotonic sequences.** We will now prove the convergence of sequence $(c^k)_{k\in\mathbb{N}}$.

Theorem 2. *Sequence $(c^k)_{k\in\mathbb{N}}$ defined by (6)–(7) converges towards a critical point of $J_{\Delta T}$.*

In order to prove this result, let us define the shifted cost functional:

$$\widetilde{J}_{\Delta T}(c) = J_{\Delta T}(c) - \ell_{c^0}.$$

In case there exists k_1 such that $\widetilde{J}_{\Delta T}(c^{k_1}) = 0$, the monotonicity of the algorithm implies that $J_{\Delta T}(c^{k_1}) = J_{\Delta T}(c^{k_1+1}) = J_{\Delta T}(c^{k_1+2}) = \ldots$ and according to (8) the sequence $(c^k)_{k\in\mathbb{N}}$ is constant for $k \geq k_1$. We then assume that $\widetilde{J}_{\Delta T}(c^k) \neq 0$ for all $k \in \mathbb{N}$.
The proof follows from the next sequence of inequalities, that holds for all $k \geq k_0$, for some $k_0 \in \mathbb{N}$, $\widetilde{\theta} \in (0, 1/2]$, $\widetilde{\kappa} > 0$, $\nu > 0$:

$$\begin{align*}
(\widetilde{J}_{\Delta T}(c^k))^{\widetilde{\theta}} - (\widetilde{J}_{\Delta T}(c^{k+1}))^{\widetilde{\theta}} &\geq \frac{\widetilde{\theta}}{(\widetilde{J}_{\Delta T}(c^k))^{1-\widetilde{\theta}}}\big(J_{\Delta T}(c^k) - J_{\Delta T}(c^{k+1})\big) \\
&\geq \frac{\widetilde{\theta}\alpha}{2\lambda(\widetilde{J}_{\Delta T}(c^k))^{1-\widetilde{\theta}}}\|c^{k+1} - c^k\|_2^2 \tag{13} \\
&\geq \frac{\widetilde{\kappa}\widetilde{\theta}\alpha}{\lambda\|\nabla J_{\Delta T}(c^k)\|_1}\|c^{k+1} - c^k\|_2^2 \tag{14} \\
&\geq \frac{\widetilde{\kappa}\widetilde{\theta}\alpha}{\lambda\nu}\|c^{k+1} - c^k\|_2. \tag{15}
\end{align*}$$

Given $q \in \mathbb{N}$, Inequality (15) leads to:

$$\begin{aligned}(\widetilde{J}_{\Delta T}(c^k))^{\widetilde{\theta}} - (\widetilde{J}_{\Delta T}(c^{k+q}))^{\widetilde{\theta}} &\geq \frac{\widetilde{\kappa}\widetilde{\theta}\alpha}{\lambda} \sum_{l=k}^{k+q-1} \|c^{l+1} - c^l\|_2 \\ &\geq \frac{\widetilde{\kappa}\widetilde{\theta}\alpha}{\lambda\nu} \|c^{k+q} - c^k\|_2.\end{aligned}$$

Since $\left((\widetilde{J}_{\Delta T}(c^k))^{\widetilde{\theta}}\right)_{k\in\mathbb{N}}$ is a Cauchy sequence, we conclude that the sequence $(c^k)_{k\in\mathbb{N}}$ is also Cauchy.

Let us explain each of the previous inequalities. The first one just follows from the concavity of $x \mapsto x^{\widetilde{\theta}}$. Inequality (13) follows from (8). Inequality (14) follows from the so-called Łojasiewicz inequality which in our case reads

$$d(c, C_{c^0}) < \widetilde{\sigma}, \qquad \|\nabla J_{\Delta T}(c)\|_1 \geq \widetilde{\kappa} |\widetilde{J}_{\Delta T}(c)|^{1-\widetilde{\theta}}, \tag{16}$$

where C_{c^0} is the limit points set of the sequence $(c^k)_{k\in\mathbb{N}}$. The proof that $d(c^k, C_{c^0}) \to 0$ and of (16) is given in Appendix. Finally, Inequality (15) follows from the next lemma:

Lemme 2. *The gradient of $J_{\Delta T}$ is then given by:*

$$\nabla J_{\Delta T}(c) \cdot \delta c = \Delta T \sum_{n=0}^{N-1} (Re < iBy_n, p_n > + \alpha c_n)\delta c_n, \tag{17}$$

where y and p are associated with c. Moreover, there exists $\nu \geq 0$ such that:

$$\|\nabla J_{\Delta T}(c^k)\|_1 \leq \nu(\|c^{k+} - c^k\|_2). \tag{18}$$

Proof. Equation (17) follows from the results of Chapter 1. Let us focus on one coefficient of $\nabla J_{\Delta T}(c^k)$, we find:

$$\begin{aligned}Re < iBy_n^k, p_n^k > + \alpha c_n^k &= Re < i\widetilde{B}(c_n^{k+1} - c_n^k) y_n^{k+1}, p_n^k > + \alpha c_n^k \\ &\quad + Re < i(B - \widetilde{B}(c_n^{k+1} - c_n^k)) y_n^{k+1}, p_n^k > \\ &\quad - Re < iB(y_n^{k+1} - y_n^k), p_n^k > .\end{aligned} \tag{19}$$

