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HIGHER ORDER TRIANGULAR FINITE ELEMENTS WITH MASS LUMPING FOR THE WAVE EQUATION*

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Abstract. In this article, we construct new higher order finite element spaces for the approximation of the two-dimensional (2D) wave equation. These elements lead to explicit methods after time discretization through the use of appropriate quadrature formulas which permit mass lumping. These formulas are constructed explicitly. Error estimates are provided for the corresponding semidiscrete problem. Finally, higher order finite difference time discretizations are proposed and various numerical results are shown.

Key words. finite elements, mass lumping, wave equation

AMS subject classifications. 65M60, 65M12, 65M15, 65C20, 73D25

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1. Introduction.

Solving the wave equation in the time domain by numerical methods is a delicate but fundamental problem for modeling numerous physical phenomena such as acoustic, elastic, or electromagnetic waves. For such phenomena, the wave equation serves as a model problem. The numerical approximation of this equation is essential, in particular, for two cases for which analytical methods do not apply:

- heterogeneous media,
- domains of arbitrary shape.

Finite difference methods (FDMs) [1], [7] are obviously less well adapted to handling domains of complicated shape than are finite element methods (FEMs), even in homogeneous media. On the other hand, the use of FEMs presents a major drawback, the presence of a mass matrix which must be inverted at each time step, and it is absolutely fundamental for the efficiency of the method to overcome this problem. For low order Lagrange elements, namely, P_1 elements, a solution is given by the so-called mass lumping procedure (see [6], [23]) which is closely related to the use of quadrature rules for the numerical evaluation of integrals over a triangle in two dimensions or a tetrahedron in three dimensions. The solution is much less obvious in the case of higher order finite elements. Yet, it is now a commonly admitted fact that, to obtain a good approximation to the solution of a realistic problem, higher order elements are necessary (see, for instance, [1], [7], [8], [9]). The question posed is thus: how can mass lumping be carried out for higher order elements? This question has already been addressed in [21], [22], [9], [10], [11], and [33] for the case of Q_k elements, i.e., elements constructed on quadrilateral meshes in two dimensions. It is shown that for $k \geq 3$, one must modify the usual locations of the degrees of freedom in the elements (here we are referring to Lagrange-type finite elements), since they must coincide with the Gauss–Lobatto quadrature points in order to lump the mass matrix. This approach corresponds, in fact, to spectral finite elements which are generally constructed for quadrilateral meshes; cf. [24], [31]. The purpose of this article is to construct a class of H^1-conforming P_2 and P_3 triangular finite elements

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for which mass lumping can be carried out and which will have the third or fourth order accuracy of Q_2 or Q_3 quadrilateral finite elements [6]. How to extend this procedure to triangular meshes is, however, not evident: one must find a quadrature rule equivalent to that of Gauss–Lobatto. Such a formula cannot be obtained by a simple tensor product using the one-dimensional (1D) formula, as in the case of quadrilaterals. Moreover, as we shall see later, the quadrature rules that one would normally apply to triangles [23] result in some negative (for P_3) or zero (for P_2) weights which then lead to unstable schemes.

To overcome this difficulty, we construct finite element spaces, $\tilde{P}_k$, $k = 2, 3$, and propose new quadrature rules with positive weights, thereby ensuring the stability of the resulting schemes. In the case $k = 2$, we have since learned, these finite elements are a scalar version of finite elements that have been used for the Stokes problem; cf. [14], [25], or [20]. In the case $k = 3$, the more interesting case which corresponds in terms of accuracy to the fourth order schemes that are now more and more often employed for wave propagation problems, these finite elements seem to be new.

This article is organized as follows: section 2 is devoted to preliminary material about quadrature formulas in triangles. In section 3, the difficulties linked to the use of classical P_2 and P_3 elements are discussed in more detail. In section 4, the $\tilde{P}_k$ spaces for $k = 2$ and $k = 3$ are presented. In section 5, error estimates for the semidiscretized problem are established, using a method based on the Laplace transform in time. In section 6, the fully discretized methods are constructed with particular attention being paid to higher order time discretization. The stability of the schemes is analyzed. Finally, in section 7, numerical results illustrating the theoretical part of the paper are given.

2. Preliminaries concerning symmetric quadrature formulas in a triangle. For a better understanding of the next two sections, we briefly recall in this section some notions related to quadrature formulas in several variables (see also [16], [13]).

- In what follows, $x = (x_1, x_2)$ will denote a variable in $\mathbb{R}^2$ and K a triangle in $\mathbb{R}^2$ whose vertices are S_1, S_2, and S_3. The barycentric coordinates of x with respect to S_1, S_2, and S_3 are $(\lambda_1(x), \lambda_2(x), \lambda_3(x))$; cf. [28].
- To any function of *three* variables $f(\lambda_1, \lambda_2, \lambda_3)$ is associated a function $\tilde{f}(x)$ of *two* variables defined by $\widetilde{f}(x) = f(\lambda_1(x), \lambda_2(x), \lambda_3(x))$. We shall often use the same notation both for f and for $\widetilde{f}$.
- The space of polynomials in *two* variables of degree less than or equal to k is denoted by P_k, and the space of *homogeneous* polynomials in *three* variables of degree k is denoted by P_k^{hom}. The space P_k can be identified with the space P_k^{hom} via the mapping $f \to \widetilde{f}$.
- Let $\mathcal{S}_3$ be the group of permutations on $\{1, 2, 3\}$. For x in K and σ in $\mathcal{S}_3$, let $x_\sigma \in K$ be defined by

  $$\forall\, i \in \{1, 2, 3\} \quad \lambda_i(x_\sigma) = \lambda_{\sigma(i)}(x),$$

  and for a function f, let f_σ be the function defined by

  $$f_\sigma(x) = f(x_\sigma).$$

Note that, if f is an element of $L^1(K)$, one can write

$$I_K(f) \equiv \int_K f(x)dx = \int_{\mathbb{R}^3_+} f(\lambda_1, \lambda_2, \lambda_3)\, \delta(1 - \lambda_1 - \lambda_2 - \lambda_3)\, d\lambda_1 d\lambda_2 d\lambda_3, \tag{2.1}$$

where $\delta(1-\lambda_1-\lambda_2-\lambda_3)$ denotes the Dirac distribution supported by the plane $\lambda_1+\lambda_2+\lambda_3=1$. Formula (2.1) shows that the linear form $f \to I_K(f)$ is symmetric (simply use the change of variable $\mu_i = \lambda_{\sigma(i)}$):

$$\forall\,\sigma \in S_3, \quad \int_K f\,dx = \int_K f_\sigma\,dx. \tag{2.2}$$

In what follows, a quadrature formula on K will be defined by a finite set of quadrature points $\mathcal{Q} = \{x_i\}$, belonging to K, and a corresponding set of weights $\{\omega_i\}$ belonging to $\mathbb{R}$. Then, for any $f \in C(K)$, an approximation $I_K^{\text{app}}(f)$ of $I_K(f)$ is defined by

$$I_K^{\text{app}}(f) = \text{mes}(K)\sum_i \omega_i f(x_i). \tag{2.3}$$

In light of (2.2), it is natural to ask that the linear form $I_K^{\text{app}}(f)$ be symmetric:

$$\forall\,\sigma \in S_3 \quad \forall\, f \in C(K), \quad I_{K(f)}^{\text{app}} = I_K^{\text{app}}(f_\sigma). \tag{2.4}$$

Thus we shall consider only symmetric quadrature formulas, constructed as follows:

(a) Let $\mathcal{Q}$ as a symmetric set of quadrature points:

$$x \in \mathcal{Q} \Rightarrow x_\sigma \in \mathcal{Q} \quad \forall\,\sigma \in S_3 \tag{2.5}$$

and define an equivalence relation on $\mathcal{Q}$ by

$$x\,\mathcal{R}\,y \Leftrightarrow \exists\,\sigma \in S_3 \;/\; y = x_\sigma. \tag{2.6}$$

(b) Let $\{C_\alpha,\ 1 \le \alpha \le M\}$ be the set of equivalence classes of $\mathcal{R}$ and define a symmetric quadrature formula by

$$I_K^{\text{app}}(f) = \text{mes}(K)\sum_{\alpha=1}^{M}\omega_\alpha\left[\sum_{x\in C_\alpha} f(x)\right]. \tag{2.7}$$

Note that such a formula can be deduced from the equivalent formula for the reference triangle $\widehat{K}$.

