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OPTIMIZATION OF BATHYMETRY FOR LONG WAVES WITH SMALL AMPLITUDE*

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Abstract. This paper deals with bathymetry-oriented optimization in the case of long waves with small amplitude. Under these two assumptions, the free-surface incompressible Navier–Stokes system can be written as a wave equation, where the bathymetry appears as a parameter in the spatial operator. Looking then for time-harmonic fields and writing the bathymetry, i.e., the bottom topography, as a perturbation of a flat bottom, we end up with a heterogeneous Helmholtz equation with an impedance boundary condition. In this way, we study a PDE-constrained optimization problem for a Helmholtz equation in heterogeneous media whose coefficients are only bounded with bounded variation. We provide necessary condition for a general cost function to have at least one optimal solution. We also prove the convergence of a finite element approximation of the solution to the considered Helmholtz equation as well as the convergence of the discrete optimum toward the continuous ones. We end this paper with some numerical experiments to illustrate the theoretical results and show that some of their assumptions are necessary.

Key words. PDE-constrained optimization, time-harmonic wave equation, bathymetry optimization, shallow water modeling, Helmholtz equation

AMS subject classifications. 35J05, 35J20, 65N30, 49Q10, 49Q12, 78A40, 78A45

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1. Introduction.

Despite the fact that the bathymetry can be inaccurately known in many situations, wave propagation models strongly depend on this parameter to capture the flow behavior, which emphasizes the importance of studying inverse problems concerning its reconstruction from free surface flows. In recent years a considerable literature has grown around this subject. A review from Sellier identifies different techniques applied for bathymetry reconstruction [45, section 4.2], which rely mostly on the derivation of an explicit formula for the bathymetry, numerical resolution of a governing system, or data assimilation methods [33, 47].

An alternative is to use the bathymetry as the control variable of a PDE-constrained optimization problem, an approach used in coastal engineering due to mechanical constraints associated with building structures and their interaction with sea waves. For instance, among the several aspects to consider when designing a harbor, building defense structures is essential for protection against wave impact. These can be optimized to locally minimize wave energy by studying its interaction with the reflected waves [34]. Bouharguane and Mohammadi [11, 40] consider a time-dependent approach to studying the evolution of sand motion on the seabed, which could also

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allow these structures to change over time. In this case, the proposed functionals are minimized using sensitivity analysis, a technique broadly applied in geosciences.

From a mathematical point of view, solving these kinds of problems is mostly numerical. For instance, a theoretical approach applied to the modeling of surfing pools can be found in [20, 42], where the goal is to maximize locally the energy of the prescribed wave. The former proposes determining a bathymetry, whereas the latter sets the shape and displacement of an underwater object along a constant depth.

In this paper, we address the determination of a bathymetry from an optimization problem, where the Helmholtz equation with a first-order absorbing boundary condition acts as a constraint. Even though this equation is limited to describing waves of small amplitude, it is often used in engineering due to its simplicity, which leads to explicit solutions when a flat bathymetry is assumed. To obtain such a formulation, we rely on two asymptotic approximations of the free-surface incompressible Navier–Stokes equations. The first one is based on a long-wave theory approach and reduces the Navier–Stokes system to the Saint-Venant equations. The second one considers waves of small amplitude from which the Saint-Venant model can be approximated by a wave equation involving the bathymetry in its spatial operator. Finally, it is when considering a time-harmonic solution of this wave equation that we get a Helmholtz equation with spatially varying coefficients. Regarding the assumptions on the bathymetry to be optimized, we assume the latter to be a perturbation of a flat bottom with a compactly supported perturbation, which can thus be seen as a scatterer. Moreover, we make very few assumptions about the regularity of the bathymetry, which is assumed to be nonsmooth and possibly discontinuous [29, 38, 49]. We therefore end up with a constraint equation given by a time-harmonic wave equation, namely a Helmholtz equation, with nonsmooth coefficients.

It is worth noting that our bathymetry optimization problem aims at finding some parameters in our PDE that minimize a given cost function and can thus be seen as a parametric optimization problem (see, e.g., [4, 2, 30]). Similar optimization problems can also be encountered when trying to identify some parameters in the PDE from measurements (see, e.g., [14, 9]). Nevertheless, all the aforementioned references deal with real elliptic and coercive problems. Since the Helmholtz equation is, unfortunately, a complex and noncoercive PDE, these results do not apply.

We also emphasize that the PDE-constrained optimization problem studied in the present paper falls into the class of so-called topology optimization problems. For practical applications involving Helmholtz-like equations as constraints, we refer the reader to [48, 10], where the shape of an acoustic horn is optimized to have better transmission efficiency, and to [35, 16, 15], where the topology optimization of photonic crystals with several different cost functions is considered. Although there are many applied and numerical studies of topology optimization problems involving the Helmholtz equation, there are only a few theoretical studies, as pointed out in [31, p. 2].

Regarding the theoretical results from [31], the authors proved the existence of an optimal solution to their PDE-constrained optimization problem as well as the convergence of the discrete optimum toward the continuous ones. Note that in this paper, a relative permittivity is considered as an optimization parameter and that the latter appears as a multiplication operator in the Helmholtz differential operator. Since in the present study the bathymetry is assumed to be nonsmooth and is involved in the principal part of our heterogeneous Helmholtz equation, we cannot rely on the theoretical results proved in [31] to study our optimization problem.

This paper is organized as follows: Section 2 presents the two approximations of the free-surface incompressible Navier–Stokes system, namely the long-wave theory approach and the reduction to waves with small amplitude, that lead us to consider a Helmholtz equation in heterogeneous media, where the bathymetry plays the role of a scatterer. Under suitable assumptions on the cost functional and the admissible set of bathymetries, in section 3 we prove the continuity of the control-to-state mapping and the existence of an optimal solution in addition to the continuity and boundedness of the resulting wave presented in section 4. The discrete optimization problem is discussed in section 5, where we study the convergence to the discrete optimal solution as well as the convergence of a finite element approximation. Finally, we present some numerical results in section 6.

2. Derivation of the wave model.

We start from the Navier–Stokes equations to derive the governing PDE. However, due to its complexity, we introduce two approximations [37]: a small relative depth (*long-wave theory*) combined with an infinitesimal wave amplitude (*small amplitude wave theory*). Asymptotic analysis on the relative depth shows that the vertical component of the depth-averaged velocity is negligible, obtaining the Saint-Venant equations. After neglecting its convective inertia terms and linearizing around the sea level, Saint-Venant equations result in a wave equation which depends on the bathymetry. Since a variable sea bottom can be seen as an obstacle, we reformulate the equation as a *scattering problem* involving the Helmholtz equation.

2.1. From Navier–Stokes system to Saint-Venant equations.

For $t \geq 0$, we define the time-dependent region,

$$\Omega_t = \{(x, z) \in \Omega \times \mathbb{R} \mid -z_b(x) \leq z \leq \eta(x, t)\},$$

where Ω is a bounded open set with Lipschitz boundary, $\eta(x, t)$ represents the water level, and $-z_b(x)$ is the bathymetry, a time-independent and negative function. The water height is denoted by $h = \eta + z_b$. See Figure 1.

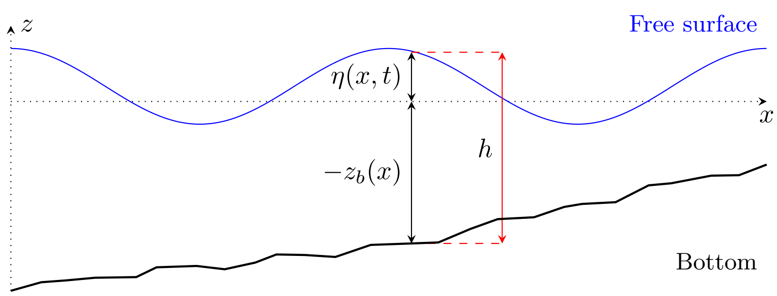


Fig. 1. *Schematic representation of the notation.*

In what follows, we consider an incompressible fluid of constant density (assumed to be equal to 1) governed by the Navier–Stokes system

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = \operatorname{div}(\sigma_T) + \mathbf{g} & \text{in } \Omega_t, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega_t, \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega_0, \end{cases} \tag{2.1}$$

where $\mathbf{u} = (u, v, w)^\top$ denotes the velocity of the fluid, $\mathbf{g} = (0, 0, -g)^\top$ is the gravity, and σ_T is the total stress tensor given by

$$\sigma_T = -p\mathbb{I} + \mu\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right),$$

with p the pressure and μ the coefficient of viscosity.

To complete (2.1), we require suitable boundary conditions. Given the outward normals

$$n_s = \frac{1}{\sqrt{1+|\nabla\eta|^2}}\begin{pmatrix}-\nabla\eta\\1\end{pmatrix},\ n_b = \frac{1}{\sqrt{1+|\nabla z_b|^2}}\begin{pmatrix}\nabla z_b\\1\end{pmatrix}$$

to the free surface and bottom, respectively, we recall that the velocity of these two must be equal to that of the fluid:

$$\begin{cases}\dfrac{\partial\eta}{\partial t} - \mathbf{u}\cdot n_s = 0 & \text{on } (x,\eta(x,t),t),\\ \mathbf{u}\cdot n_b = 0 & \text{on } (x,-z_b(x),t).\end{cases} \tag{2.2}$$

On the other hand, the stress at the free surface is continuous, whereas at the bottom we assume a no-slip condition,

$$\begin{cases}\sigma_T\cdot n_s = -p_a n_s & \text{on } (x,\eta(x,t),t),\\ (\sigma_T n_b)\cdot t_b = 0 & \text{on } (x,-z_b(x),t),\end{cases} \tag{2.3}$$

with p_a the atmospheric pressure and t_b a unitary tangent vector to n_b.

A long-wave theory approach can then be developed to approximate the previous model by a Saint-Venant system [25]. Denote by H the relative depth and L the characteristic dimension along the horizontal axis; this approach is based on the approximation $\varepsilon := \frac{H}{L} \ll 1$, leading to a hydrostatic pressure law for the nondimensionalized Navier–Stokes system and a vertical integration of the remaining equations. For the sake of completeness, details of this derivation in our case are given in the appendix. For a two-dimensional system (2.1), the resulting system is then

$$\frac{\partial\eta}{\partial t}\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial\eta}{\partial x}\right|^2} + \frac{\partial(h_\delta\overline{u})}{\partial x} = 0, \tag{2.4}$$

$$\frac{\partial(h_\delta\overline{u})}{\partial t} + \delta\frac{\partial(h_\delta\overline{u}^2)}{\partial x} = -h_\delta\frac{\partial\eta}{\partial x} + \delta u(x,\delta\eta,t)\frac{\partial\eta}{\partial t}\left(\sqrt{1+(\varepsilon\delta)^2\left|\frac{\partial\eta}{\partial x}\right|^2} - 1\right) + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta\varepsilon), \tag{2.5}$$

where $\delta := \frac{A}{H}$, $h_\delta = \delta\eta + z_b$, and $\overline{u}(x,t) := \frac{1}{h_\delta(x,t)}\int_{-z_b}^{\delta\eta} u(x,z,t)dz$. If $\varepsilon \to 0$, we recover the classical derivation of the one-dimensional Saint-Venant equations.