Let us estimate each term of this decomposition. Definition (6) of the algorithm implies that:

$$Re < i\widetilde{B}(c_n^{k+1} - c_n^k) y_n^k, p_n^{k-1} > + \alpha c_n^k = -\frac{\alpha}{\delta}(c_n^{k+1} - c_n^k). \tag{20}$$

Then, thanks to (5), we obtain:

$$Re < i(B - \widetilde{B}(c_n^{k+1} - c_n^k)) y_n^{k+1}, p_n^k > \leq \Delta T \|B\|_*^2 |c_n^{k+1} - c_n^k|. \tag{21}$$

The last term is estimated by a discrete Gronwall inequality as follows:

$$\begin{aligned}y_{n+1}^{k+1} - y_{n+1}^k =& e^{iA\Delta T}\left(e^{i(c_n^{k+1} - c_n^k)B\Delta T} - Id\right) e^{ic_n^k B\Delta T} y_n^{k+1} \\ &+ e^{iA\Delta T} e^{ic_n^k B\Delta T}(y_n^{k+1} - y_n^k),\end{aligned}$$

which gives, thanks to (11) and the fact that the matrices involved are unitary:

$$\|y_{n+1}^{k+1} - y_{n+1}^k\|_2 \leq |c_n^{k+1} - c_n^k| \|B\|_* \Delta T + \|y_n^{k+1} - y_n^k\|_2.$$

Using that $\|p_N^k - p_N^{k-1}\|_2 = 0$, we get

$$\|y_n^{k+1} - y_n^k\|_2 \leq \|c^{k+1} - c^k\|_1 \|B\|_*.$$

We then obtain (18) with

$$\nu = \frac{\alpha}{\delta} + (T + \Delta T)\|B\|_*^2.$$

□

4. To go further

The greater part of this lecture is inspired from

J. Salomon, "Convergence of the time-discretized monotonic schemes", ESAIM Math. Model. Numer. Anal., pp. 77–93 (2007).

To see other convergence analysis based on the Łojasiewicz inequality, we refer to:

- J. Bolte and H. Attouch, *"On the convergence of the proximal point algorithm for nonsmooth functions involving analytic features"*, Mathematical Programming volume 116, pp. 516(2009).
- A. Levitt, *"Convergence of gradient-based algorithms for the Hartree-Fock equations"*, ESAIM Math. Model. Numer. Anal. 46.06 (2012).
- M. Lewin and S. Paul, *"A numerical perspective on Hartree-Fock-Bogoliubov theory"*, ESAIM: M2AN, 48(1):53–86, (2014).
- E. Cancès, V. Ehrlacher and T. Lelièvre, *"Greedy algorithms for highdimensional eigenvalue problems"*, Constr. Approx. 40, pp. 387–423 (2014).

Appendix

The previous computations allows us to define set C of the critical points of $J_{\Delta T}$:

$$C = \Big\{c/\ \forall n = 0, \dots, N-1,\ Re < iBy_n, p_n > +\alpha c_n = 0\Big\}, \tag{22}$$

4.1. Critical points and limit points.

We first establish a relation between C and C_{c^0}.

Lemme 3. *Keeping the previous notations:*

$$C_{c^0} \subset C,$$

where C is the set of critical points, defined by (22).

Proof. : Consider a convergent subsequence $(c^{k_\ell})_{\ell\in\mathbb{N}}$ of $(c^k)_{k\in\mathbb{N}}$ and its limit c^∞. Denote by y^{c^∞} and p^{c^∞} the corresponding state and adjoint state. By means of continuity and since $\|c^{k_\ell} - c^{k_\ell-1}\|_2 \to 0$ (see (8)), we find the following limits:

$$\begin{array}{rcl} c^{k_\ell-1} & \to & c^\infty, \\ B^*(c_n^{k_\ell} - c_n^{k_\ell-1}) & \to & B, \\ y^{k_\ell} & \to & y^{c^\infty}, \\ p^{k_\ell-1} & \to & p^{c^\infty}. \end{array}$$

When n tends to $+\infty$, Equation (6) becomes:

$$\forall j = 0, \dots, N-1, \quad c_n^\infty = -\frac{1}{\alpha} Re < iBy_n, p_n >,$$

which is the desired conclusion. □

Thanks to (10), a standard argument of compactness shows that:

$$d(c^k, C_{c^0}) \to 0, \tag{23}$$

where $d(c^k, C_{c^0})$ is the distance associated to $\|.\|_2$ between c^k and set C_{c^0}. Furthermore, Equation (8) then implies:

$$J_{\Delta T}(C_{c^0}) = \ell_{c^0}. \tag{24}$$

Finally, note that since C is compact (see (22)), C_{c^0} is also compact.