The following question arises: for a given set P of polynomials, what conditions must a quadrature formula satisfy in order that it be exact in P?

In fact, we shall consider only symmetric finite dimensional spaces of polynomials P:

$$\forall\, p \in P \quad \forall\,\sigma \in \mathcal{S}_3, \quad p_\sigma \in P,$$

and for such a space P we choose a basis $\mathcal{B}(P)$ having the same symmetry properties. Note in particular that P_k is symmetric. Let us define an equivalence relation in $\mathcal{B}(P)$ by

$$r\,\widetilde{\mathcal{R}}\,s \Leftrightarrow \exists\,\sigma \in S_3 \;/\; s = r_\sigma, \tag{2.8}$$

and let $\{\widetilde{C}_\beta, 1 \le \beta \le \widetilde{M}\}$ be the set of equivalence classes of $\widetilde{\mathcal{R}}$. The following result is immediate.

Lemma 2.1. *A symmetric quadrature formula $I_K^{\text{app}}(f)$ is exact for any element of P if and only if it is exact for one representative of each equivalence class $\widetilde{C}_\beta$.*

This lemma indicates that $\widetilde{M}$ is the number of degrees of freedom needed for the quadrature formula in order that it be exact for any element of P. Note, however, that this does not necessarily mean that M (the number of equivalence classes in $\mathcal{Q}$) must be equal to $\widetilde{M}$ (the number of equivalence classes in $\mathcal{B}(P)$) since the location of the quadrature points as well as the weights are degrees of freedom for the quadrature formula.

Consider, for instance, the equivalence classes $\widetilde{C}_\beta$ for the case $P = P_k$. (We denote by N_k the dimension of P_k and write $\widetilde{M}_k$ for $\widetilde{M}$.) Since, in two dimensions, each class is made up of one, three, or six elements, we have

$$\widetilde{M}_k \le N_k \le 6\widetilde{M}_k. \tag{2.9}$$

For the first five values of k, one has

(i) $k = 1,\quad N_1 = 3,\quad \widetilde{M}_1 = 1,\quad C_1 = \{\lambda_1, \lambda_2, \lambda_3\},$

(ii) $k = 2,\quad N_2 = 6,\quad \tilde{M}_2 = 2,\quad \begin{array}{l} C_1 = \{\lambda_1^2, \lambda_2^2, \lambda_3^2\}, \\ C_2 = \{\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3\}, \end{array}$

(iii) $k = 3,\quad N_2 = 10,\quad \widetilde{M}_2 = 3,\quad \begin{array}{l} C_1 = \{\lambda_1^3, \lambda_2^3, \lambda_3^3\}, \\ C_2 = \{\lambda_1^2\lambda_2, \lambda_1^2\lambda_3, \lambda_2^2\lambda_1, \lambda_2^2\lambda_3, \lambda_3^2\lambda_1, \lambda_3^2\lambda_2\}, \\ C_3 = \{\lambda_1\lambda_2\lambda_3\}, \end{array}$

(iv) $k = 4,\quad N_4 = 15,\quad \widetilde{M}_4 = 4,\quad \begin{array}{l} C_1 = \{\lambda_1^4, \lambda_2^4, \lambda_3^4\}, \\ C_2 = \{\lambda_1^3\lambda_2, \lambda_1^3\lambda_3, \lambda_2^3\lambda_1, \lambda_2^3\lambda_3, \lambda_3^3\lambda_1, \lambda_3^3\lambda_2\}, \\ C_3 = \{\lambda_1^2\lambda_2^2, \lambda_1^2\lambda_3^2, \lambda_2^2\lambda_3^2\}, \\ C_4 = \{\lambda_1^2\lambda_2\lambda_3, \lambda_2^2\lambda_1\lambda_3, \lambda_3^2\lambda_1\lambda_2\}, \end{array}$

(v) $k = 5,\quad N_5 = 21,\quad \widetilde{M}_5 = 5,\quad \begin{array}{l} C_1 = \{\lambda_1^5, \lambda_2^5, \lambda_3^5\}, \\ C_2 = \{\lambda_1^4\lambda_2, \lambda_1^4\lambda_3, \lambda_2^4\lambda_1, \lambda_2^4\lambda_3, \lambda_3^4\lambda_1, \lambda_3^4\lambda_2\}, \\ C_3 = \{\lambda_1^3\lambda_2^2, \lambda_1^3\lambda_3^2, \lambda_2^3\lambda_1^2, \lambda_2^3\lambda_3^2, \lambda_3^3\lambda_1^2, \lambda_3^3\lambda_2^2\}, \\ C_4 = \{\lambda_1^3\lambda_2\lambda_3, \lambda_2^3\lambda_1\lambda_3, \lambda_3^3\lambda_1\lambda_2\}. \end{array}$

Note that, in each case, $\widetilde{M}_k$ is much less than N_k: the number of degrees of freedom needed for the quadrature formula is much less than the dimension of the space of polynomials one wishes to integrate exactly. Contrary to what one might think, it is not true that $\widetilde{M}_k = k$: one can check that $\widetilde{M}_7 = 8$ and $\widetilde{M}_8 = 9$. One can obtain an exact formula for $\widetilde{M}_k$ by noticing that the numbers of triplets (p, q, r) of integers satisfying $0 \le p \le q \le r$ and $p + q + r = k$ is the same as the number of possible decompositions of k as a sum of integers between 1 and 3. Using the theory of generating functions (see [30]), one obtains

$$\widetilde{M}_k = \frac{k^2}{12} + \frac{k}{2} + \frac{47}{72} + \frac{(-1)^n}{8} + \frac{2}{9}\cos\left(\frac{2k\pi}{3}\right). \tag{2.10}$$

In particular, for large k, one has

$$\widetilde{M}_k \sim \frac{k(k+1)}{12} = \frac{N_k}{6},$$

a result suggested by the fact that in the set of polynomials of arbitrary degree each equivalence class has "generically" six elements.

3. Difficulties related to classical finite elements.

3.1. The mass lumping problem. We consider the following model problem:

$$\text{(3.1)}\qquad \begin{cases} \text{Find } u : \Omega \times \mathbb{R}^+ \to \mathbb{R} \quad \text{so that} \\ \dfrac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0, \quad (x,t) \in \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad (x,t) \in \Gamma = \partial\Omega \times \mathbb{R}^+, \end{cases}$$

where Ω denotes a bounded polygonal subset of $\mathbb{R}^2$. Let $\mathcal{T}_h$ be a triangulation of Ω that satisfies the usual properties of finite element meshes, and let $V_h \subset H_0^1(\Omega)$ be an approximation subspace of Lagrange finite elements subordinate to $\mathcal{T}_h$. The problem to be solved can be written (omitting the initial conditions) as follows:

$$\text{(3.2)}\qquad \begin{cases} \text{Find } u_h(t) : \mathbb{R}^+ \to V_h \quad \text{so that} \\ \dfrac{d^2}{dt^2} \displaystyle\int_\Omega u_h(x,t) v_h(x) dx + \int_\Omega \nabla u_h(x,t) \nabla v_h(x) dx = 0 \quad \forall\, v_h \in V_h. \end{cases}$$