2.2. Small amplitudes. With respect to the classical Saint-Venant formulation, passing to the limit $\delta \to 0$ is equivalent to neglecting the convective acceleration terms and linearizing the system (2.4),(2.5) around the sea level $\eta = 0$. In order to do this, we rewrite the derivatives as

$$\frac{\partial(h_\delta\overline{u})}{\partial t} = h_\delta\frac{\partial\overline{u}}{\partial t} + \delta\frac{\partial\eta}{\partial t}\overline{u},\ \frac{\partial(h_\delta\overline{u})}{\partial x} = \delta\frac{\partial(\eta\overline{u})}{\partial x} + \frac{\partial(z_b\overline{u})}{\partial x},$$

and then, taking $\varepsilon, \delta \to 0$ in (2.4)–(2.5) yields

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \dfrac{\partial (z_b \overline{u})}{\partial x} = 0, \\ -\dfrac{\partial (z_b \overline{u})}{\partial t} + z_b \dfrac{\partial \eta}{\partial x} = 0. \end{cases}$$

Finally, after differentiating the first equation with respect to t and replacing the second in the new expression, we obtain the wave equation for a variable bathymetry. All previous computations hold for the two- and three-dimensional system (2.1). In this case, we obtain

$$\frac{\partial^2 \eta}{\partial t^2} - \operatorname{div}(g z_b \nabla \eta) = 0. \tag{2.6}$$

2.3. Helmholtz formulation. Equation (2.6) defines a time-harmonic field, whose solution has the form $\eta(x,t) = \operatorname{Re}\{\psi_{tot}(x) \mathrm{e}^{-\mathrm{i}\omega t}\}$, where the amplitude ψ_{tot} satisfies

$$\omega^2 \psi_{tot} + \operatorname{div}(g z_b \nabla \psi_{tot}) = 0. \tag{2.7}$$

We wish to rewrite the equation above as a scattering problem. Since a variable bottom $z_b(x) := z_0 + \delta z_b(x)$ (with z_0 a constant describing a flat bathymetry and δz_b a perturbation term) can be considered as an obstacle, we thus assume that δz_b has a compact support in Ω and that ψ_{tot} satisfies the so-called Sommerfeld radiation condition. In a bounded domain as Ω, we impose the latter thanks to an impedance boundary condition (also known as a first-order absorbing boundary condition), which ensures the existence and uniqueness of the solution [41, p. 108]. We then reformulate (2.7) as

$$\begin{cases} \operatorname{div}((1+q)\nabla \psi_{tot}) + k_0^2 \psi_{tot} = 0 & \text{in } \Omega, \\ \nabla(\psi_{tot} - \psi_0) \cdot \hat{n} - \mathrm{i}k_0(\psi_{tot} - \psi_0) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where we have introduced the parameter $q(x) := \frac{\delta z_b(x)}{z_0}$, which is assumed to be compactly supported in Ω, $k_0 := \frac{\omega}{\sqrt{g z_0}}$, $\hat{n}$ is the unit normal to $\partial\Omega$, and $\psi_0(x) = \mathrm{e}^{\mathrm{i}k_0 x \cdot \vec{d}}$ is an incident plane wave propagating in the direction $\vec{d}$ (such that $|\vec{d}| = 1$).

Decomposing the total wave as $\psi_{tot} = \psi_0 + \psi_{sc}$, where ψ_{sc} represents an unknown scattered wave, we obtain the Helmholtz formulation

$$\begin{cases} \operatorname{div}((1+q)\nabla \psi_{sc}) + k_0^2 \psi_{sc} = -\operatorname{div}(q \nabla \psi_0) & \text{in } \Omega, \\ \nabla \psi_{sc} \cdot \hat{n} - \mathrm{i}k_0 \psi_{sc} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Its structure will be useful in proving the existence of a minimizer for a PDE-constrained functional, as discussed in the next section.

3. Description of the optimization problem. We are interested in studying the problem of a cost functional constrained by the weak formulation of a Helmholtz equation. The considered PDE intends to generalize the equations considered so far, whereas the cost function indirectly affects the choice of the set of admissible controls. These can be discontinuous since they are included in the space of functions of bounded variations. In this framework, we treat the continuity and regularity of the associated control-to-state mapping and discuss the existence of an optimal solution to the optimization problem.

3.1. Weak formulation. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. We consider the general Helmholtz equation,

$$\begin{cases} -\mathrm{div}\,((1+q)\nabla\psi) - k_0^2\psi = \mathrm{div}\,(q\nabla\psi_0) & \text{in } \Omega, \\ (1+q)\nabla\psi\cdot\hat{n} - \mathrm{i}k_0\psi = g - q\nabla\psi_0\cdot\hat{n} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where g is a source term. We assume that $q \in L^\infty(\Omega)$ and that there exists $\alpha > 0$ such that

$$\text{for a.a. } x \in \Omega,\ 1 + q(x) \geq \alpha. \tag{3.2}$$

Remark 3.1. Here we have generalized the models described in the previous section: If q has a fixed compact support in Ω, we have that the total wave ψ_{tot} satisfying (2.8) is a solution to (3.1) with $g = \nabla\psi_0\cdot\hat{n} - \mathrm{i}k_0\psi_0$ and no right-hand side; whereas the scattered wave ψ_{sc} satisfying (2.9) is a solution to (3.1) with $g = 0$. All the proofs obtained in this broader setting still hold true for both problems.

A weak formulation for (3.1) is given by

$$a(q;\psi,\phi) = b(q;\phi)\ \forall\phi \in H^1(\Omega), \tag{3.3}$$

where

$$\begin{aligned} a(q;\psi,\phi) &:= \int_\Omega \left((1+q)\nabla\psi\cdot\nabla\overline{\phi} - k_0^2\psi\overline{\phi}\right)\,dx - \mathrm{i}k_0\int_{\partial\Omega}\psi\overline{\phi}\,d\sigma, \\ b(q;\phi) &:= -\int_\Omega q\nabla\psi_0\cdot\nabla\overline{\phi}\,dx + \langle g,\overline{\phi}\rangle_{H^{-1/2},H^{1/2}}. \end{aligned} \tag{3.4}$$

Note that thanks to the Cauchy–Schwarz inequality, the sesquilinear form a is continuous,

$$\begin{aligned} |a(q;\psi,\phi)| &\leq C(\Omega,q,\alpha)(1 + \|q\|_{L^\infty(\Omega)})\,\|\psi\|_{1,k_0}\,\|\phi\|_{1,k_0}\,, \\ \|\psi\|_{1,k_0}^2 &:= k_0^2\,\|\psi\|_{L^2(\Omega)}^2 + \alpha\,\|\nabla\psi\|_{L^2(\Omega)}^2\,, \end{aligned}$$

where $C(\Omega,q,\alpha) > 0$ is a generic constant. In addition, taking $\phi = \psi$ in the definition of a, it satisfies a Gårding inequality

$$\mathrm{Re}\{a(q;\psi,\psi)\} + 2k_0^2\,\|\psi\|_{L^2(\Omega)}^2 \geq \|\psi\|_{1,k_0}^2\,, \tag{3.5}$$

and the well-posedness of problem (3.3) follows from the Fredholm alternative. Finally, uniqueness holds for any $q \in L^\infty(\Omega)$ satisfying (3.2) owning to [27, Theorems 2.1 and 2.4].

Remark 3.2. We briefly show here that (3.3) has a unique solution. We emphasize that only the uniqueness has to be proved since the Fredholm alternative then ensures the existence. We consider $\psi \in H^1(\Omega)$ such that $a(q;\psi,\phi) = 0$ for all $\phi \in H^1(\Omega)$. Since $\mathrm{Im}\{a(q;\psi,\psi)\} = -k_0\,\|\psi\|_{L^2(\partial\Omega)}^2$, we obtain that $\psi|_{\partial\Omega} = 0$, and the boundary condition $(1+q)\nabla\psi\cdot\hat{n} - \mathrm{i}k_0\psi = 0$ then gives $(1+q)\nabla\psi\cdot\hat{n} = 0$. The unique continuation property [1] which holds since $\Omega \subset \mathbb{R}^2$ then proves that $\psi = 0$.

Regarding the case $\Omega \subset \mathbb{R}^3$, we cannot conclude using the unique continuation property unless q satisfy additional smoothness assumptions. We refer the reader to [27, 28] for further discussion and results on the existence and uniqueness of a solution to the Helmholtz equation with variable coefficients.

3.2. Continuous optimization problem. We are interested in solving the following PDE-constrained optimization problem:

$$\begin{aligned} &\text{Minimize } J(q,\psi) \\ &\text{subject to } (q,\psi) \in U_\Lambda \times H^1(\Omega), \text{ where } \psi \text{ satisfies (3.3).} \end{aligned} \tag{3.6}$$

We now define the set U_Λ of admissible q. We wish to find optimal q that can have discontinuities, and thus we cannot look for q in some Sobolev spaces that are continuously embedded into $C^0(\overline{\Omega})$, even if such regularity is useful for proving existence of minimizers (see, e.g., [4, Chapter VI] and [7, Theorem 4.1]). To find an optimal q satisfying (3.2) and having possible discontinuities, we follow [14] and introduce the following set:

$$U_\Lambda = \{q \in BV(\Omega) \mid \alpha - 1 \leq q(x) \leq \Lambda \text{ for a.a. } x \in \Omega\}.$$

Above, $\Lambda \geq \max\{\alpha-1, 0\}$, and $BV(\Omega)$ is the set of functions with bounded variations [3], that is, functions whose distributional gradient belongs to the set $\mathcal{M}_{\mathrm{b}}(\Omega, \mathbb{R}^N)$ of bounded Radon measures. Note that the piecewise constant functions over Ω belong to U_Λ.

Some useful properties of $BV(\Omega)$ can be found in [3] and are recalled below for the sake of completeness. This is a Banach space for the norm (see [3, Proposition 3.2, p. 120])

$$\|q\|_{BV(\Omega)} := \|q\|_{L^1(\Omega)} + |Dq|(\Omega),$$

where D is the distributional gradient, and

$$|Dq|(\Omega) = \sup\left\{ \int_\Omega q \operatorname{div}(\varphi)\, dx \;\middle|\; \varphi \in \mathcal{C}_{\mathrm{c}}^1(\Omega, \mathbb{R}^2) \text{ and } \|\varphi\|_{L^\infty(\Omega)} \leq 1 \right\} \tag{3.7}$$

is the variation of q (see [3, Definition 3.4, p. 119]).

The weak* convergence in $BV(\Omega)$, denoted by

$$q_n \rightharpoonup q, \text{ weak* in } BV(\Omega),$$

means that

$$q_n \to q \text{ in } L^1(\Omega) \text{ and } Dq_n \rightharpoonup Dq \text{ in } \mathcal{M}_{\mathrm{b}}(\Omega, \mathbb{R}^N),$$

where $Dq_n \rightharpoonup Dq$ in $\mathcal{M}_b(\Omega, \mathbb{R}^N)$ means that

$$\lim_{n\to+\infty} \int_\Omega \psi \cdot dDq_n = \int_\Omega \psi \cdot dDq \quad \forall \psi \in \mathcal{C}^0(\Omega, \mathbb{R}^N).$$

Also, the continuous embedding $BV(\Omega) \subset L^1(\Omega)$ is compact. Finally, we recall that the application $q \in BV(\Omega) \mapsto |Dq|(\Omega) \in \mathbb{R}^+$ is lower semicontinuous with respect to the weak* topology of BV. Hence, for any sequence $q_n \rightharpoonup q$ in $BV(\Omega)$, one has

$$|Dq|(\Omega) \leq \liminf_{n\to+\infty} |Dq_n|(\Omega).$$

The set U_Λ is a closed, weakly* closed, and convex subset of $BV(\Omega)$. We will also consider the next set of admissible parameters,

$$U_{\Lambda,\kappa} = \{q \in U_\Lambda \mid |Dq|(\Omega) \leq \kappa\},$$

which possesses the aforementioned properties. Note that choosing U_Λ or $U_{\Lambda,\kappa}$ affects the convergence analysis of the discrete optimization problem, a topic discussed in section 5.

Remark 3.3. In this paper, we are interested in computing either the total wave satisfying (2.8) or the scattered wave solution to (2.9). Since this requires working with q having a fixed compact support in Ω, we also introduce the set of admissible parameters,

$$\widetilde{U}_\varepsilon := \{q \in U \ | q(x) = 0 \text{ for a.a. } x \in \mathcal{O}_\varepsilon\}, \ \mathcal{O}_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq \varepsilon\},$$

which is a set of bounded functions with bounded variations that have a fixed support in Ω. We emphasize that this set is a convex, closed, and weak-$*$ closed subset of $BV(\Omega)$. As a consequence, all the theorems we prove also hold for this set of admissible parameters.

3.3. Continuity of the control-to-state mapping. In this section, we establish the continuity of the application $q \in U \mapsto \psi(q) \in H^1(\Omega)$, where $\psi(q)$ satisfies problem (3.3). We assume that $U \subset BV(\Omega)$ is a given weakly* closed set satisfying

$$\forall q \in U, \text{ for a.a. } x \in \Omega, \ \alpha - 1 \leq q(x) \leq \Lambda.$$

Note that both U_Λ, $U_{\Lambda,\kappa}$ and $\widetilde{U}_\varepsilon$ (see Remark 3.3) also satisfy these two assumptions. The next result considers the dependence of the stability constant with respect to the optimization parameter q.

