Université Paris-Dauphine Année 2020-2021

MIDO-M2 Mathématiques et Applications / Mathématiques Appliquées et Théoriques

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Examen - Introduction au calcul scientifique et à l'analyse numérique- Année 2020-2021

Justify **EACH** *answer.*

Exercise 1. Finite elements

We consider the elliptic problem on $\Omega = (0,1)$.

$$\begin{aligned} -u'' &= f \text{ on } (0,1) \\ u(0) &= 0 \\ u'(1) &= 1. \end{aligned}$$

Part 1 : functional analysis

1. To which boudary conditions correspond this problem ?
2. Explain how to get a weak formulation, and give this weak formulation.
3. Which functional space fits with the weak formulation ?

Part 2 : finite elements In order to solve this system numerically, we use the finite element described as follows

— we choose as a mesh the points $x_i = \frac{i}{N}$ for $i = 0, \cdots, N$,
— on an element $[x_i, x_{i+1}]$, we choose the nodal points $x_i,\ x_{i+1}$,
— we choose as a fintie element space P_1, i.e. piecewise affine functions,

1. Give a variational formulation of the problem.
2. Give an example of basis obtained when using P_1 (Give an explicit formula to define an arbitrary element of this basis).
3. Describe a linear system obtained with this method (Write the coefficient with integral, that will not be computed at this step).

Exercise 2. Optimal Control

Given $t \mapsto y(t)$ a real valued function $\forall t \in [0,T],\ y(t) \in \mathbb{R}$, consider the optimal control problem :

Find $c \in L^2(0,T)$ *minimizing*

$$J(c) = \frac{1}{2}|y(T) - y_{cible}|^2 + \frac{\alpha}{2}\int_0^T c^2(t)dt,$$

under te constraint

$$\begin{aligned} y'(t) - c(t)\exp(-y(t)) &= 0, \\ y(t=0) &= 0. \end{aligned}$$

wher the parameters T, α and y_{cible} are supposed to be known. **We assume that** $y_{cible} > 0$.

1. Give the Lagrangian $\mathcal{L}(y, c, p)$ of the system.
2. Give the optimality conditions, i.e. the equation corresponding to :
   (a) $\partial_p \mathcal{L}(y, c, p) = 0, \ \forall t \in [0, T]$.
   (b) $\partial_y \mathcal{L}(y, c, p) = 0, \ \forall t \in [0, T]$.
   (c) $\partial_{y(T)} \mathcal{L}(y, c, p) = 0$
   (d) $\partial_c \mathcal{L}(y, c, p) = 0, \ \forall t \in [0, T]$.
3. Suppose that the optimality system has a solution that we denote by (y, p, c). The following questions aims at solving explictely the solution of our problem. **We denote by** $\log$ **the natural logarithm, i.e., the function such that** $\exp(\log(x)) = x$ **for all** $x \in \mathbb{R}$**.**
   (a) Denote by d the primitive[1] function c which cancels in $t = 0$. Explain why $d > -1$ and show that
   $$y(t) = \ln(1 + d(t)),$$
   where
   (b) Show that there exists a constant $\kappa \in \mathbb{R}$ such that
   $$p(t) = \kappa(d(t) + 1), \ \forall t \in [0, T].$$
   (c) Deduce from the previous results that the optimal control c is constant.
   (d) Which equation satisfies κ ?
   (e) Discuss the number of solutions of this equation.

Exercise 3. Reduced basis
Given an Hilbert space H, consider a variational problem :
find $u^\star \in H$ *such that*
$$a(u^\star, v) = f(v), \ \forall \ v \in H.$$
We assume that a and f statisfy the assumption of the Lax-Milgram Theorem. **We denote by** α **and** γ **the coercivity constants associated with** a.
Consider goal-oriented application, where only a scalar function $\ell(u^\star)$, is actually important. **We assume linear dependence between** $u^\star$ **and this quantity of interest, i.e.** ℓ **is assumed to be linear.** Consider now $u \in H$ and introduce the residual
$$r(v) = a(u, v) - f(v), \ \forall \ v \in H. \tag{1}$$
We want to investigate approximations of $\ell(u^\star)$ by introducing a mapping $\varphi(r) = \ell(u)$, **where** u **and** r **are connected by** (1)**, seen as a constraint equation.**
Part 1 : results on φ

1. Show that φ is well-defined as a function of r, i.e., prove that if r is fixed, then, $\ell(u)$ is uniquely defined.
2. Show that
$$\varphi(r) = \varphi(0) + \varphi'(0) \cdot r. \tag{2}$$
3. Show that $\varphi(0) = \ell(u^\star)$.

1. meaning that the derivative of d is c

Part 2 : Lagrangian

Since $\varphi(r) = \ell(u)$ and $\varphi(0) = \ell(u^\star)$, it remains to compute $\varphi'(0)$ by standard adjoint methods. Introduce the Lagrangian

$$\mathcal{L}(r, u, p) = \ell(u) - (\langle p, r\rangle - a(u, p) + f(p)) \,.$$

1. Show that $\partial_u \mathcal{L}(r, u, p) = 0$ can be writen in the variational form :

$$\ell(v) \quad = \quad -a(v, p) \quad \forall v \in H.$$

2. Show that $\varphi'(0) = \partial_r \mathcal{L}(r, u, p_0) = -p_0$, where u, p_0 are such that $\partial_u \mathcal{L}(r, u, p_0) = 0$ and $\partial_{p_0} \mathcal{L}(r, u, p_0) = 0$.
3. Using Eq. (2) show that :

$$\ell(u) = \ell(u^\star) + r(p_0).$$

4. Given now an arbitrary $p \in H$, show that :

$$\ell(u^\star) = \ell(u) - r(p) + r(p - p_0).$$

Part 3 : Duality

The quantity $p - p_0$ can be expressed (or at least, estimated) in an a posteriori way, that is without using p_0. In this way, we introduce the dual residual

$$r^{dual}(v) := \ell(v) - a(v, p)$$

1. Show that

$$r^{dual}(v) = a(v, p_0 - p),$$

2. Deduce that

$$\alpha \|p_0 - p\| \leqslant \|r^{dual}\| \leqslant \gamma \|p_0 - p\|,$$

   where α and γ are the coercivity and continuity constants associated with a.
3. Show that

$$\|\ell(u^\star) - (\ell(u) - r(p))\| \leqslant \frac{\|r\| \cdot \|r^{dual}\|}{\alpha}.$$

4. What is the interest of this result ?