Let $\{a_i\}$ be the set of the degrees of freedom associated with V_h and let $\{w_i\}$ be the corresponding Lagrange basis. If U_h denotes the vector whose components are the coordinates of u_h in the basis $\{w_i\}$, then (3.2) is equivalent to the ordinary differential system

$$\text{(3.3)}\qquad M_h \frac{d^2 U_h}{dt^2}(t) + K_h U_h(t) = 0,$$

where the mass matrix M_h and the stiffness matrix K_h are given, respectively, by

$$\text{(3.4)}\qquad \begin{cases} M_{ij} = \displaystyle\int_\Omega w_i(x)\, w_j(x)\, dx = (w_i, w_j), \\ K_{ij} = \displaystyle\int_\Omega \nabla w_i(x)\, \nabla w_j(x)\, dx. \end{cases}$$

The problem addressed in this article is the fact that the mass matrix M_h, although it approximates the identity operator, is not diagonal, due to the fact that two different basis functions are not necessarily orthogonal in L^2:

$$i \neq j \quad \not\Rightarrow \quad (w_i, w_j) = 0.$$

If now, instead of computing the integrals defining M_{ij} exactly, we evaluate them approximately using, in each triangle K of $\mathcal{T}_h$, a quadrature formula like those in section 2, we in fact replace the continuous L^2 inner product (w_i, w_j) by the discrete inner product $(w_i, w_j)_h$, defined by

$$\text{(3.5)}\qquad \forall\, (u,v) \in C(\overline{\Omega}) \quad (u,v)_h = \sum_l \omega_{l,h} u(\hat{a}_l) v(\hat{a}_l),$$

where $\{\hat{a}_l\}$ is the union over all the triangles of $\mathcal{T}_h$ of all the quadrature points and $\{\omega_{l,h}\}$ is the set of all the appropriate weights. Mass lumping will be achieved if two

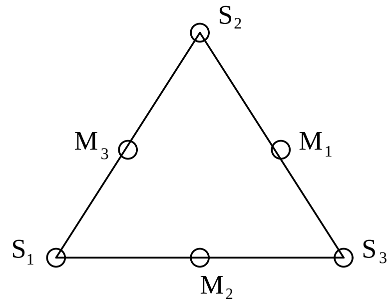

FIG. 3.1. P_2 *finite element.*

different basis functions are now orthogonal for the discrete inner product $(.,.)_h$ or equivalently if the new approximation to the mass matrix is diagonal. All the developments in the remainder of the paper are based on the following trivial observation.

LEMMA 3.1. *If the nodes of the finite element space V_h and the quadrature points coincide, i.e., if*

$$\{a_i\} = \{\hat{a}_l\}, \tag{3.6}$$

then one has mass lumping.

Indeed, it suffices to remark that if $i \neq j$, the product $w_i w_j$ necessarily vanishes at each point $\hat{a}_l$. For P_1 Lagrange elements, with the trapezoidal quadrature rule,

$$I_K^{\text{app}}(f) = \frac{\text{mes}(K)}{3}(f(S_1) + f(S_2) + f(S_3)), \tag{3.7}$$

Lemma 3.1 guarantees that the approximate mass matrix is diagonal. For higher order elements, such as P_2 or P_3, the use of condition (3.6) is complicated by the fact that one does not want to lose accuracy by applying numerical integration. In other words, the quadrature formula must also be sufficiently accurate. This is, in fact, the case for the trapezoidal rule coupled to P_1 elements. The general conditions to be met in order to ensure sufficient accuracy have been known for more than 20 years. First derived for application to elliptic problems (see [6], [19]), these conditions were then generalized to parabolic problems [27] and to hyperbolic problems (see [2], [3]). For standard P_k elements the rule is

In each triangle K, the quadrature formula $I_K^{\text{app}}(f)$ must be exact for P_{2k-2}.

In the next two sections it is shown that satisfying this condition (and thereby preserving the accuracy of P_2 and P_3 Lagrange elements) poses an obstacle to mass lumping.

3.2. The case of P_2 elements. For P_2 elements, we seek, in accordance with the above condition, a symmetric quadrature formula exact for P_2. The degrees of freedom of the element form a symmetric set of quadrature points as illustrated in Figure 3.1.

There are two equivalence classes of points, $\{S_1, S_2, S_3\}$ and $\{M_1, M_2, M_3\}$, and two equivalence classes of polynomials in $P_2(K)$. The two parameters for the adequate quadrature formula will be the weights ω_s (s for summit) corresponding to $\{S_1, S_2, S_3\}$ and ω_e (e for edge) corresponding to $\{M_1, M_2, M_3\}$:

$$I_K^{\text{app}}(f) = \text{mes}(K)\left\{\omega_s\left(f(S_1) + f(S_2) + f(S_3)\right) + \omega_e\left(f(M_1) + f(M_2) + f(M_3)\right)\right\}.$$

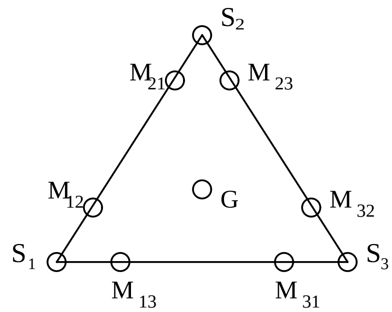

FIG. 3.2. P_3 *finite element.*

The unique well-known solution is

$$\omega_s = 0 \quad \text{and} \quad \omega_e = \frac{1}{3}.$$

Unfortunately, as the weights corresponding to the vertices of the triangle are zero, the corresponding basis functions have a "discrete norm" $|w_i|_h^2$ equal to 0! Thus, some terms of the new diagonal mass matrix M_h are equal to zero, so that this matrix is not invertible.

3.3. The case of P_3 elements. In the case of P_3 elements, we are looking for a quadrature formula exact for P_4 which is made up of four equivalence classes. Therefore, we need a priori four degrees of freedom to define the quadrature formula. We choose a set of degrees of freedom similar in its structure to that of the classical P_3 finite elements. In fact, we must consider the following symmetric set $(\mathcal{Q})$ of quadrature points which can be divided into three equivalence classes:

- the vertices $\{S_1, S_2, S_3\}$,
- the six edge points $\{M_{12}(\alpha), M_{21}(\alpha), M_{13}(\alpha), M_{31}(\alpha), M_{23}(\alpha), M_{32}(\alpha)\}$,
- the center of the triangle $\{G\}$,

where α denotes a real parameter between 0 and 1 and $M_{ij}(\alpha)$ is the barycenter of S_i and S_j with respective weights α and $(1-\alpha)$ (see Figure 3.2). We recall that the classical location of the degrees of freedom in $\mathbb{R}^3$ corresponds to $\alpha = 1/3$. It is easy to see that $(\mathcal{Q})$ is P_3 unisolvent: if P belongs to P_3 and P vanishes on $(\mathcal{Q})$, then it is zero at four distinct points of each edge. Therefore being a polynomial of degree 3, it is necessarily proportional to the bubble function, but as $p(G) = 0$, we have $p \equiv 0$.

By allowing the interior edge nodes to move in a symmetric way along the boundary, we get the desired fourth parameter. We thus look for a quadrature formula of the form

$$I_K^{\text{app}}(f) = \text{mes}(K) \left\{ \omega_s \sum_{j=1}^{3} f(S_j) + \omega_\alpha \sum_{\substack{i \neq j \\ i,j=1}}^{3} f(M_{ij}(\alpha)) + \omega_G f(G) \right\} \tag{3.8}$$

with four parameters (ω_s, ω_α, ω_G, and α). It can be shown (the details are omitted) that there exists a unique choice that makes formula (3.8) exact in P_4:

$$\alpha = \frac{3-\sqrt{3}}{6}, \quad \omega_s = -\frac{1}{60}, \quad \omega_\alpha = \frac{1}{10}, \quad \omega_G = \frac{9}{20}.$$

The corresponding quadrature formula, which can be found in [23], has the major drawback that one of the weights, the one associated with the summits (ω_s), is strictly negative. As a consequence, the corresponding semidiscrete scheme is unstable and any time discretization would lead to an unconditionally unstable scheme. Formula (3.8) is therefore not applicable in our context.