THEOREM 3.4. *Assume that $q \in U$ and $\psi \in H^1(\Omega)$. Then there exists a constant $C_{\mathrm{s}}(k_0) > 0$ that does not depend on q such that*

$$\|\psi\|_{1,k_0} \leq C_{\mathrm{s}}(k_0) \sup_{\|\phi\|_{1,k_0}=1} |a(q;\psi,\phi)|, \tag{3.8}$$

where the constant $C_{\mathrm{s}}(k_0) > 0$ only depends on the wavenumber and on Ω. In addition, if ψ is the solution to (3.3)*, then it satisfies the bound*

$$\|\psi\|_{1,k_0} \leq C_{\mathrm{s}}(k_0) C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} \left(\|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right), \tag{3.9}$$

where $C(\Omega) > 0$ only depends on the domain.

Proof. The existence and uniqueness of a solution to problem (3.3) follows from [27, Theorems 2.1 and 2.4].

The proof of (3.8) proceeds by contradiction assuming this inequality to be false. Therefore, we suppose there exist sequences $(q_n)_n \subset U$ and $(\psi_n)_n \subset H^1(\Omega)$ such that $\|q_n\|_{BV(\Omega)} \leq M$, $\|\psi_n\|_{1,k_0} = 1$ and

$$\lim_{n\to+\infty} \sup_{\|\phi\|_{1,k_0}=1} |a(q_n;\psi_n,\phi)| = 0. \tag{3.10}$$

The compactness of the embeddings $BV(\Omega) \subset L^1(\Omega)$ and $H^1(\Omega) \subset L^2(\Omega)$ yields the existence of a subsequence (still denoted (q_n,ψ_n)) such that

$$\psi_n \rightharpoonup \psi_\infty \text{ in } H^1(\Omega), \quad \psi_n \to \psi_\infty \text{ in } L^2(\Omega), \quad \text{and } q_n \to q_\infty \in U \text{ in } L^1(\Omega). \tag{3.11}$$

Compactness of the trace operator implies that $\lim_{n\to+\infty} \psi_n|_{\partial\Omega} = \psi_\infty|_{\partial\Omega}$ holds strongly in $L^2(\partial\Omega)$, and thus from (3.11) we get

$$\lim_{n\to+\infty} \int_\Omega k_0^2 \psi_n \overline{\phi}\, dx + \mathrm{i}k_0 \int_{\partial\Omega} \psi_n \overline{\phi}\, d\sigma = \int_\Omega k_0^2 \psi_\infty \overline{\phi}\, dx + \mathrm{i}k_0 \int_{\partial\Omega} \psi_\infty \overline{\phi}\, d\sigma \ \forall\ v \in H^1(\Omega),$$

$$\lim_{n\to+\infty} \int_\Omega \nabla\psi_n \cdot \nabla\overline{\phi}\, dx = \int_\Omega \nabla\psi_\infty \cdot \nabla\overline{\phi}\, dx.$$

We now pass to the limit in the term of a that involves q_n; see (3.4). We start from

$$(q_n \nabla \psi_n, \nabla \overline{\phi})_{L^2(\Omega)} - (q_\infty \nabla \psi_\infty, \nabla \overline{\phi})_{L^2(\Omega)} = ((q_n - q_\infty)\nabla \psi_n, \nabla \overline{\phi})_{L^2(\Omega)} + (q_\infty \nabla(\psi_n - \psi_\infty), \nabla \overline{\phi})_{L^2(\Omega)}$$

and use the Cauchy–Schwarz inequality to get

$$\begin{aligned}
&\left| \int_\Omega q_n \nabla \psi_n \cdot \nabla \overline{\phi}\, dx - \int_\Omega q_\infty \nabla \psi_\infty \cdot \nabla \overline{\phi}\, dx \right| \\
&\qquad \leq \left| ((q_n - q_\infty)\nabla \psi_n, \nabla \overline{\phi})_{L^2(\Omega)} \right| + \left| (q_\infty \nabla(\psi_n - \psi_\infty), \nabla \overline{\phi})_{L^2(\Omega)} \right| \\
&\qquad \leq \left\| \sqrt{|q_n - q_\infty|} \nabla \phi \right\|_{L^2(\Omega)} \left\| \sqrt{|q_n - q_\infty|} \nabla \psi_n \right\|_{L^2(\Omega)} \\
&\qquad\quad + \left| (q_\infty \nabla(\psi_n - \psi_\infty), \nabla \overline{\phi})_{L^2(\Omega)} \right| \\
&\qquad \leq 2 \frac{\sqrt{\Lambda}}{\sqrt{\alpha}} \|\psi_n\|_{1,k_0} \left\| \sqrt{|q_n - q_\infty|} \nabla \phi \right\|_{L^2(\Omega)} + \left| (\nabla(\psi_n - \psi_\infty), q_\infty \nabla \overline{\phi})_{L^2(\Omega)} \right|.
\end{aligned}$$

The right term above goes to 0 owing to $q_\infty \in L^\infty(\Omega)$ and (3.11). For the other term, since $q_n \to q_\infty$ strongly in L^1, we can extract another subsequence $(q_{n_k})_k$ such that $q_{n_k} \to q_\infty$ pointwise a.a. in Ω. Also, $\sqrt{|q_n - q_\infty|}|\nabla \phi|^2 \leq 2\sqrt{\Lambda}|\nabla \phi|^2 \in L^1(\Omega)$, and the Lebesgue dominated convergence theorem then yields

$$\lim_{k \to +\infty} \left\| \sqrt{|q_{n_k} - q_\infty|} \nabla \phi \right\|_{L^2(\Omega)} = 0.$$

This gives that (see also [14, equation (2.4)])

$$\lim_{k \to +\infty} (q_{n_k} \nabla \psi_{n_k}, \nabla \overline{\phi})_{L^2(\Omega)} = (q_\infty \nabla \psi_\infty, \nabla \overline{\phi})_{L^2(\Omega)} \quad \forall \phi \in H^1(\Omega). \tag{3.12}$$

Finally, gathering (3.12) together with (3.10) yields

$$0 = \lim_{k \to +\infty} a(q_{n_k}; \psi_{n_k}, \phi) = a(q_\infty, \psi_\infty, \phi) \quad \forall \phi \in H^1(\Omega),$$

and the uniqueness result [27, Theorems 2.1 and 2.4] shows that $\psi_\infty = 0$, and thus the whole sequence, actually converges to 0. To get our contradiction, it remains to show that $\|\nabla \psi_n\|_{L^2(\Omega)}$ converges to 0 as well. From the Gårding inequality (3.5), we have

$$\|\psi_n\|_{1,k_0}^2 \leq \mathrm{Re}\{a(q_n; \psi_n, \psi_n)\} + 2k_0^2 \|\psi_n\|_{L^2(\Omega)}^2 \xrightarrow[n \to +\infty]{} 0,$$

where we used (3.10) and the strong L^2 convergence of ψ_n toward $\psi_\infty = 0$. Finally one gets $\lim_{n \to +\infty} \|\psi_n\|_{1,k_0} = 0$, which contradicts $\|\psi_n\|_{1,k_0} = 1$ and gives the desired result.

Then applying (3.8) to the solution to (3.3) finally yields

$$\begin{aligned}
\|\psi\|_{1,k_0} &\leq C_{\mathrm{s}}(k_0) \sup_{\|\phi\|_{1,k_0} = 1} |a(q; \psi, \phi)| \leq C_{\mathrm{s}}(k_0) \sup_{\|\phi\|_{1,k_0} = 1} |b(q; \phi)| \\
&\leq C_{\mathrm{s}}(k_0) \sup_{\|\phi\|_{1,k_0} = 1} \Big(\|q\|_{L^\infty(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \|\phi\|_{H^{1/2}(\partial\Omega)} \Big) \\
&\leq C_{\mathrm{s}}(k_0) C(\Omega) \max\{k_0^{-1}, \alpha^{-1/2}\} \left(\|q\|_{L^\infty(\Omega)} \|\nabla \psi_0\|_{L^2(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)} \right),
\end{aligned}$$

where $C(\Omega) > 0$ comes from the trace inequality. ∎

Remark 3.5. Let us consider a more general version of problem (3.1), given by

$$\begin{cases} -\mathrm{div}\,((1+q)\nabla\psi) - k_0^2\psi = F & \text{in } \Omega, \\ (1+q)\nabla\psi\cdot\hat{n} - ik_0\psi = G & \text{on } \partial\Omega. \end{cases}$$

We emphasize that the estimation of the stability constant $C_{\mathrm{s}}(k_0)$ with respect to the wavenumber has been obtained for $(F,G)\in L^2(\Omega)\times L^2(\partial\Omega)$ for $q=0$ in [32] and for $q\in \mathrm{Lip}(\Omega)$ satisfying (3.2) in [6, 27, 28]. Since their proofs rely on Green, Rellich, and Morawetz identities, they do not extend to the case $(F,G)\in \big(H^1(\Omega)\big)'\times H^{-1/2}(\partial\Omega)$, but such cases can be tackled as done in [24, Theorem 2.5, p. 10]. The case of Lipschitz q has been studied in [13]. As a result, the dependence of the stability constant with respect to k_0, in the case where $q\in U$ and $(F,G)\in \big(H^1(\Omega)\big)'\times H^{-1/2}(\partial\Omega)$, does not seem to have been tackled so far to the best of our knowledge.

Remark 3.6 (H^1-bounds for the total and scattered waves). From Remark 3.1, we obtain that the total wave ψ_{tot} and the scattered wave ψ_{sc} are solutions to (3.3), with respective right-hand sides

$$b_{tot}(q;\phi) = \int_{\partial\Omega} (\nabla\psi_0\cdot\hat{n} - \mathrm{i}k_0\psi_0)\overline{\phi}\,d\sigma,\quad b_{sc}(q;\phi) = -\int_{\Omega} q\nabla\psi_0\cdot\nabla\overline{\phi}\,dx.$$

As a result of Theorem 3.4 and the continuity of the trace, we have

$$\begin{aligned} &\|\psi_{tot}\|_{1,k_0} \le C(\Omega)C_{\mathrm{s}}(k_0)k_0\max\{k_0^{-1},\alpha^{-1/2}\}, \\ &\|\psi_{sc}\|_{1,k_0} \le C_{\mathrm{s}}(k_0)\alpha^{-1/2}\,\|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^2(\Omega)} \le k_0C_{\mathrm{s}}(k_0)\alpha^{-1/2}\,\|q\|_{L^\infty(\Omega)}\sqrt{|\Omega|}. \end{aligned}$$

We can now prove some regularity for the control-to-state mapping.

THEOREM 3.7. *Let $(q_n)_n\subset U$ be a sequence that weakly* converges toward q_∞ in $BV(\Omega)$. Let $(\psi(q_n))_n$ be the sequence of weak solutions to problem* (3.3)*. Then $\psi(q_n)$ converges strongly in $H^1(\Omega)$ toward $\psi(q_\infty)$. In other words, the mapping*

$$q\in (U_\Lambda,\text{weak}^*)\mapsto \psi(q)\in (H^1(\Omega),\text{strong})$$

is continuous.

Proof. Since $q_n \rightharpoonup q_\infty$, weak* in $BV(\Omega)$, the sequence $(q_n)_n$ is bounded. Using the fact that U is weak* closed, we obtain that $q_\infty\in U$. Therefore, the sequence $(\psi(q_n))_n$ of the solution to problem (3.3) satisfies estimate (3.9) uniformly with respect to n. As a result, there exists a $\psi_\infty\in H^1(\Omega)$ such that the convergences (3.11) hold. Then using (3.12), we get that $a(q_n;\psi(q_n),\phi)\to a(q_\infty;\psi_\infty,\phi)$.

Since $b(q_n,\phi)\to b(q_\infty,\phi)$ for all $\phi\in H^1(\Omega)$, this proves that $a(q_\infty;\psi_\infty,\phi) = b(q;\phi)$ for all $\phi\in H^1(\Omega)$. Consequently $\psi_\infty=\psi(q_\infty)$ owing to the uniqueness of a weak solution to (3.3), and we have also proved that $\psi(q_n)\rightharpoonup\psi(q_\infty)$ in $H^1(\Omega)$.