4.2. **Łojasiewicz inequality.** The Łojasiewicz inequality enables us to estimate the variation in an analytic function by its gradient. This result is detailed in the next theorem.

Theorem 3 (Łojasiewicz inequality)**.** *Let $\Gamma : \mathbb{R}^N \to \mathbb{R}$ be an analytic function in a neighborhood of a point a in $\mathbb{R}^N$. Then there exists $\sigma > 0$ and $0 < \theta \leq \frac{1}{2}$ such that:*

$$\forall x \in \mathbb{R}^N,\ x \in B(a, \sigma) \qquad \|\nabla\Gamma(x)\| \geq |\Gamma(x) - \Gamma(a)|^{1-\theta},$$

where $B(a, \sigma)$ denotes the ball centered in a with a radius equal to σ, $\|.\|$ is a norm on $\mathbb{R}^N$.

The real number θ is called a Łojasiewicz exponent of a. A more precise result can be obtained if the Hessian matrix of Γ, denoted by $H_\Gamma(a)$, is invertible.

Lemme 4. *Suppose that $H_\Gamma(a)$ is invertible, then there exists $\sigma > 0$ and $\kappa > 0$ such that:*

$$\forall x \in \mathbb{R}^N,\ x \in B(a, \sigma) \qquad \|\nabla\Gamma(x)\| \geq \kappa|\Gamma(x) - \Gamma(a)|^{\frac{1}{2}}.$$

It is easy to check that $\widetilde{J}_{\Delta T}$ is analytic. Now, consider a in C_{c^0}. By means of (24), we find that $\widetilde{J}_{\Delta T}(a) = 0$. Consequently, Theorem 3 ensures that there exist $0 < \theta_a \leq 1/2$ and $\sigma_a > 0$ such that:

$$\forall c \in \mathbb{R}^N,\ \|c - a\|_1 < \sigma_a \qquad \|\nabla J_{\Delta T}(c)\|_1 \geq |\widetilde{J}_{\Delta T}(c)|^{1-\theta_a}.$$

The compactness of C_{c^0} enables us to extract from the set $\{B(a, \frac{\sigma_a}{2}), a \in C_{c^0}\}$ a set $F_{c^0} = \cup_{i\in I} B(a_i, \frac{\sigma_{a_i}}{2})$, where I is a finite set of indexes such that:

$$C_{c^0} \subset F_{c^0}.$$

Let us denote by $\theta' \in]0, 1/2]$, the lower bound of $\{\theta_{a_i}\}_{i\in I}$ and by σ', the lower bound of $\{\frac{\sigma_{a_i}}{2}\}_{i\in I}$. Given c such that $d(c, C_{c^0}) < \sigma'$, there exists i_c such that $c \in B(a_{i_c}, \sigma_{a_{i_c}})$ and we come to the following version of Theorem 3:

Lemme 5. *Keeping the above notations:*

$$\forall c \in \mathbb{R}^N,\ d(c, C_{c^0}) < \sigma', \qquad \|\nabla J_{\Delta T}(c)\|_1 \geq |\widetilde{J}_{\Delta T}(c)|^{1-\theta'}.$$

In case $H_{J_{\Delta T}}$ is invertible on C_{c^0}, a similar analysis can be led to obtain the corresponding version of Lemma 4.

Lemme 6. *Suppose that $H_{J_{\Delta T}}(c)$ is invertible for all $c \in C_{c^0}$,*

$$\exists\sigma'' > 0, \exists\kappa' > 0, \forall c \in \mathbb{R}^N,\ d(c, C_{c^0}) < \sigma'', \quad \|\nabla J_{\Delta T}(c)\|_1 \geq \kappa'|\widetilde{J}_{\Delta T}(c)|^{\frac{1}{2}}.$$

Summarizing the above mentioned results, we have obtained that there exist $\widetilde{\sigma} > 0$, $\widetilde{\kappa} > 0$ and $\widetilde{\theta} \in]0, \frac{1}{2}]$ such that:

$$d(c, C_{c^0}) < \widetilde{\sigma}, \qquad \|\nabla J_{\Delta T}(c)\|_1 \geq \widetilde{\kappa} |\widetilde{J}_{\Delta T}(c)|^{1-\widetilde{\theta}},$$

with $\widetilde{\theta} = 1/2$ when $H_{J_{\Delta T}}(c)$ is invertible.