REMARK 3.1. *A proof of the instability result can be found in* [33]: *some modes of the discrete solution behave as does the solution of the differential equation*

$$\frac{\partial^2 u}{\partial t^2} - \omega_h^2 u = 0,$$

which admits exponentially growing solutions in $e^{\omega_h t}$, *where* $\omega_h > 0$ *grows like* $1/h$ *when* h *goes to* 0.

4. New finite element spaces. We have seen in the previous section that the spaces P_2 and P_3 seem unsuited for mass lumping even if one plays on the choice of the basis functions. The difficulty that we have encountered is due to our failure to respect a new constraint which is a necessary condition for ensuring the stability of the semidiscrete wave equation

($\mathcal{S}$) *The modified mass operator must be positive definite.*

Of course, this stability condition will be satisfied if and only if

($\mathcal{P}$) *The weights in the quadrature formula are strictly positive.*

The idea is to construct slightly larger finite element spaces, $\widetilde{P}_2$ and $\widetilde{P}_3$, for which one can, with the additional degrees of freedom, find appropriate quadrature formulas with strictly positive weights and sufficient accuracy to preserve the accuracy obtained, respectively, with the P_2 and P_3 spaces. To preserve this accuracy, certain conditions need to be imposed on the quadrature formula. Consider a polynomial finite element space of the form

$$V_h = \{v_h \in H^1(\bar{\Omega}) / \, \forall K \in \mathcal{T}_h, \; v_h|_K \in \widetilde{P}\}, \tag{4.1}$$

where $\widetilde{P}$ is a finite dimensional subspace of polynomials satisfying

$$P_k \subset \widetilde{P} \subset P_{k'}, \quad k \leq k'. \tag{4.2}$$

Then we get the same accuracy with $\widetilde{P}$ as with standard P_k elements if (see [6], [19], or section 5)

($\mathcal{A}$) *The quadrature formula is exact in* $P_{k+k'-2}$.

Thus we shall construct spaces $\widetilde{P}_k$, $k = 2, 3$, and the associated integration formulas using the following guidelines:

(i) The space $\widetilde{P}_k$ should be as small as possible with $P_k \subset \widetilde{P}_k \subset P_{k'}$.
(ii) The set ($\mathcal{Q}$) of quadrature points should be $\widetilde{P}_k$ unisolvent.
(iii) The quadrature formula should satisfy the accuracy condition ($\mathcal{A}$).
(iv) The quadrature formula should satisfy the positivity condition ($\mathcal{P}$).
(v) The number of degrees of freedom on the boundary should be sufficiently large to ensure the H^1-conforming nature of V_h.

Conditions (iii), (iv), and (v) are purely mathematical criteria linked, respectively, to accuracy, stability, and consistency considerations. Criterion (ii) is imposed by condition (3.6) of Lemma 3.1. It implies some compatibility between the choice of the finite element space and the quadrature nodes. Criterion (i) is more concerned with efficiency and aims at minimizing the total number of degrees of freedom of the element. Both (i) and (ii) indicate that k' should be chosen as small as possible. Note that while (i), (ii), (iii), and (iv) are purely local criteria involving a single triangle, criterion (v) takes into account the problem of joining distinct adjacent elements. However, this criterion may have an influence on the others: if, by passing from P_k to $\widetilde{P}_k$, one increases the degree of the traces of the polynomials on the edges of the triangle, then one also increases the number of degrees of freedom on the boundary, which goes, of course, against (i). Moreover, the fact that these nodes must remain located on the boundary reduces the liberty of moving them for satisfying (iii) and (iv). Thus it is useful, when enlarging the P_k spaces, to make use of "bubble-type" functions, i.e., polynomials vanishing on the boundary of the triangle: the degrees of the traces of elements of the new space $\widetilde{P}_k$ thus remain less than or equal to k.

4.1. The case of P_2 elements. The idea is to introduce a new space $\widetilde{P}_2$ satisfying $P_2 \subset \widetilde{P}_2 \subset P_3$. To comply with $(\mathcal{A})$, we need to integrate P_3 exactly. Thus, as seen in section 1, we need a priori three degrees of freedom for the quadrature formula. Let us consider

$$\widetilde{P}_2 = P_2 \oplus [b], \tag{4.3}$$

where $b = \lambda_1\lambda_2\lambda_3$ is the "bubble" function and where $[v_1, v_2, \ldots, v_m]$ denotes the subspace generated by the vectors $\{v_1, v_2, \ldots, v_m\}$.

The dimension of $\widetilde{P}_2$ is 7 so that criterion (i) has been satisfied in an optimal way. Thus we need seven Lagrange interpolation points and hence seven quadrature points. These are chosen to be the three vertices $\{S_1, S_2, S_3\}$, the midpoints of the three edges $\{M_1, M_2, M_3\}$ and the center of gravity G; see Figure 4.1. These constitute a symmetric set of quadrature points $(\mathcal{Q})$, in the sense of section 2, with three equivalence classes. It is immediate to check that this set of points is $\widetilde{P}_2$ unisolvent (criterion (ii)), the bubble function b being, up to a multiplicative constant, the basis function associated with G. Moreover, as b vanishes on the edges of K, the degree of the trace of any element of $\widetilde{P}_2$ on any edge of K remains less than or equal to two. The trace on any edge of such a function is thus entirely determined by its values at the two vertices and at the midpoint (criterion (v)). It remains to determine the symmetric quadrature formula satisfying criteria (iii) and (iv). We know that this quadrature formula must be of the form

$$\begin{aligned} E(f) = \mathrm{mes}(K)\{&\omega_s(f(S_1) + f(S_2) + f(S_3)) \\ &+ \omega_e(f(M_1) + f(M_2) + f(M_3)) + \omega_G f(G)\}. \end{aligned} \tag{4.4}$$

LEMMA 4.1. *There exists a unique quadrature formula of the form* (4.4) *satisfying criteria* (iii) *and* (iv). *It is obtained by setting*

$$\omega_s = \frac{1}{20}, \qquad \omega_e = \frac{2}{15}, \qquad \omega_G = \frac{9}{20}. \tag{4.5}$$

Proof. Note that if such a formula exists the weights are necessarily given by

$$\left\{\begin{aligned} \omega_s &= \frac{3}{\mathrm{mes}(K)} \int_K \lambda_1(\lambda_1 - \frac{1}{2}) \left(\lambda_1 - \frac{1}{3}\right) dx, \\ \omega_a &= \frac{12}{\mathrm{mes}(K)} \int_K \lambda_1(1 - \lambda_1) \left(\lambda_1 - \frac{1}{3}\right) dx, \\ \omega_G &= \frac{27}{\mathrm{mes}(K)} \int_K \lambda_1\lambda_2\lambda_3 dx. \end{aligned}\right. \tag{4.6}$$

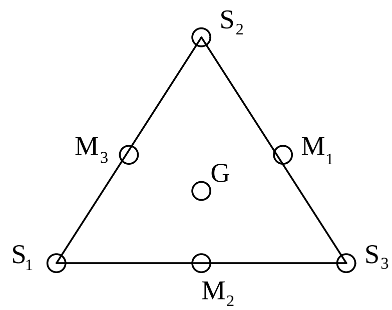

Fig. 4.1. *The degrees of freedom for* $\widetilde{P}_2$.