We now show that $\psi(q_n)\to\psi(q_\infty)$ strongly in H^1. To see this, we start by noting that, up to extracting a subsequence (still denoted by q_n), we can use (3.12) to get that

$$\lim_{n\to+\infty} b(q_n;\psi(q_n)) = b(q_\infty;\psi(q_\infty)).$$

Since $\psi(q_n),\psi(q_\infty)$ satisfy the variational problem (3.3), we infer

$$\lim_{n\to+\infty} a(q_n;\psi(q_n),\psi(q_n)) = a(q_\infty;\psi(q_\infty),\psi(q_\infty)), \tag{3.13}$$

where the whole sequence actually converges owing to the uniqueness of the limit. Then using the fact that $\psi(q_n) \rightharpoonup \psi(q_\infty)$ in $H^1(\Omega)$ together with (3.13), one gets

$$\begin{aligned}\left\|\sqrt{1+q_n}\nabla\psi(q_n)\right\|^2_{L^2(\Omega)} &= a(q_n;\psi(q_n),\psi(q_n)) + k_0\left\|\psi(q_n)\right\|^2_{L^2(\Omega)} + \mathrm{i}k_0\left\|\psi(q_n)\right\|^2_{L^2(\partial\Omega)}\\ &\xrightarrow[n\to+\infty]{} a(q_\infty;\psi(q_\infty),\psi(q_\infty)) + k_0\left\|\psi(q_\infty)\right\|^2_{L^2(\Omega)} + \mathrm{i}k_0\left\|\psi(q_\infty)\right\|^2_{L^2(\partial\Omega)}\\ &= \left\|\sqrt{1+q_\infty}\nabla\psi(q_\infty)\right\|^2_{L^2(\Omega)}.\end{aligned}$$

To show that $\lim_{n\to+\infty}\left\|\nabla\psi(q_n)\right\|^2_{L^2(\Omega)} = \left\|\nabla\psi(q_\infty)\right\|^2_{L^2(\Omega)}$, note that

$$\nabla\psi(q_n) = \frac{\sqrt{1+q_n}\nabla\psi(q_n)}{\sqrt{1+q_n}}.$$

Using the same arguments as those proving (3.12), we have a subsequence (using the same notation) such that $q_n \to q_\infty$ pointwise a.e. in Ω, and thus $\sqrt{1+q_n}^{-1} \to \sqrt{1+q_\infty}^{-1}$ pointwise a.e. in Ω. Due to Lebesgue's dominated convergence theorem and $\sqrt{1+q_n}\nabla\psi(q_n) \to \sqrt{1+q_\infty}\nabla\psi(q_\infty)$ strongly in $L^2(\Omega)$, we have

$$\nabla\psi(q_n) = \frac{\sqrt{1+q_n}\nabla\psi(q_n)}{\sqrt{1+q_n}} \to \frac{\sqrt{1+q_\infty}\nabla\psi(q_\infty)}{\sqrt{1+q_\infty}} = \nabla\psi(q_\infty) \text{ strongly in } L^2(\Omega).$$

The latter, together with the weak H^1-convergence, shows that $\psi(q_n) \to \psi(q_\infty)$ strongly in H^1. ∎

3.4. Existence of optimal solution in U_Λ. We are now in a position to prove the existence of a minimizer for problem (3.6).

THEOREM 3.8. *Assume that the cost function $(q,\psi) \in U_\Lambda \mapsto J(q,\psi) \in \mathbb{R}$ satisfies:*

(A1) *There exist $\beta > 0$ and J_0 such that*

$$J(q,\psi) = J_0(q,\psi) + \beta|Dq|(\Omega),$$

where $|Dq|(\Omega)$ is defined in as (3.7).

(A2) *For all $(q,\psi) \in U_\Lambda \times H^1(\Omega)$, $J_0(q,\psi) \geq m > -\infty$.*

(A3) *$(q,\psi) \mapsto J_0(q,\psi)$ is lower semicontinuous with respect to the (weak*, weak) topology of $BV(\Omega) \times H^1(\Omega)$.*

Then the optimization problem (3.6) *has at least one optimal solution in $U_\Lambda \times H^1(\Omega)$.*

Proof. The existence of a minimizer for problem (3.6) can be obtained with a standard technique by combining Theorem 3.7 with weak-compactness arguments as done in [14, Lemma 2.1], [7, Theorem 4.1], and [31, Theorem 1]. We still give the proof for the sake of completeness.

We introduce the following set:

$$\mathcal{A} = \left\{(q,\psi) \in U_\Lambda \times H^1(\Omega) \;\middle|\; a(q;\psi,\phi) = b(q;\phi)\ \forall\phi \in H^1(\Omega)\right\}.$$

The existence and uniqueness of the solution to problem (3.3) ensure that $\mathcal{A}$ is nonempty. In addition, combining assumptions (A1) and (A2), we obtain that $J(q,\psi)$ is bounded from below on $\mathcal{A}$. We thus have a minimizing sequence $(q_n,\psi_n) \in \mathcal{A}$ such that

$$\lim_{n\to+\infty} J(q_n,\psi_n) = \inf_{(q,\psi)\in\mathcal{A}} J(q,\psi).$$

Theorem 3.4 and (A1) then give that the sequence $(q_n, \psi_n) \in BV(\Omega) \times H^1(\Omega)$ is uniformly bounded with respect to n and thus admits a subsequence that converges towards (q^*, ψ^*) in the (weak*, weak) topology of $BV(\Omega) \times H^1(\Omega)$. Now using Theorem 3.7 and the weak* lower semicontinuity of $q \mapsto |Dq|(\Omega)$, we end up with $(q^*, \psi^*) \in \mathcal{A}$ and

$$J(q^*, \psi^*) \leq \liminf_{n \to +\infty} J(q_n, \psi_n) = \inf_{(q,\psi)\in\mathcal{A}} J(q, \psi). \qquad \square$$

It is worth noting that the penalization term $\beta \|q\|_{BV(\Omega)}$ has been introduced only to obtain a uniform bound in the BV-norm for the minimizing sequence.

3.5. Existence of optimal solution in $U_{\Lambda,\kappa}$. We show here the existence of an optimal solution to problem (3.6) for $U = U_{\Lambda,\kappa}$. Note that any $q \in U_{\Lambda,\kappa}$ is actually bounded in BV since

$$\|q\|_{BV(\Omega)} \leq 2\max(\Lambda, \kappa, |\alpha - 1|).$$

With this property at hand, we can get a result similar to Theorem 3.8 without adding a penalization term in the cost function, and hence $\beta = 0$.

THEOREM 3.9. *Assume that the cost function $(q, \psi) \in U_{\Lambda,\kappa} \mapsto J(q, \psi) \in \mathbb{R}$ satisfies* (A2) *and* (A3) *given in Theorem* 3.8 *and that $\beta = 0$. Then the optimization problem* (3.6) *with $U = U_{\Lambda,\kappa}$ has at least one optimal solution.*

Proof. We introduce the following nonempty set:

$$\mathcal{A} = \left\{(q, \psi) \in U_{\Lambda,\kappa} \times H^1(\Omega) \mid a(q; \psi, \phi) = b(q; \phi) \ \forall \phi \in H^1(\Omega)\right\}.$$

From (A2), $J(q, \psi)$ is bounded from below on $\mathcal{A}$. We thus have a minimizing sequence $(q_n, \psi_n) \in \mathcal{A}$ such that

$$\lim_{n\to+\infty} J(q_n, \psi_n) = \inf_{(q,\psi)\in\mathcal{A}} J(q, \psi).$$

Since $(q_n)_n \subset U_{\Lambda,\kappa}$, it satisfies $\|q_n\|_{BV(\Omega)} \leq 2\max(\Lambda, \kappa, |\alpha - 1|)$ and thus admits a convergent subsequence toward some $q \in U_{\Lambda,\kappa}$. Theorem 3.7 then gives that $\psi(q_n) \to \psi(q)$ strongly in $H^1(\Omega)$, and we can finish by following the proof of Theorem 3.8. $\square$

4. Boundedness/continuity of solution to Helmholtz problem. In this section, we prove that even if the parameter q is not smooth enough for the solution to (3.1) to be in $H^s(\Omega)$ for some $s > 1$, we can still have a continuous solution. In order to prove such regularity for ψ, we are going to rely on the De Giorgi–Nash–Moser theory [26, Chapter 8.5], [36, Chapters 3.13 and 7.2] and, more precisely, on [43, Proposition 3.6], which reads as follows.

THEOREM 4.1. *Consider the elliptic problem associated with the inhomogeneous Neumann boundary condition given by*

$$\begin{cases} \mathcal{L}v := \operatorname{div}(A(x)\nabla v) = f_0 - \displaystyle\sum_{j=1}^{N} \frac{\partial f_j}{\partial x_j}, \\ \nabla v \cdot \hat{n} = h + \displaystyle\sum_{j=1}^{N} f_j n_j, \end{cases} \tag{4.1}$$

where $A \in L^\infty(\Omega, \mathbb{R}^{N\times N})$ satisfy the standard ellipticity condition $A(x)\xi \cdot \xi \geq \gamma|\xi|^2$ for almost all $x \in \Omega$. Let $p > N$, and assume that $f_0 \in L^{p/2}(\Omega)$, $f_j \in L^p(\Omega)$ for all

$j = 1, \dots, N$ and $h \in L^{p-1}(\partial\Omega)$. Then the weak solution v to (4.1) *satisfies*

$$\|v\|_{C^0(\Omega)} \leq C(N, p, \Omega, \gamma) \left(\|v\|_{L^2(\Omega)} + \|f_0\|_{L^{p/2}(\Omega)} + \sum_{j=1}^{N} \|f_j\|_{L^p(\Omega)} + \|h\|_{L^{p-1}(\partial\Omega)} \right).$$

4.1. C^0-bound for the general Helmholtz problem. Using Theorem 4.1, we can prove some L^∞-bound for the weak solution to the Helmholtz equation with bounded coefficients.

THEOREM 4.2. *Assume that $q \in L^\infty(\Omega)$ and that it satisfies* (3.2) *and $g \in L^2(\partial\Omega)$. Then the solution to problem* (3.3) *satisfies*

$$\|\psi\|_{C^0(\Omega)} \leq \widetilde{C}(\Omega)\widetilde{C}_{\mathrm{s}}(k_0, \alpha) \left(\|q\|_{L^\infty(\Omega)} \|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)} \right), \tag{4.2}$$

where

$$\widetilde{C}_{\mathrm{s}}(k_0, \alpha) = 1 + \left((1 + k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \max\{k_0^{-1}, \alpha^{-1/2}\} C_{\mathrm{s}}(k_0),$$

and $\widetilde{C}(\Omega) > 0$ does not depend on k or q.

Proof. We cannot readily apply Theorem 4.1 to the weak solution of problem (3.1) since it involves a complex valued operator. We therefore consider the problem satisfied by $\nu = \mathrm{Re}\{u\}$ and $\zeta = \mathrm{Im}\{u\}$, which is given by

$$\begin{cases} -\mathrm{div}\,((1+q)\nabla\nu) - k_0^2\nu = \mathrm{div}\,(q\nabla\,\mathrm{Re}\{\psi_0\}) & \text{in } \Omega, \\ -\mathrm{div}\,((1+q)\nabla\zeta) - k_0^2\zeta = \mathrm{div}\,(q\nabla\,\mathrm{Im}\{\psi_0\}) & \text{in } \Omega, \\ (1+q)\nabla\nu \cdot \hat{n} = \mathrm{Re}\{g\} - k_0\zeta - q\nabla\,\mathrm{Re}\{\psi_0\} \cdot \hat{n} & \text{on } \partial\Omega, \\ (1+q)\nabla\zeta \cdot \hat{n} = \mathrm{Im}\{g\} + k_0\nu - q\nabla\,\mathrm{Im}\{\psi_0\} \cdot \hat{n} & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

Since problem (4.3) is equivalent to problem (3.1), we get that the weak solution $(\nu, \zeta) \in H^1(\Omega)$ to (4.3) satisfies the inequality (3.9). Assuming that $g \in L^2(\partial\Omega)$ and using the continuous Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$, the (compact) embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$, the facts that $q \in L^\infty(\Omega)$ satisfies (3.2) and that ψ_0 is smooth, we get the next regularities

$$f_{0,1} = k_0^2\nu \in L^6(\Omega),\ f_{j,1} = q\frac{\partial\,\mathrm{Re}\{\psi_0\}}{\partial x_j} \in L^\infty(\Omega),\ h_1 = \mathrm{Re}\{g\} - k_0\zeta \in L^2(\partial\Omega),$$
$$f_{0,2} = k_0^2\zeta \in L^6(\Omega),\ f_{j,2} = q\frac{\partial\,\mathrm{Im}\{\psi_0\}}{\partial x_j} \in L^\infty(\Omega),\ h_2 = \mathrm{Im}\{g\} + k_0\nu \in L^2(\partial\Omega).$$

Now applying Theorem 4.1 to (4.3) twice with $p = 3$ and $N = 2$, one gets C^0-bounds for ν and ζ,

$$\|\nu\|_{C^0(\Omega)} \leq C(2, 3, \Omega, \gamma) \left(\|\nu\|_{L^2(\Omega)} + \|f_{0,1}\|_{L^{3/2}(\Omega)} + \sum_{j=1}^{2} \|f_{j,1}\|_{L^3(\Omega)} + \|h_1\|_{L^2(\partial\Omega)} \right),$$
$$\|\zeta\|_{C^0(\Omega)} \leq C(2, 3, \Omega, \gamma) \left(\|\zeta\|_{L^2(\Omega)} + \|f_{0,2}\|_{L^{3/2}(\Omega)} + \sum_{j=1}^{2} \|f_{j,2}\|_{L^3(\Omega)} + \|h_2\|_{L^2(\partial\Omega)} \right).$$

Some computations with the Hölder and multiplicative trace inequalities then give

$$
\begin{aligned}
(\|\nu\|_{L^2(\Omega)} + \|\zeta\|_{L^2(\Omega)}) &\leq 2\,\|\psi\|_{L^2(\Omega)}\,,\\
\|f_{0,1}\|_{L^{3/2}(\Omega)} + \|f_{0,2}\|_{L^{3/2}(\Omega)} &\leq k_0^2\,\|\psi\|_{L^{3/2}(\Omega)} \leq |\Omega|^{1/6} k_0^2\,\|\psi\|_{L^2(\Omega)}\,,\\
\|f_{j,l}\|_{L^3(\Omega)} &\leq \|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^\infty(\Omega)}\,,\ j=1,2,\\
\|h_1\|_{L^2(\partial\Omega)} + \|h_2\|_{L^2(\partial\Omega)} &\leq \|g\|_{L^2(\partial\Omega)} + k_0\,\|\psi\|_{L^2(\partial\Omega)}\\
&\leq \|g\|_{L^2(\partial\Omega)} + k_0 C(\Omega)\sqrt{\|\psi\|_{L^2(\Omega)}\,\|\psi\|_{H^1(\Omega)}}.
\end{aligned}
$$

Then using Young's inequality yields

$$
\begin{aligned}
k_0\sqrt{\|\psi\|_{L^2(\Omega)}\,\|\psi\|_{H^1(\Omega)}} &\leq C\left(\|\psi\|_{H^1(\Omega)} + k_0^2\,\|\psi\|_{L^2(\Omega)}\right)\\
&\leq C\left(\|\nabla\psi\|_{L^2(\Omega)} + k_0^2\,\|\psi\|_{L^2(\Omega)}\right),
\end{aligned}
$$

where $C > 0$ is a generic constant. We obtain the bound

$$
\begin{aligned}
\|\psi\|_{C^0(\Omega)} &= \|\nu\|_{C^0(\Omega)} + \|\zeta\|_{C^0(\Omega)}\\
&\leq \widetilde{C}(\Omega)\Big(\left(1+k_0^2\right)\|\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} + \|q\|_{L^\infty(\Omega)}\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big).
\end{aligned}
$$

Using the definition of $\|\psi\|_{1,k_0}$ on the estimate above, we get

$$
\begin{aligned}
\|\psi\|_{C^0(\Omega)} \leq \widetilde{C}(\Omega)\Big(&\left((1+k_0^2)k_0^{-1} + \alpha^{-1/2}\right)\|\psi\|_{1,k_0}\\
&+ \|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)}\Big).
\end{aligned} \tag{4.4}
$$

To apply the a priori estimate (3.9), we recall that the $H^{-1/2}$-norm can be replaced by a L^2-norm (since $g \in L^2(\partial\Omega)$), and then

$$
\begin{aligned}
\|\psi\|_{1,k_0} &\leq C(\Omega)\max\{k_0^{-1}, \alpha^{-1/2}\}C_{\mathrm{s}}(k_0)\left(\|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right)\\
&\leq C(\Omega)\max\{k_0^{-1}, \alpha^{-1/2}\}C_{\mathrm{s}}(k_0)\max\{1, \sqrt{|\Omega|}\}\left(\|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right).
\end{aligned}
$$

Finally, combining the latter expression with (4.4), we obtain that the weak solution to the Helmholtz equation satisfies

$$
\begin{aligned}
\|\psi\|_{C^0(\Omega)} \leq \widetilde{C}(\Omega)&\left(1 + \left((1+k_0^2)k_0^{-1} + \alpha^{-1/2}\right)\max\{k_0^{-1}, \alpha^{-1/2}\}C_{\mathrm{s}}(k_0)\right)\\
&\times\left(\|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{L^2(\partial\Omega)}\right),
\end{aligned}
$$

where $\widetilde{C}(\Omega) > 0$. ∎

Remark 4.3.

1. For the one-dimensional Helmholtz problem, the a priori estimate (3.9) and the continuous embedding $H^1(I) \subset C^0(I)$ directly give the continuity of u over a given interval I,

$$
\|\psi\|_{C^0(I)} \leq C\,\|\psi\|_{1,k_0} \leq C(k_0)\left(\|q\|_{L^\infty(\Omega)}\,\|\nabla\psi_0\|_{L^\infty(\Omega)} + \|g\|_{H^{-1/2}(\partial\Omega)}\right).
$$

   Note that we do not need to assume that $g \in L^2(\partial\Omega)$.

2. For the two-dimensional Helmholtz problem with $q = 0$, we can get the above $\mathcal{C}^0$-estimate from the embedding $H^2(\Omega) \hookrightarrow \mathcal{C}^0(\overline{\Omega})$ since

$$\|\psi\|_{C^0(\Omega)} \leq C \, \|\psi\|_{H^2(\Omega)}$$

for a generic constant C. We can then see that with respect to k_0, the estimate (4.2) actually has the same dependence as the H^2-estimate in [32, Proposition 3.6, p. 677].

4.2. C^0-bounds for the total and scattered waves. Thanks to Remark 3.1 and following the proof of Theorem 4.2, these bounds can be roughly obtained by setting $g = \nabla\psi_0 \cdot \hat{n} - \mathrm{i}k_0\psi_0$ and omitting the L^∞-norms in (4.4) for the total wave ψ_{tot}, and by simply setting $g = 0$ in the case the scattered wave ψ_{sc}. Using then the H^1-bounds from Remark 3.6, we actually get

$$\|\psi_{tot}\|_{\mathcal{C}^0(\Omega)} \leq \widetilde{C}(\Omega)k_0 \left(\left((1+k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \max\{k_0^{-1}, \alpha^{-1/2}\} C_{\mathrm{s}}(k_0) + 1 \right),$$
$$\|\psi_{sc}\|_{\mathcal{C}^0(\Omega)} \leq \widetilde{C}(\Omega)k_0 \left(\left((1+k_0^2)k_0^{-1} + \alpha^{-1/2} \right) \alpha^{-1/2} C_{\mathrm{s}}(k_0) + 1 \right) \|q\|_{L^\infty(\Omega)}.$$

We emphasize that the previous estimates show that the scattered wave ψ_{sc} vanishes in Ω if $q \to 0$. This is expected since, if $q = 0$, there is no obstacle to scatter the incident wave, which amounts to saying that $\psi_{tot} = \psi_0$.

5. Discrete optimization problem and convergence results. This section is devoted to the finite element discretization of the optimization problem (3.6). We consider a quasi-uniform family of triangulations (see [23, Definition 1.140, p. 76]) $\{\mathcal{T}_h\}_{h>0}$ of Ω and the corresponding finite element spaces

$$\mathcal{V}_h = \left\{ \phi_h \in \mathcal{C}(\overline{\Omega}) \mid \phi_h|_T \in \mathbb{P}_1(T) \; \forall T \in \mathcal{T}_h \right\}.$$

Note that thanks to Theorem 4.2, the solution to the general Helmholtz equation (3.1) is continuous, which motivates us to use continuous piecewise linear finite elements. We are going to look for a discrete optimal bathymetry that belongs to some finite element spaces $\mathcal{K}_h$, and thus we introduce the following set of discrete admissible parameters:

$$U_h = U \cap \mathcal{K}_h.$$

The full discretization of the optimization problem (3.6) then reads

$$\text{find } q_h^* \in U_h \text{ such that } \widetilde{J}(q_h^*) \leq \widetilde{J}(q_h) \quad \forall q_h \in U_h, \tag{5.1}$$

where $\widetilde{J}(q_h) = J(q_h, \psi_h(q_h))$ is the reduced cost functional and $\psi_h := \psi_h(q_h) \in \mathcal{V}_h$ satisfies the discrete Helmholtz problem

$$a(q_h; \psi_h, \phi_h) = b(q_h; \phi_h) \quad \forall \phi_h \in \mathcal{V}_h. \tag{5.2}$$

The existence of the solution to problem (5.2) will be discussed in the next subsection.

Before giving the definition of $\mathcal{K}_h$, we would like to discuss briefly the strategy for proving that the discrete optimal solution converges toward the continuous ones. To achieve this, we need to pass to the limit in inequality (5.1). Since J is only lower semicontinuous with respect to the weak* topology of BV, we can only pass to the limit on one side of the inequality, and the continuity of J will be needed to pass to the limit on the other side to keep this inequality valid as $h \to 0$.

We discuss first the case $U = U_\Lambda$ for which Theorem 3.8 gives the existence of optimal q but only if $\beta > 0$. Since we have to pass to the limit in (5.1), we need the fact that $\lim_{h\to 0} |Dq_h|(\Omega) = |Dq|(\Omega)$. Since the total variation is only continuous with respect to the strong topology of BV, we have to approximate any $q \in U_\Lambda$ by some $q_h \in U_h$ such that

$$\lim_{h\to 0} \|q - q_h\|_{BV(\Omega)} = 0.$$

However, from [5, Example 4.1, p. 8] there exists an example of a BV-function v that cannot be approximated by piecewise constant function v_h over a given mesh in such a way that $\lim_{h\to 0} |Dv_h|(\Omega) = |Dv|(\Omega)$. Nevertheless, if one considers an adapted mesh that depends on a given function $v \in BV(\Omega) \cap L^\infty(\Omega)$, we get the existence of a piecewise constant function on this specific mesh that strongly converges in BV toward v (see [8, Theorem 4.2, p. 11]). As a result, when considering $U = U_\Lambda$, we use the following discrete set of admissible parameters:

$$\mathcal{K}_{h,1} = \{q_h \in L^\infty(\Omega) \mid q_h|_T \in \mathbb{P}_1(T)\ \forall T \in \mathcal{T}_h\}.$$

Note that from Theorem [8, Theorem 4.1 and Remark 4.2, p. 10], the set $U_h = U_\Lambda \cap \mathcal{K}_{h,1}$ defined above has the required density property, and hence we introduce it as a discrete set of admissible parameters.

In the case $U = U_{\Lambda,\kappa}$, we need the density of U_h not for the strong topology of BV but only for the weak* topology. The discrete set of admissible parameters is then going to be $U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}$, with

$$\mathcal{K}_{h,0} = \{q_h \in L^\infty(\Omega) \mid q_h|_T \in \mathbb{P}_0(T)\ \forall T \in \mathcal{T}_h\}.$$

We show below the convergence of the discrete optimal solution to the continuous one for both cases highlighted above.

5.1. Convergence of the finite element approximation. We prove here some useful approximation results for any U_h defined above. We have the following convergence result whose proof can be found in [24, Lemma 4.1, p. 22] (see also [27, Theorem 4.1, p. 10]).