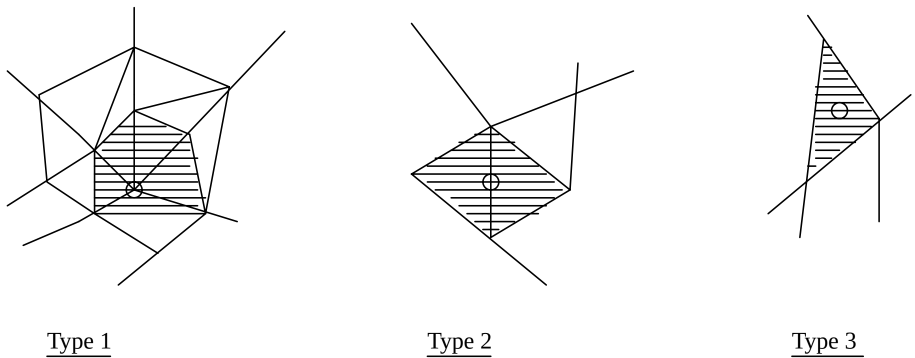

Fig. 4.2. *The three types of basis functions in* V_h.

Note also that the integrands in (4.6) are polynomials of degree 3 each of which vanishes at the nodes of two of the three equivalence classes of $(\mathcal{Q})$. The computation of the integrals leads to (4.5). To conclude, it suffices to check that the formula is exact for one representative of each of the three equivalence classes of $\widetilde{P}_2$. □

Remark 4.1. *In fact, formula* (4.4), (4.5) *is well known as Simpson's rule* (*see, for instance,* [23], [34]).

We observe that all the quadrature weights are strictly positive as desired. We can now consider the space

$$V_h = \{v \in H^1(\Omega)/\ \forall\, K \in \mathcal{T}_h,\ v/_K \in \widetilde{P}_2\} \tag{4.7}$$

as an approximation space for $H^1(\Omega)$. V_h clearly admits three types of basis functions (see Figure 4.2) associated, respectively, with a vertex, an edge, or a triangle so that $\dim V_h = N_s + N_e + N_t$, where N_s, N_e, and N_t are, respectively, the number of nodes, edges, and triangles of $\mathcal{T}_h$.

4.2. The case of P_3 elements. We wish here to construct a symmetric subspace $\widetilde{P}_3$ such that $P_3 \subset \widetilde{P}_3 \subset P_4$. We then need a quadrature formula which integrates P_5 exactly. This requires a priori a symmetric formula with five parameters. The symmetric set $(\mathcal{Q})$ of quadrature points that we shall use is made up of three equivalence classes:

- three vertices $\{S_1, S_2, S_3\}$.
- six boundary points $\{M_{12}(\alpha), M_{21}(\alpha), M_{13}(\alpha), M_{31}(\alpha), M_{23}(\alpha), M_{32}(\alpha)\}$.

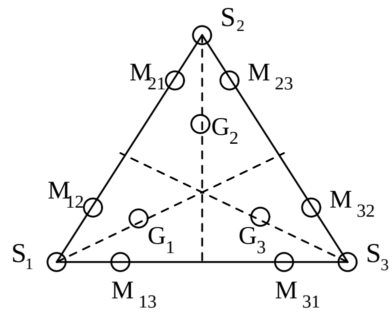

FIG. 4.3. *The set of quadrature nodes.*

- three interior points $\{G_1(\beta), G_2(\beta), G_3(\beta)\}$.

Here, α and β denote two real parameters between 0 and 1, and (see Figure 4.3)

- $G_i(\beta)$ has barycentric coordinates $\lambda_i = \beta$ and $\lambda_j = \frac{1-\beta}{2}$ for $j \neq i$.
- $M_{ij}(\alpha)$ is the barycenter of S_i and S_j with weights α and $(1-\alpha)$, respectively.

We have five parameters to play with in order to define our quadrature formula; the three weights associated with the three equivalence classes of $(\mathcal{Q})$ and the two "location" parameters α and β. Note that, in comparison with the classical P_3 elements (section 3.2), we have gained one parameter (namely, β) by splitting the center of gravity G into three interior points $G_1(\beta)$, $G_2(\beta)$, and $G_3(\beta)$.

It remains to find the right space $\widetilde{P}_3$. We put

$$\widetilde{P}_3 = P_3 \oplus bP_1. \tag{4.8}$$

By construction, $\widetilde{P}_3$ satisfies criteria (i) and (v). Concerning criterion (ii), we have the following lemma which in addition gives the Lagrange basis of $\widetilde{P}_3$.

LEMMA 4.2. *If $0 < \alpha < 1/2$, $0 < \beta < 1/2$, and $\beta \neq 1/3$, the set $(\mathcal{Q})$ is $\widetilde{P}_3$ unisolvent. Moreover,*

(i) *the basis function associated with G_i is*

$$w_i^G = b\left(\lambda_i - \frac{1-\beta}{2}\right).$$

(ii) *The basis function associated with M_{ij} is*

$$w_{ij} = p_{ij} - \frac{8b}{\beta(1-\beta)^2(3\beta-1)}\left[A_{ij}\left(\lambda_i - \frac{1-\beta}{2}\right) + B_{ij}\left(\lambda_j - \frac{1-\beta}{2}\right) + C_{ij}\left(\lambda_k - \frac{1-\beta}{2}\right)\right],$$

where we have set

$$p_{ij} = \frac{\lambda_i\lambda_j}{\alpha(1-\alpha)(2\alpha-1)}(\alpha\lambda_i - (1-\alpha)\lambda_j + (1-2\alpha)\lambda_k),$$
$$A_{ij} = p_{ij}(G_i), \quad B_{ij} = p_{ij}(G_j), \quad C_{ij} = p_{ij}(G_k).$$

(iii) *The basis function associated with S_i is*

$$w_i^S = p_i - \frac{8b}{\beta(1-\beta)^2(3\beta-1)}\left[A_i\left(\lambda_i - \frac{1-\beta}{2}\right) + B_i\sum_{l\neq i}\left(\lambda_l - \frac{1-\beta}{2}\right)\right],$$

where p_i and A_i are given by

$$p_i = \lambda_i \left(\sum_l \lambda_l^2 - \frac{1-2\alpha+2\alpha^2}{\alpha(1-\alpha)} \lambda_j \lambda_k + \frac{2-7\alpha+7\alpha^2}{\alpha(1-\alpha)} \lambda_i (\lambda_j + \lambda_k) \right),$$
$$A_i = p_i(G_i), \quad B_i = p_i(G_l), \quad l \neq i.$$

Proof. For the unisolvence, we note that, as $P_3 \cap bP_1 = [b]$, the dimension of $\widetilde{P}_3$ is 12, which coincides with the number of degrees of freedom. Thus it suffices to remark that any $\widetilde{p}$ of $\widetilde{P}_3$ has a unique decomposition of the form

$$\left\{ \begin{array}{l} \widetilde{p} = p + bq, \\ p \in P_3, \quad p(G) = 0, \\ q \in P_1. \end{array} \right. \tag{4.9}$$

Then, if $\widetilde{p}$ vanishes at all quadrature points, p vanishes at all boundary points of $(\mathcal{Q})$ and also at G. These points being P_3 unisolvent whenever $0 < \alpha < 1/2$, we deduce that $p = 0$. Now because $\widetilde{p}(G_j(\beta)) = 0$, $j = 1, \ldots, 3$, we have $q(G_j(\beta)) = 0$, $j = 1, \ldots, 3$. Since $q \in P_1$ and the three points $G_1(\beta)$, $G_2(\beta)$, and $G_3(\beta)$ are not collinear for $0 < \beta < 1/2$ and $\beta \neq 1/3$ we conclude that $q = 0$. The computation of the basis functions using the decomposition (4.9) is straightforward but rather tedious and will not be included here. ☐

In order to satisfy criteria (iii) and (iv), it remains to construct an adequate quadrature formula, which must have the form