THEOREM 5.1. *Let $q_h \in U_h$, and let $\psi(q_h) \in H^1(\Omega)$ be the solution to the variational problem*

$$a(q_h; \psi(q_h), \phi) = b(q_h, \phi) \quad \forall \phi \in H^1(\Omega).$$

Let $S^* : (q_h, f) \in U_h \times L^2(\Omega) \mapsto S^*(q_h, f) = \psi^* \in H^1(\Omega)$ be the solution operator associated to the following problem:

$$\text{Find } \psi^* \in H^1(\Omega) \text{ such that } a(q_h; \phi, \psi^*) = (\phi, \overline{f})_{L^2(\Omega)} \quad \forall \phi \in H^1(\Omega).$$

Denote by C_a the continuity constant of the bilinear form $a(q_h; \cdot, \cdot)$, which does not depend on h since $q_h \in U_h$, and define the adjoint approximation property by

$$\delta(\mathcal{V}_h) := \sup_{f \in L^2(\Omega)} \inf_{\phi_h \in \mathcal{V}_h} \frac{\|S^*(q_h, f) - \phi_h\|_{1,k_0}}{\|f\|_{L^2(\Omega)}}.$$

Assume that the spaces $\mathcal{V}_h$ satisfy

$$2C_a k_0 \delta(\mathcal{V}_h) \le 1; \tag{5.3}$$

then the solution $\psi_h(q_h)$ to problem (5.2) satisfies

$$\|\psi(q_h) - \psi_h(q_h)\|_{1,k_0} \le 2C_a \inf_{\phi_h \in \mathcal{V}_h} \|\psi(q_h) - \phi_h\|_{1,k_0}.$$

We emphasize that the above error estimates in fact imply the existence and uniqueness of a solution to the discrete problem (5.2) (see [39, Theorem 3.9]). In the case $q \in \mathcal{C}^{0,1}(\Omega)$, where Ω is a convex Lipschitz domain, assumption (5.3) has been discussed in [27, Theorem 4.3, p. 11] and roughly amounts to saying that (5.3) holds if $k_0^2 h$ is small enough. Since the proof relies on H^2-regularity for a Poisson problem, we cannot readily extend the argument here since we can only expect to have $\psi \in H^1(\Omega)$ and the fact that S^* also depends on the meshsize. We can still show that (5.3) is satisfied for small enough h.

Lemma 5.2. *Assume that $q_h \in U_h$ weak* converges toward $q \in BV(\Omega)$. Then* (5.3) *is satisfied for small enough h.*

Proof. Note first that Theorem 3.7 also holds for the adjoint problem, and thus

$$\lim_{h\to 0} \|S^*(q_h, f) - S^*(q, f)\|_{1,k_0} = 0.$$

Using the density of smooth functions in H^1 and the properties of the piecewise linear interpolant [23, Corollary 1.122, p. 66], we have that

$$\lim_{h\to 0} \left(\sup_{f\in L^2(\Omega)} \inf_{\phi_h \in \mathcal{V}_h} \frac{\|S^*(q, f) - \phi_h\|_{1,k_0}}{\|f\|_{L^2(\Omega)}} \right) = 0,$$

and thus a triangular inequality shows that (5.3) holds for small enough h. □

We can now prove a discrete counterpart to Theorem 3.7.

Theorem 5.3. *Let $(q_h)_h \subset U_h$ be a sequence that weakly* converges toward q in $BV(\Omega)$. Let $(\psi_h(q_h))_h$ be the sequence of discrete solutions to problem* (5.2)*. Then $\psi(q_h)$ converges, as h goes to* 0*, strongly in $H^1(\Omega)$ toward $\psi(q)$ satisfying problem* (3.3)*.*

Proof. For h small enough, Lemma 5.2 ensures that (5.3) holds, and a triangular inequality then yields

$$\begin{aligned}
\|\psi_h(q_h) - \psi(q)\|_{1,k_0} &\le \|\psi_h(q_h) - \psi(q_h)\|_{1,k_0} + \|\psi(q_h) - \psi(q)\|_{1,k_0} \\
&\le 2C_a \inf_{\phi_h\in\mathcal{V}_h} \|\psi(q_h) - \phi_h\|_{1,k_0} + \|\psi(q_h) - \psi(q)\|_{1,k_0} \\
&\le (1 + 2C_a) \|\psi(q_h) - \psi(q)\|_{1,k_0} + 2C_a \inf_{\phi_h\in\mathcal{V}_h} \|\psi(q) - \phi_h\|_{1,k_0}.
\end{aligned}$$

Theorem 3.7 gives that the first term above goes to zero as $h \to 0$. For the second one, we can use the density of smooth functions in H^1 to get that it goes to zero as well. □

5.2. Convergence of the discrete optimal solution: Case $U_h = U_\Lambda \cap \mathcal{K}_{h,1}$. We are now in a position to prove the convergence of a discrete optimal design toward a continuous one in the case when

$$U = U_\Lambda,\ U_h = U_\Lambda \cap \mathcal{K}_{h,1}.$$

Hence the set of discrete controls is composed of piecewise linear functions on $\mathcal{T}_h$.

Theorem 5.4. *Assume that* (A1)-(A2)-(A3) *from Theorem* 3.8 *hold and that the cost function $J_0 : (q, \psi) \in U_\Lambda \times H^1(\Omega) \mapsto J_0(q, \psi) \in \mathbb{R}$ is continuous with respect to the* (weak*, strong) *topology of $BV(\Omega) \times H^1(\Omega)$. Let $(q_h^*, \psi_h(q_h^*)) \in U_{\Lambda,h} \times \mathcal{V}_h$ be an* *optimal pair of* (5.1). *Then the sequence* $(q_h^*)_h \subset U_\Lambda$ *is bounded, and there exists a subsequence (same notation used) and* $q^* \in U_\Lambda$ *such that* $q_h^* \rightharpoonup q^*$ *weakly** *in* $BV(\Omega)$, $\psi(q_h^*) \to \psi(q^*)$ *strongly in* $H^1(\Omega)$, *and*

$$\widetilde{J}(q^*) \leq \widetilde{J}(q) \quad \forall q \in U_\Lambda.$$

Hence any accumulation point of $(q_h^*, \psi_h(q_h^*))$ *is an optimal pair for problem* (3.6).

Proof. Let $q_\Lambda \in U_{\Lambda,h}$ be given as

$$q_\Lambda(x) = \Lambda \quad \forall x \in \Omega.$$

Then $Dq_\Lambda = 0$. Since $\psi_h(q_\Lambda)$ is well defined and converges toward $\psi(q_\Lambda)$ strongly in H^1 (see Theorem 5.4), we have that

$$\widetilde{J}(q_\Lambda) = J(q_\Lambda, \psi_h(q_\Lambda)) = J_0(q_\Lambda, \psi_h(q_\Lambda)) \xrightarrow[h\to 0]{} J_0(q_\Lambda, \psi(q_\Lambda)).$$

As a result, using the fact that $(q_h^*, \psi_h(q_h^*))$ is an optimal pair to problem (5.2), we get that

$$\beta |D(q_h^*)|(\Omega) \leq -J_0(q_h^*, \psi_h(q_h^*)) + J(q_\Lambda, \psi_h(q_\Lambda)) \leq -m + J_0(q_\Lambda, \psi_h(q_\Lambda)),$$

and thus the sequence $(q_h^*)_h \subset U_{\Lambda,h} \subset U_\Lambda$ is bounded in $BV(\Omega)$ uniformly with respect to h. We can then assume that it has a subsequence that converges and denote by $q^* \in U_\Lambda$ its weak* limit, and Theorem 5.3 then shows that $\psi_h(q_h^*) \to \psi(q^*)$ strongly in $H^1(\Omega)$. The lower semicontinuity of J ensures that

$$J(q^*, \psi(q^*)) = \widetilde{J}(q^*) \leq \liminf_{h\to 0} \widetilde{J}(q_h^*) = \liminf_{h\to 0} J(q_h^*, \psi_h(q_h^*)).$$

Now let $q \in U_\Lambda$; using the density of smooth functions in BV, one gets that there exists a sequence $q_h \in U_{\Lambda,h}$ such that $\|q_h - q^*\|_{BV(\Omega)} \to 0$ (see also [5, Remark 4.2, p. 10]). From Theorem 5.3, one gets $\psi_h(q_h) \to \psi(q)$ strongly in $H^1(\Omega)$, and the continuity of J ensures that $\widetilde{J}(q_h) \to \widetilde{J}(q)$. Since $\widetilde{J}(q_h^*) \leq \widetilde{J}(q_h)$ for all $q_h \in U_{\Lambda,h}$, by passing to the inf-limit one gets that

$$\widetilde{J}(q^*) \leq \liminf_{h\to 0} \widetilde{J}(q_h^*) \leq \liminf_{h\to 0} \widetilde{J}(q_h) = \widetilde{J}(q) \quad \forall q \in U_\Lambda,$$

and the proof is complete. ∎

5.3. Convergence of the discrete optimal solution: Case $\boldsymbol{U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}}$. We are now in a position to prove the convergence of a discrete optimal design toward a continuous one in the case when

$$U = U_{\Lambda,\kappa}, \ U_h = U_{\Lambda,\kappa} \cap \mathcal{K}_{h,0}.$$

Hence the set of discrete controls is composed of piecewise constant functions on $\mathcal{T}_h$ that satisfy

$$\forall q_h \in U_h, \ \|q_h\|_{BV(\Omega)} \leq 2\max(\Lambda, \kappa, |\alpha - 1|).$$

We can compute explicitly the previous norm by integrating by parts the total variation (see, e.g., [5, Lemma 4.1, p. 7]). This reads

$$\forall q_h \in U_h, \ |Dq_h|(\Omega) = \sum_{F\in\mathcal{F}^i} |F||[q_h]|_F|,$$

where $\mathcal{F}^i$ is the set of interior faces, and $|[q_h]|_F$ is the jump of q_h on the interior face $F = \partial T_1 \cap \partial T_2$, meaning that $|[q_h]|_F = |q_h|_{T_1} - |q_h|_{T_2}$, where $|\cdot|_{T_i}$ denotes the value of the finite element function on the face T_i. Note then that any $q_h \in U_h$ can only have either a finite number of discontinuity or jumps that are not too large.

THEOREM 5.5. *Assume that* $\beta = 0$ *and* (A2)–(A3) *from Theorem* 3.8 *hold and that the cost function* $J : (q, \psi) \in U_\Lambda \times H^1(\Omega) \mapsto J(q, \psi) \in \mathbb{R}$ *is continuous with respect to the* (weak*, strong) *topology of* $BV(\Omega) \times H^1(\Omega)$. *Let* $(q_h^*, \psi_h(q_h^*)) \in U_h \times \mathcal{V}_h$ *be an optimal pair of* (5.1). *Then the sequence* $(q_h^*)_h \subset U_{\Lambda,\kappa}$ *is bounded, and there exists* $q^* \in U_{\Lambda,\kappa}$ *such that* $q_h^* \rightharpoonup q^*$ *weakly* in* $BV(\Omega)$, $\psi(q_h^*) \to \psi(q^*)$ *strongly in* $H^1(\Omega)$, *and*

$$\widetilde{J}(q^*) \leq \widetilde{J}(q) \quad \forall q \in U_\Lambda.$$

Hence any accumulation point of $(q_h^*, \psi_h(q_h^*))$ *is an optimal pair for problem* (3.6).

Proof. Since $(q_h^*)_h$ belongs to U_h, it satisfies $\|q_h\|_{BV(\Omega)} \leq 2\max(\Lambda, \kappa, |\alpha - 1|)$ and thus is bounded uniformly with respect to h. We denote by $q^* \in U_{\Lambda,\kappa}$ the weak* limit of a converging subsequence. Theorem 5.4 then shows that $\psi_h(q_h^*)$ converges strongly in $H^1(\Omega)$ toward $\psi(q^*)$.

Now let $q \in U_{\Lambda,\kappa}$; using the density of smooth functions in BV, one gets that there exists a sequence $q_h \in U_h$ such that $q_h \rightharpoonup q$ weak* in $BV(\Omega)$ (see also [5, Introduction]). From Theorem 5.3, one gets $\psi_h(q_h) \to \psi(q)$ strongly in $H^1(\Omega)$, and the continuity of J ensures that $\widetilde{J}(q_h) \to \widetilde{J}(q)$. The proof can then be done as in the proof of Theorem 5.4. ∎

6. Numerical experiments.

In this section, we tackle numerically the optimization problem (3.6), when it is constrained to the total amplitude ψ_{tot} described by (2.8). We focus on two examples: a *damping problem*, where the computed bathymetry optimally reduces the magnitude of the incoming waves; and an *inverse problem*, in which we recover the bathymetry from the observed magnitude of the waves.

In what follows, we consider an incident plane wave $\psi_0(x) = \mathrm{e}^{\mathrm{i}k_0 x\cdot\vec{d}}$ propagating in the direction $\vec{d} = (0 \;\; 1)^\top$, with

$$k_0 = \frac{\omega_0}{\sqrt{g z_0}}, \quad \omega_0 = \frac{2\pi}{T_0}, \quad T_0 = 20, \quad g = 9.81, \quad z_0 = 3.$$

For the space domain, we set $\Omega = [0, L]^2$, where $L = \frac{10\pi}{k_0}$. We also impose an L^∞-constraint on the variable q, namely that $q \geq -0.9$.

6.1. Numerical methods.

We discretize the space domain by using a structured triangular mesh of 8192 elements, that is, a space step of $\Delta x = \Delta y = 8.476472$.

For the discretization of ψ_{sc}, we use a $\mathbb{P}^1$-finite element method. The optimized parameter q is discretized through a $\mathbb{P}^0$-finite element method. Hence, on each triangle, the approximation of ψ_{sc} is determined by three nodal values, located at the edges of the triangle, and the approximation of q is determined by one nodal value, placed at the center of gravity of the triangle.

On the other hand, we perform the optimization through a subspace trust-region method, based on the interior-reflective Newton method described in [18, 17]. Each iteration involves the solution of a linear system using the method of preconditioned conjugate gradients, for which we supply the Hessian multiply function. The computations are achieved with MATLAB (version 9.4.0.813654 (R2018a)).

Remark 6.1. The next numerical experiment aims at going further than the previous analysis. As a consequence, the considered setting does not meet all the assumptions of Theorem 5.4 (as well as those of Theorem 5.5; see section 6.3) which states the convergence of the optimum of the discretized/discrete problem toward the optimum of the continuous one. Indeed, regarding Theorem 5.4, the optimization parameters shall be unbounded functions, and we omit the penalization term $\beta|Dq|(\Omega)$ with $\beta > 0$ in the considered cost functions.

6.2. Example 1: A wave damping problem. We first consider the minimization of the cost functional

$$J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0} |\psi_{tot}(x, y)|^2 dxdy,$$

where $\Omega_0 = [\frac{L}{6}, \frac{5L}{6}]^2$ is the domain in which the waves are to be damped. The bathymetry is only optimized on a subset $\Omega_q = [\frac{L}{4}, \frac{3L}{4}]^2 \subset \Omega_0$.

The results are shown in Figure 2 for the bathymetry and Figure 3 for the wave. We observe that the optimal bathymetry we obtain is highly oscillating. In our experiments, this oscillation remained at every level of space discretization we tested. This could be related to the fact that in all our results, $q \in BV(\Omega)$. Note also that the damping is more efficient over Ω_q. This fact is consistent with the results of the next experiment.

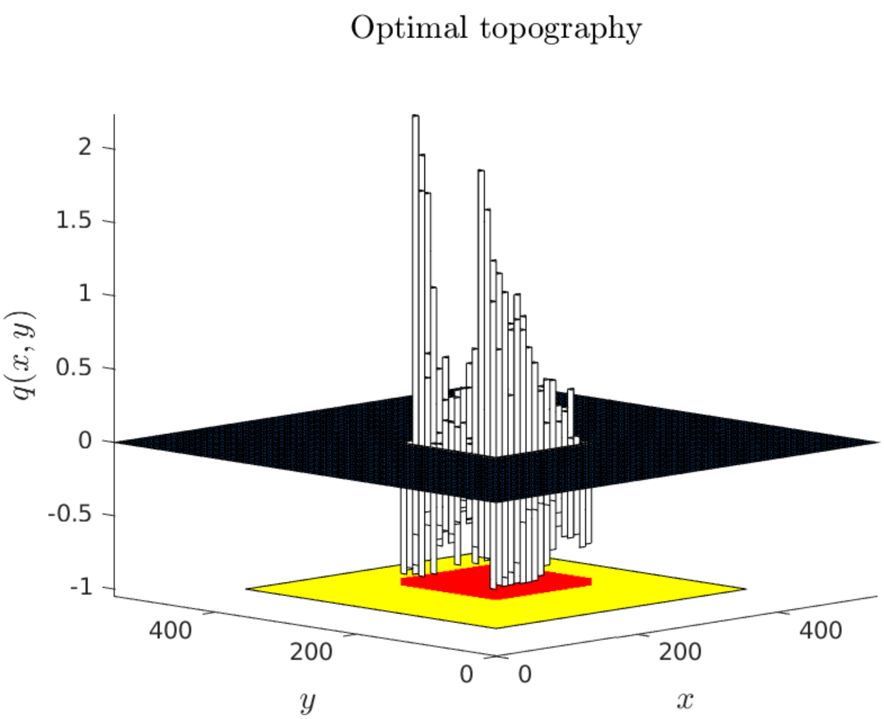
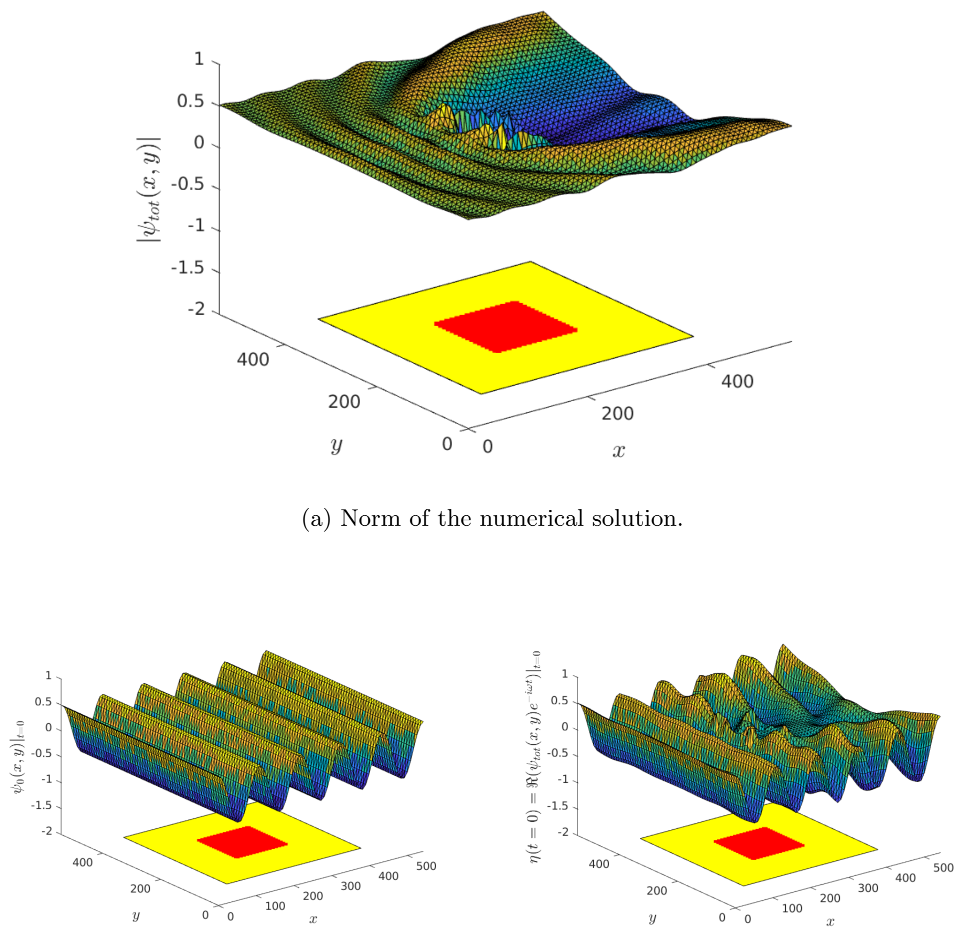


Fig. 2. *Optimal bathymetry for a wave damping problem. The yellow part represents Ω_0, and the red part corresponds to the nodal points associated with q. The black plane corresponds to the level of the flat bathymetry. (See online version for color.)*



(a) Norm of the numerical solution.

(b) Real part of the incident wave.

(c) Real part of the numerical solution.

Fig. 3. *Numerical solution of a wave damping problem. The yellow part represents* Ω_0*, and the red part corresponds to the nodal points associated with* q*. (See online version for color.)*

6.3. Example 2: An inverse problem. Many inverse problems associated with the Helmholtz equation have been studied in the literature. For example, we refer the reader to [19, 22, 46] and the references therein. Note that in most of these papers the inverse problem rather consists of determining the location of a scatterer or its shape, which often means that $q(x,y)$ is assumed to be constant inside and outside the scatterer. On the contrary, the inverse problem we consider in this section consists of determining a full real-valued function.

Given the bathymetry

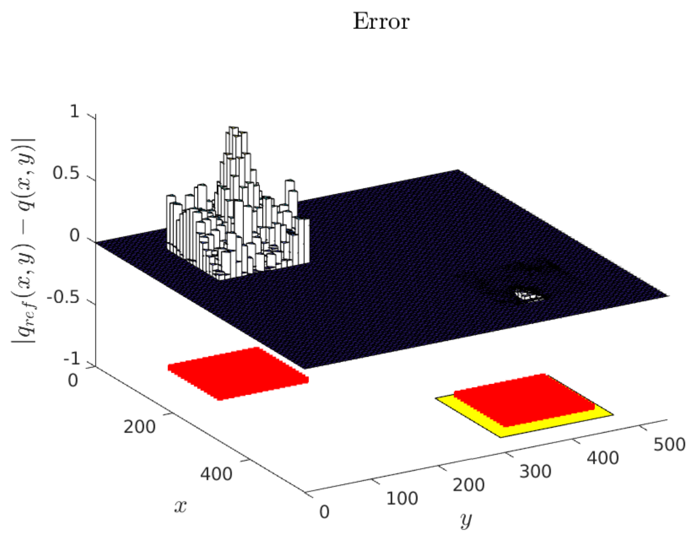
$$q_{ref}(x,y) := \mathrm{e}^{-\tau\left((x-\frac{L}{4})^2+(y-\frac{L}{4})^2\right)} + \mathrm{e}^{-\tau\left((x-\frac{3L}{4})^2+(y-\frac{3L}{4})^2\right)},$$

where $\tau = 10^{-3}$, we try to reconstruct it on the domain $\Omega_q = [\frac{L}{8}, \frac{3L}{8}]^2 \cup [\frac{5L}{8}, \frac{7L}{8}]^2$ by minimizing the cost functional

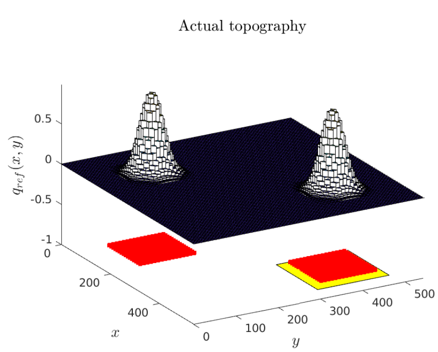
$$J(q, \psi_{tot}) = \frac{\omega_0^2}{2} \int_{\Omega_0} |\psi_{tot}(x, y) - \psi_{ref}(x, y)|^2 dxdy,$$

where ψ_{ref} is the amplitude associated with q_{ref}, and $\Omega_0 = [\frac{3L}{4} - \delta, \frac{3L}{4} + \delta]^2$, $\delta = \frac{L}{6}$. Note that in this case, Ω_q is not contained in Ω_0.

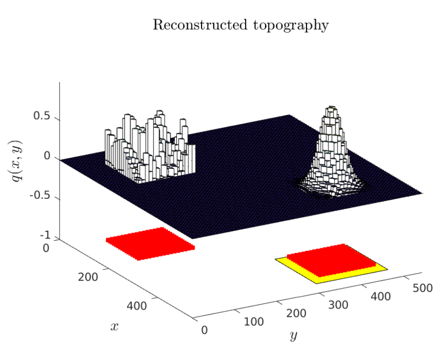
In Figure 4, we observe that the part of the bathymetry that does not belong to the observed domain Ω_0 is not recovered by the procedure. On the contrary, the bathymetry is well reconstructed in the part of the domain corresponding to Ω_0.



(a) Reconstruction error.



(b) Actual bathymetry.



(c) Reconstructed bathymetry.

FIG. 4. *Detection of a bathymetry from a wavefield. The yellow part represents Ω_0, and the red part corresponds to the nodal points associated with q. (See online version for color.)*

In this example, the assumptions of Theorem 5.5 are also relaxed. Indeed, though we look for a bounded and piecewise constant q_h, we do not demand that $|Dq_h|(\Omega) \leq \kappa$

for some $\kappa > 0$. Nevertheless, we have observed in our numerical experiments that $|Dq_h|(\Omega) = \mathcal{O}(h^{-s})$ for some $s > 0$. This result is reported in Figure 5.