(4.10)
$$I_K(f) = \text{mes}(K) \left\{ \omega_s \sum_{j=1}^3 f(S_j) + \omega_\alpha \sum_{i \neq j \ i,j=1}^3 f(M_{ij}(\alpha)) + \omega_\beta \sum_{j=1}^3 f(G_j(\beta)) \right\}.$$

LEMMA 4.3. *The unique quadrature formula of the form* (4.10) *which integrates P_5 exactly has quadrature points determined by*

$$\begin{aligned} \beta &= \frac{1}{3} + \frac{2}{21}\sqrt{7} \simeq 0.5853, \\ \alpha &= \frac{42 + 21\sqrt{7} - \sqrt{21\left(35 + 16\sqrt{7}\right)}}{84 + 42\sqrt{7}} \simeq 0.2935 \end{aligned} \tag{4.11}$$

and strictly positive weights given by

$$\begin{aligned} \omega_s &= 2\frac{919\sqrt{7} + 2471}{124080\sqrt{7} + 330960} \simeq 0.0148, \\ \omega_\alpha &= 2\frac{\sqrt{7}\left(2+\sqrt{7}\right)^4}{25280 + 9520\sqrt{7}} \simeq 0.0488, \\ \omega_\beta &= 2\frac{147 + 42\sqrt{7}}{400\sqrt{7} + 1280} \simeq 0.2208. \end{aligned} \tag{4.12}$$

Proof. We give a rather constructive proof. First note that the polynomial

$$p_1 = \lambda_1 \lambda_2 \lambda_3 (\lambda_1 - \beta) \left(\lambda_1 - \frac{1-\beta}{2} \right)$$

has degree 5 and vanishes at all quadrature points. Therefore, if the formula exists, β is necessarily a root of the quadratic equation

$$f_1(\beta) = \int_K \lambda_1\lambda_2\lambda_3(\lambda_1 - \beta)\left(\lambda_1 - \frac{1-\beta}{2}\right) dx = 0. \tag{4.13}$$

It is clear that this equation has one real solution between 1/3 and 1 since

$$f_1(1/3) = \int_K \lambda_1\lambda_2\lambda_3(\lambda_1 - 1/3)^2 dx > 0$$

and

$$\forall\, \beta \geq 1 \quad f_1(\beta) < 0 \quad (p_1 < 0 \text{ in } K).$$

Computing this solution one obtains the value given in (4.11). The weight ω_β is then defined by

$$3\omega_\beta \beta \frac{(1-\beta)^2}{4}(\mathrm{mes}(K)) = \int_K \lambda_1\lambda_2\lambda_3 dx. \tag{4.14}$$

In the same way, the polynomial

$$p_2 = \lambda_1(1-\lambda_1)(\lambda_1 - \alpha)(\lambda_1 - 1 + \alpha)$$

has degree 4 and vanishes at all quadrature points except the $G_j(\beta)$'s. Therefore, α is defined as the solution of the quadratic equation

$$\begin{aligned} &\int_K \lambda_1(1-\lambda_1)(\lambda_1-\alpha)(\lambda_1 - 1 + \alpha) dx \\ &\quad = \omega_\beta(\mathrm{mes}(K))\left\{\beta(1-\beta)(\beta-\alpha)(\beta+\alpha-1)\right. \\ &\qquad \left. + (1-\beta)\left(\frac{1+\beta}{2}\right)\left(\frac{1-\beta}{2} - \alpha\right)\left(\frac{1-\beta}{2} - 1 + \alpha\right)\right\}. \end{aligned} \tag{4.15}$$

Selecting the unique solution of this equation which is smaller than 1/2 we obtain the value of α. Once α and β are known, ω_α and ω_β can be found by solving a 2×2 linear system whose equations are obtained by taking f constant and then f linear in the equation

$$I_K(f) = \int_K (f) \quad \forall\, f \in P_5.$$

To complete the proof, it suffices to check a posteriori that the formula is exact for one representative of each class of P_5. □

REMARK 4.2. *The computations of α, β, ω_α, ω_β, and ω_s have been carried out with the help of MAPLE.*

Of course, the most important result is that the weights ω_s, ω_α, ω_β are strictly positive. We can now construct a finite element space for $H^1(\Omega)$ as

$$V_h = \{v \in C^0(\bar{\Omega}) /\, \forall\, K \in \mathcal{T}_h,\ v/_K \in \widetilde{P}_3\}. \tag{4.16}$$

Once again there are three types of basis functions. The difference between $\widetilde{P}_2$ and this case is that there is one basis function of type 1 per node (as for $\widetilde{P}_2$) but two basis functions of type 2 per edge and three basis functions of type 3 by triangle.

5. Convergence and error estimates. In this section, the problem of estimating the error due to numerical integration is dealt with. The aim is to extend the results obtained by Ciarlet and Raviart (cf. [6, section 4.4.1] and [28]) for elliptic problems to the case of the wave equation (or more generally second order hyperbolic equations). The case of parabolic evolution equations was studied by Raviart [27]. The error analysis for second order hyperbolic equations without numerical integration is due to Dupont [16]. He obtained results directly in the time domain using an elliptic projection and energy estimates. The effect of numerical integration was first studied by Baker and Dougalis [3] by essentially following the lines of the proof of Dupont. The authors were not aware of this latter work when they performed the analysis presented in this section. The method given here differs from that of Baker and Dougalis in that it uses the Laplace transform in time and thus reduces the study of a hyperbolic problem to that of an infinite family of elliptic type problems parametrized by the Laplace variable s ($s \in \mathbb{C}$). The main difficulty is to control the constants appearing in the "elliptic" error estimates, more precisely, if $s = \eta + i\omega$ where η ($\eta > 0$) is fixed and ω varies in $\mathbb{R}$, to control these constants when $|\omega| \to +\infty$. The advantages of this approach are that it leads to simpler proofs (at least from the point of view of the authors) than those in [3], that it yields somewhat better estimates in that less regularity is required of the solution to obtain the optimal order of convergence, and that the dependence on time of the constants in the error estimates is given explicitly. For the sake of clarity, the presentation is restricted to the case of the wave equation with constant coefficients and with zero initial data:

$$(5.1) \qquad \begin{cases} \text{Find } u : \Omega \times \mathbb{R}^+ \to \mathbb{R} \quad \text{so that} \\[4pt] \dfrac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = f(x,t), \quad (x,t) \in \Omega \times \mathbb{R}^+, \\[4pt] u(x,0) = 0, \quad \dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega, \\[4pt] \dfrac{\partial u}{\partial n}(x,t) = 0, \quad (x,t) \in \Gamma = \partial\Omega \times \mathbb{R}^+. \end{cases}$$

The case of variable coefficients could be treated in much the same way modulo appropriate regularity assumptions and the case of nonzero initial data could be treated by using an adequate extension operator. Further, in the interest of simplicity and conciseness, only H^1-type estimates, estimates in the *energy norm*, will be given. Standard duality arguments could be used to obtain L^2-estimates.

As the demonstration for our error estimates does not depend on the fact that the elements are $\widetilde{P}_3$, we shall use a slightly more general setting: suppose that k and k' are positive integers with $3 \le k \le k'$, and that $\widetilde{P}_k$ is a space of polynomials with $P_k \subset \widetilde{P}_k \subset P_{k'}$. (The case $k \le 2$ could be treated in a similar way but with slightly different regularity assumptions.) Suppose that the regular affine family of finite elements [6, pp. 87, 124] $\{\widetilde{V}_h^k\}_{h \in \mathcal{H}}$ is such that, for each h in $\mathcal{H}$,

$$\widetilde{V}_h^k = \{v \in C^0(\bar{\Omega}) : \forall\, K \in \mathcal{T}_h, v_{|K} \in \widetilde{P}_k(K)\}.$$

Suppose also that the quadrature formula

$$\oint_\Omega f\,dx = \sum_{K \in \mathcal{T}_h} \oint_K f\,dx = \sum_{K \in \mathcal{T}_h} \operatorname{mes}(K) \oint_{\widehat{K}} f \circ F_K\,d\hat{x},$$

where $F_K : \widehat{K} \longrightarrow K$ is the affine mapping of the reference element $\widehat{K}$ onto the element K, has positive weights and is exact for piecewise polynomials of degree no more than $k + k' - 2$.

We use the standard notation for Sobolev spaces, norms, and seminorms: if $\mathcal{O}$ is an open subset of $\mathbb{R}^2$ or of $\mathbb{R}^2 \times (0, T)$, then $H^m(\mathcal{O})$ is the space of functions in $L^2(\mathcal{O})$, all of whose derivatives of order up to and including m belong to $L^2(\mathcal{O})$. The norm on $H^m(\mathcal{O})$ is denoted $\|\cdot\|_{m,\mathcal{O}}$ or simply $\|\cdot\|_m$, and $|\cdot|_{m,\mathcal{O}}$ or simply $|\cdot|_m$ is the usual seminorm

$$\|v\|_m^2 = \sum_{|\alpha| \le m} \int_{\mathcal{O}} \left| \frac{\partial^\alpha v}{\partial y^\alpha} \right|^2 dy \quad \text{and} \quad |v|_m^2 = \sum_{|\alpha| = m} \int_{\mathcal{O}} \left| \frac{\partial^\alpha v}{\partial y^\alpha} \right|^2 dy.$$

5.1. A family of auxiliary problems. We begin by studying a family of elliptic problems indexed by a complex parameter s and related to our original problem via the Laplace transform. These auxiliary problems concern complex-valued functions, which we shall denote by boldfaced characters. The space of square summable complex-valued functions will be denoted by $\mathbf{L}^2(\Omega)$ and will be equipped with the usual scalar product and norm

$$(\mathbf{u}, \mathbf{v})_{0,\Omega} = \int_\Omega \mathbf{u}\overline{\mathbf{v}} dx, \qquad \|\mathbf{u}\|_{0,\Omega}^2 = (\mathbf{u}, \mathbf{u})_{0,\Omega}. \tag{5.2}$$

Similarly, $\mathbf{C}^0(\Omega)$ will be the space of continuous, complex-valued functions in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^m(\Omega)$ the space of complex-valued functions in $\mathbf{L}^2(\Omega)$, all of whose derivatives of order up to and including m are also in $\mathbf{L}^2(\Omega)$. For $\mathbf{H}^1(\Omega)$, in addition to the standard seminorm, $|\mathbf{v}|_{1,\Omega}^2 = (\nabla \mathbf{v}, \nabla \mathbf{v})_{0,\Omega}$; we shall make use of an s-dependent scalar product on $\mathbf{H}^1(\Omega)$ and its associated norm

$$\begin{aligned} (\mathbf{u}, \mathbf{v})_{1,s} &= (\nabla \mathbf{u}, \nabla \mathbf{v})_{0,\Omega} + |s|^2 (\mathbf{u}, \mathbf{v})_{0,\Omega}, \\ \|\mathbf{v}\|_{1,s}^2 &= (\mathbf{v}, \mathbf{v})_{1,s} = |\mathbf{v}|_{1,\Omega}^2 + |s|^2 \, \|\mathbf{v}\|_{0,\Omega}^2, \end{aligned} \tag{5.3}$$

where s is a complex parameter, in practice the Laplace variable, with a strictly positive real part

$$s = \eta + i\omega, \quad \eta > 0 \text{ (fixed)}, \quad \omega \in \mathbb{R}.$$

For each s, we are interested in the solution $\mathbf{u} = \mathbf{u}_s$ in $\mathbf{H}^1(\Omega)$ of the following problem:

$$\begin{aligned} -\Delta \mathbf{u} + s^2 \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \frac{\partial \mathbf{u}}{\partial n} &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $\mathbf{f} = \mathbf{f}_s$ belongs to $\mathbf{L}^2(\Omega)$. The corresponding variational problem is

$$\begin{aligned} &\mathbf{u} \in \mathbf{V} = \mathbf{H}^1(\Omega), \\ &a(s; \mathbf{u}\, \mathbf{v}) = \bar{s}\, (\mathbf{f}, \mathbf{v})_{0,\Omega} \quad \forall \mathbf{v} \in \mathbf{V}, \end{aligned} \tag{5.4}$$

where $a(s; \cdot, \cdot) : \mathbf{V}^2 \longrightarrow \mathbb{C}$ is defined by

$$a(s; \mathbf{u}, \mathbf{v}) = \bar{s} (\nabla \mathbf{u}, \nabla \mathbf{v})_{0,\Omega} + |s|^2 s (\mathbf{u}, \mathbf{v})_{0,\Omega}.$$

One shows easily that $a(s;\cdot,\cdot)$ has the following coercivity and continuity properties (for $s=\eta+i\omega$):

$$\eta\,\|\mathbf{v}\|_{1,s}^2 \le |a(s;\mathbf{v},\mathbf{v})| \qquad \forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),$$
$$|a(s;\mathbf{u},\mathbf{v})| \le |s|\,\|\mathbf{u}\|_{1,s}^2\,\|\mathbf{v}\|_{1,s}^2 \quad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{H}^1(\Omega),$$

which guarantee the existence and uniqueness of the solution of (5.4). Note that the coercivity constant does not depend on ω, whereas the continuity constant is equivalent to $|\omega|$ when ω tends towards ∞: this is the source of difficulty for our analysis as for the analysis of any hyperbolic problem.

To write the finite element problem

$$\begin{aligned} &\mathbf{u}_h \in \widetilde{\mathbf{V}}_h^k,\\ &a_h(s;\mathbf{u}_h,\mathbf{v}_h) = \bar{s}\,(\mathbf{f},\mathbf{v}_h)_{0,\Omega} \quad \forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h^k, \end{aligned} \tag{5.5}$$

we define the bilinear form $a_h(s;\cdot,\cdot):\widetilde{\mathbf{V}}_h^k\times\widetilde{\mathbf{V}}_h^k\longrightarrow\mathbb{C}$ by

$$a_h(s;\mathbf{u_h},\mathbf{v}_h) = \bar{s}(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)_{0,\Omega} + |s|^2 s(\mathbf{u}_h,\mathbf{v}_h)_h,$$

where

$$(\mathbf{u}_h,\mathbf{v}_h)_h = \oint_\Omega \mathbf{u}_h\,\overline{\mathbf{v}}_h\,dx.$$

It is the positivity of the weights in the quadrature formula that guarantees the continuity and coercivity of the forms $a_h(s;\cdot,\cdot)$. More precisely, because the weights are positive, $\|\cdot\|_h^2=(\cdot,\cdot)_h$ determines a norm on $\widetilde{\mathbf{V}}_h^k$, a norm equivalent to the norm $\|\cdot\|_{0,\Omega}^2$ as $\widetilde{\mathbf{V}}_h^k$ is finite dimensional. Thus, for each h, $h>0$, (5.5) has a unique solution. To obtain error estimates though, we need to know that the constants of continuity and coercivity are independent of the parameter h:

$$C\,\eta\,\|\mathbf{v}_h\|_{1,s}^2 \le |a_h(s;\mathbf{v_h},\mathbf{v}_h)| \qquad \forall\,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h^k,$$
$$|a_h(s,\mathbf{u_h},\mathbf{v_h})| \le C\,|s|\,\|\mathbf{u}_h\|_{1,s}^2\,\|\mathbf{v}_h\|_{1,s}^2 \quad \forall\,\mathbf{u}_h,\mathbf{v}_h\in\widetilde{\mathbf{V}}_h^k$$

with C independent of h as well as of s. This, however, is implied by the fact that the constants in the norm equivalence

$$C_1\|\mathbf{v}_h\|_{0,\Omega}^2 \le \|\mathbf{v}_h\|_h^2 \le C_2\|\mathbf{v}_h\|_{0,\Omega}^2$$

are independent of h, a fact that can be demonstrated in a straightforward manner by using a reference element.