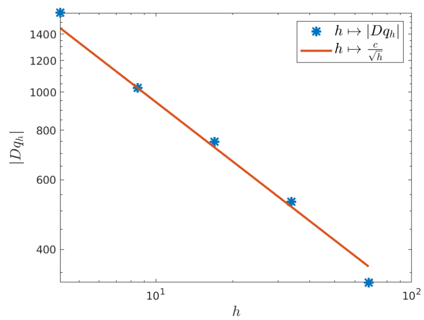


FIG. 5. *Norm of $Dq_h(\Omega)$ (blue stars) for various values of h. (See online version for color.)*

It is worth noting that these numerical results show that imposing an upper bound on $|Dq_h|$ (either using a penalization term in the cost function or imposing it in the admissible set) is crucial to proving the existence of optimal bathymetry (see Theorems 3.8 and 3.9).

Appendix A. Derivation of Saint-Venant system. For the sake of completeness, and following the standard procedure described in [25] (see also [12, 44]), we derive the Saint-Venant equations from the Navier–Stokes system. For simplicity of presentation, system (2.1) is restricted to two dimensions, but a more detailed derivation of the three-dimensional case can be found in [21]. Since our analysis focuses on the shallow water regime, we introduce the parameter $\varepsilon := \frac{H}{L}$, where H denotes the relative depth, and L is the characteristic dimension along the horizontal axis. The importance of the nonlinear terms is represented by the ratio $\delta := \frac{A}{H}$, with A the maximum vertical amplitude. We then use the change of variables

$$x' := \frac{x}{L},\ z' := \frac{z}{H},\ t' := \frac{C_0}{L}t$$

and

$$u' := \frac{u}{\delta C_0},\ w' := \frac{w}{\delta\varepsilon C_0},\ \eta' := \frac{\eta}{A},\ z_b' := \frac{z_b}{H},\ p' := \frac{p}{gH},$$

where $C_0 = \sqrt{gH}$ is the characteristic dimension for the horizontal velocity. Assuming that the viscosity and atmospheric pressure are constants, we define their respective dimensionless versions by

$$\mu' := \frac{\mu}{C_0 L},\ p_a' := \frac{p_a}{gH}.$$

Dropping primes after rescaling, the dimensionless system (2.1) reads

$$\delta\frac{\partial u}{\partial t}+\delta^2\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+2\delta\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \delta\frac{\partial}{\partial z}\left(\mu\Big(\frac{1}{\varepsilon^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\Big)\right), \tag{A.1}$$

$$\varepsilon^2\delta\left(\frac{\partial w}{\partial t}+\delta\Big(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\Big)\right)=-\frac{\partial p}{\partial z}-1 + \delta\frac{\partial}{\partial x}\left(\mu\Big(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\Big)\right)+2\delta\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial z}\right), \tag{A.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{A.3}$$

The boundary condition in (2.2) remains similar and reads

$$\begin{cases}-\delta u\dfrac{\partial \eta}{\partial x}+w=\dfrac{\partial \eta}{\partial t}\sqrt{1+(\varepsilon\delta)^2\left|\dfrac{\partial \eta}{\partial x}\right|^2} & \text{on } (x,\delta\eta(x,t),t),\\ u\dfrac{\partial z_b}{\partial x}+w=0 & \text{on } (x,-z_b(x),t).\end{cases} \tag{A.4}$$

However, the rescaled boundary conditions in (2.3) are now given by

$$\left(p-2\delta\mu\frac{\partial u}{\partial x}\right)\frac{\partial \eta}{\partial x}+\mu\left(\frac{1}{\varepsilon^2}\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=p_a\frac{\partial \eta}{\partial x} \qquad \text{on } (x,\delta\eta(x,t),t), \tag{A.5}$$

$$\delta^2\mu\left(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\right)\frac{\partial \eta}{\partial x}+\left(p-2\delta\mu\frac{\partial w}{\partial z}\right)=p_a \qquad \text{on } (x,\delta\eta(x,t),t), \tag{A.6}$$

and at the bottom $(x,-z_b(x),t)$,

$$\varepsilon\left(p-2\delta\mu\frac{\partial u}{\partial x}\right)\frac{\partial z_b}{\partial x}+\delta\mu\left(\frac{1}{\varepsilon}\frac{\partial u}{\partial z}+\varepsilon\frac{\partial w}{\partial x}\right) -\delta\mu\Big(\frac{\partial u}{\partial z}+\varepsilon^2\frac{\partial w}{\partial x}\Big)\left(\frac{\partial z_b}{\partial x}\right)^2+\varepsilon\left(2\delta\mu\frac{\partial w}{\partial z}-p\right)\frac{\partial z_b}{\partial x}=0. \tag{A.7}$$

To derive the Saint-Venant equations, we use an asymptotic analysis in ε. In addition, we assume a small viscosity coefficient

$$\mu=\varepsilon\mu_0.$$

A first simplification of the system consists of deriving an explicit expression for p, known as the *hydrostatic pressure*. Indeed, after rearranging the terms of order ε^2 in (A.2) and integrating in the vertical direction, we get

$$p(x,z,t)=\mathcal{O}(\varepsilon^2\delta)+(\delta\eta-z)+\varepsilon\delta\mu_0\left(\frac{\partial u}{\partial x}+2\frac{\partial w}{\partial z}-\frac{\partial u}{\partial x}(x,\eta,t)\right) + p(x,\delta\eta,t)-2\varepsilon\delta\mu_0\frac{\partial w}{\partial z}(x,\eta,t). \tag{A.8}$$

To compute explicitly the last term, we combine (A.5) with (A.6) to obtain

$$p(x,\delta\eta,t) - 2\varepsilon\delta\mu_0 \frac{\partial w}{\partial z}(x,\delta\eta,t) = p_a\left(1 - (\varepsilon\delta)^2\Big(\frac{\partial \eta}{\partial x}\Big)^2\right) + (\varepsilon\delta)^2\left(p - 2\varepsilon\mu_0\frac{\partial u}{\partial x}(x,\eta,t)\right)\left(\frac{\partial \eta}{\partial x}\right)^2,$$

which can be combined with (A.8) to obtain

$$p(x,z,t) = (\delta\eta - z) + p_a + \mathcal{O}(\varepsilon\delta). \tag{A.9}$$

As a second approximation, we integrate vertically (A.3) and (A.1). We introduce $h_\delta = \delta\eta + z_b$. Due to the Leibniz integral rule and the boundary conditions in (A.4), integrating the mass equation (A.3) gives

$$\int_{-z_b}^{\delta\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)dz = 0,$$

$$\frac{\partial}{\partial x}\left(\int_{-z_b}^{\delta\eta} u dz\right) - \delta u(x,\delta\eta,t)\frac{\partial \eta}{\partial x} - u(x,-z_b,t)\frac{\partial z_b}{\partial x} + w(x,\delta\eta,t) - w(x,-z_b,t) = 0,$$

$$\frac{\partial \eta}{\partial t}\sqrt{1 + (\varepsilon\delta)^2\left|\frac{\partial \eta}{\partial x}\right|^2} + \frac{\partial (h_\delta \overline{u})}{\partial x} = 0.$$

To treat the momentum equation (A.1), we note that (A.3) allows us to rewrite the convective acceleration terms as

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z}.$$

Its integration, combined with the boundary conditions in (A.4), leads to

$$\begin{aligned}\int_{-z_b}^{\delta\eta}\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)dz &= \frac{\partial}{\partial x}\left(\int_{-z_b}^{\delta\eta} u^2 dz\right) - \delta u^2(x,\delta\eta,t)\frac{\partial \eta}{\partial x} - u^2(x,-z_b,t)\frac{\partial z_b}{\partial x} \\ &\quad + u(x,\delta\eta,t)\cdot w(x,\delta\eta,t) - u(x,-z_b,t)\cdot w(x,-z_b,t) \\ &= \frac{\partial (h_\delta \overline{u^2})}{\partial x} + u(x,\delta\eta,t)\frac{\partial \eta}{\partial t}\sqrt{1 + (\varepsilon\delta)^2\left|\frac{\partial \eta}{\partial x}\right|^2},\end{aligned}$$

where we have introduced the depth-averaged velocity

$$\overline{u}(x,t) := \frac{1}{h_\delta(x,t)}\int_{-z_b}^{\delta\eta} u(x,z,t)dz.$$

The vertical integration of the left-hand side of (A.1) then brings

$$\int_{-z_b}^{\delta\eta}\left[\delta\frac{\partial u}{\partial t} + \delta^2\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right)\right]dz = \delta\frac{\partial (h_\delta\overline{u})}{\partial t} + \delta^2\frac{\partial (h_\delta\overline{u^2})}{\partial x} + \delta^2 u(x,\delta\eta,t)\frac{\partial \eta}{\partial t}\left(\sqrt{1 + (\varepsilon\delta)^2\left|\frac{\partial \eta}{\partial x}\right|^2} - 1\right).$$

To deal with the term $h_\delta \overline{u^2}$, we start from (A.9), which shows that $\frac{\partial p}{\partial x} = \mathcal{O}(\delta)$. Plugging this expression into (A.1) yields

$$\frac{\partial^2 u}{\partial z^2} = \mathcal{O}(\varepsilon).$$

From boundary conditions (A.5) and (A.7), we obtain

$$\frac{\partial u}{\partial z}(x, \delta\eta, t) = \mathcal{O}(\varepsilon^2), \ \frac{\partial u}{\partial z}(x, z_b, t) = \mathcal{O}(\varepsilon).$$

Consequently, $u(x, z, t) = u(x, 0, t) + \mathcal{O}(\varepsilon)$, and then $u(x, z, t) - \overline{u}(x, t) = \mathcal{O}(\varepsilon)$. Hence, we have the approximation

$$h_\delta \overline{u^2} = h_\delta \overline{u}^2 + \int_{-z_b}^{\delta\eta} (\overline{u} - u)^2 dz = h_\delta \overline{u}^2 + \mathcal{O}(\varepsilon^2),$$

and finally,

$$\begin{aligned}\int_{-z_b}^{\delta\eta} \left[\delta \frac{\partial u}{\partial t} + \delta^2 \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right)\right] dz = \delta \frac{\partial (h_\delta \overline{u})}{\partial t} + \delta^2 \frac{\partial (h_\delta \overline{u}^2)}{\partial x} + \mathcal{O}(\varepsilon^2 \delta^2) \\ + \delta^2 u(x, \delta\eta, t) \frac{\partial \eta}{\partial t}\left(\sqrt{1 + (\varepsilon\delta)^2 \left|\frac{\partial \eta}{\partial x}\right|^2} - 1\right).\end{aligned} \tag{A.10}$$

We then integrate the right-hand side of (A.1) to obtain

$$\begin{aligned}\int_{-z_b}^{\delta\eta} \left[-\frac{\partial p}{\partial x} + \delta \frac{\mu_0}{\varepsilon} \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) + \varepsilon\delta\mu_0 \left(2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right)\right)\right] dz \\ = -\delta h_\delta \frac{\partial \eta}{\partial x} + \mathcal{O}(\varepsilon\delta) + \delta \left[\frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) - \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t)\right].\end{aligned}$$

Combining this expression with (A.10), we get the following vertical integration of the momentum equation:

$$\frac{\partial \eta}{\partial t} \sqrt{1 + (\varepsilon\delta)^2 \left|\frac{\partial \eta}{\partial x}\right|^2} + \frac{\partial (h_\delta \overline{u})}{\partial x} = 0, \tag{A.11}$$

$$\begin{aligned}\frac{\partial (h_\delta \overline{u})}{\partial t} + \delta \frac{\partial (h_\delta \overline{u}^2)}{\partial x} = -h_\delta \frac{\partial \eta}{\partial x} + \left[\frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) - \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t)\right] \\ + \delta u(x, \delta\eta, t) \frac{\partial \eta}{\partial t}\left(\sqrt{1 + (\varepsilon\delta)^2 \left|\frac{\partial \eta}{\partial x}\right|^2} - 1\right) + \mathcal{O}(\varepsilon).\end{aligned} \tag{A.12}$$

The convergence of (A.12) is guaranteed by the boundary equations (A.5) and (A.7), from which we get

$$\frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, \delta\eta, t) = \mathcal{O}(\varepsilon\delta), \ \frac{\mu_0}{\varepsilon} \frac{\partial u}{\partial z}(x, -z_b, t) = \mathcal{O}(\varepsilon).$$

Hence, we have the system (2.4),(2.5).

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