5.1.1. Estimates in the $\|\cdot\|_{1,s}$-norm. To estimate the error in the $\|\cdot\|_{1,s}$-norm we proceed just as in the classical case for elliptic problems: we use Strang's lemma, which in the present context yields the following lemma.

LEMMA 5.1 (Strang's lemma). *If* $\mathbf{u}$ *is the solution of* (5.4) *and* $\mathbf{u}_h$ *the solution of* (5.5), *then there is a constant* C *independent of* h *and of* s *such that*

(5.6)
$$\|\mathbf{u}-\mathbf{u}_h\|_{1,s} \le C \inf_{\mathbf{v}_h\in\widetilde{\mathbf{V}}_h^k}\left[\left(1+\frac{|s|}{\eta}\right)\|\mathbf{u}-\mathbf{v}_h\|_{1,s} + \frac{1}{\eta}\sup_{\mathbf{w}_h\in\widetilde{\mathbf{V}}_h^k}\frac{|(a-a_h)(s;\mathbf{v}_h,\mathbf{w}_h)|}{\|\mathbf{w}_h\|_{1,s}}\right].$$

The proof is just the same as in the usual case [6], only particular attention must be paid to the dependence of the constants on s and on η.

The infimum in (5.6) is now bounded from above by taking $\mathbf{v}_h = \Pi_h \mathbf{u}$, where Π_h is the interpolation operator from $\mathbf{C}^0(\Omega)$ (or any of its subspaces $\mathbf{H}^l(\Omega)$, $l \geq 2$) onto $\widetilde{\mathbf{V}}_h^k$ that interpolates a function $\mathbf{v} \in \mathbf{C}^0(\Omega)$ at the quadrature points. Using the result [6, Theorem 3.2.1]

(5.7)
$$\sum_{K \in \mathcal{T}_h} |\mathbf{v} - \Pi_h \mathbf{v}|_{m,K} \leq C h^{l-m} \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{l,K} \quad \forall \mathbf{v} \in \mathbf{H}^l(\Omega), \quad 2 \leq l \leq k+1, \quad m \leq l,$$

first with $l = \lambda$ and $m = 0$, then with $l = \lambda + 1$ and $m = 1$, we obtain that

(5.8)
$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_{1,s} \leq C h^{\lambda} \left(|s|\, |\mathbf{v}|_{\lambda,\Omega} + |\mathbf{v}|_{\lambda+1,\Omega} \right) \quad \forall \mathbf{v} \in \mathbf{H}^{\lambda+1}(\Omega), \quad 2 \leq \lambda \leq k.$$

In order to estimate $|(a - a_h)(s; \Pi_h \mathbf{u}, \mathbf{w}_h)|$, we define the continuous bilinear form $E_h : \widetilde{\mathbf{V}}_h^k \times \widetilde{\mathbf{V}}_h^k \longrightarrow \mathbb{R}$ by

$$E_h(\mathbf{v}_h, \mathbf{w}_h) = \sum_{K \in \mathcal{T}_h} \left(\int_K \mathbf{v}_h \mathbf{w}_h - \oint_K \mathbf{v}_h \mathbf{w}_h \right),$$

so that $(a - a_h)(s; \mathbf{v}_h, \mathbf{w}_h) = s|s|^2 E_h(\mathbf{v}_h, \mathbf{w}_h)$. Note that $E_h(\mathbf{v}_h, \mathbf{w}_h) = 0$ if the sum of the degrees of the polynomials $\mathbf{v}_h$ and $\mathbf{w}_h$ is less than or equal to $k + k' - 2$.

LEMMA 5.2. *There is a constant C such that for $\mathbf{v}_h, \mathbf{w}_h \in \widetilde{\mathbf{V}}_h^k$ and $1 \leq p, q \leq k-1$,*

$$|E_h(\mathbf{v}_h, \mathbf{w}_h)| \leq C\, h^{p+q} \left(\sum_{K \in \mathcal{T}_h} |\mathbf{v}_h|_{p,K}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} |\mathbf{w}_h|_{q,K}^2 \right)^{\frac{1}{2}}.$$

Proof. The demonstration makes use of the reference element $\widehat{K}$ for the family of triangulations $\{\mathcal{T}_h\}_{h \in \mathcal{H}}$: for each K in $\mathcal{T}_h$, let F_K denote the affine map from $\widehat{K}$ onto K, and, for $\mathbf{v}_h \in \widetilde{\mathbf{V}}_k^h$, let $\mathbf{v}_K \in \widetilde{\mathbf{P}}_k(K)$ denote the restriction of $\mathbf{v}_h$ to K and let $\widehat{\mathbf{v}}_K$ denote the induced map $\mathbf{v}_K \circ F_K$ in $\widetilde{\mathbf{P}}_k(\widehat{K})$. Thus

(5.9)
$$\begin{aligned} E_h(\mathbf{v}_h, \mathbf{w}_h) &= \sum_{K \in \mathcal{T}_h} \left(\int_K \mathbf{v}_h \mathbf{w}_h - \oint_K \mathbf{v}_h \mathbf{w}_h \right) \\ &= \sum_{K \in \mathcal{T}_h} E_K(\mathbf{v}_K, \mathbf{w}_K) \\ &= \sum_{K \in \mathcal{T}_h} \operatorname{mes}(K)\, \widehat{E}(\widehat{\mathbf{v}}_K, \widehat{\mathbf{w}}_K), \end{aligned}$$

where we have used the notation

$$E_K(\mathbf{v}_K, \mathbf{w}_K) = \int_K \mathbf{v}_K \mathbf{w}_K - \oint_K \mathbf{v}_K \mathbf{w}_K \quad \text{and} \quad \widehat{E}(\widehat{\mathbf{v}}_K, \widehat{\mathbf{w}}_K) = \int_{\widehat{K}} \widehat{\mathbf{v}}_K \widehat{\mathbf{w}}_K - \oint_{\widehat{K}} \widehat{\mathbf{v}}_K \widehat{\mathbf{w}}_K.$$

Now let $\widehat{\Pi}_j$ denote the L^2-projection of $\mathbf{L}^2(\widehat{K})$ onto $\mathbf{P}_j(\widehat{K})$. Since the quadrature rule is exact for polynomials of degree less than or equal to $k + k' - 2$ and since $p - 1 + k'$ and $q - 1 + k'$ as well as $p - 1 + q - 1$ are all less than or equal to $k + k' - 2$ we have

(5.10)
$$\begin{aligned} |\widehat{E}(\widehat{\mathbf{v}}_K, \widehat{\mathbf{w}}_K)| &= |\widehat{E}(\widehat{\mathbf{v}}_K - \widehat{\Pi}_{p-1} \widehat{\mathbf{v}}_K, \widehat{\mathbf{w}}_K - \widehat{\Pi}_{q-1} \widehat{\mathbf{w}}_K)| \\ &\leq C\, |\widehat{\mathbf{v}}_K - \widehat{\Pi}_{p-1} \widehat{\mathbf{v}}_K|_{0,\widehat{K}} |\widehat{\mathbf{w}}_K - \widehat{\Pi}_{q-1} \widehat{\mathbf{w}}_K|_{0,\widehat{K}}, \end{aligned}$$

where we have used the continuity of $\widehat{E}$ (in the $\mathbf{H}^{0,\infty}(\widehat{K})$-norm) and the equivalence of norms on the finite dimensional space $\widetilde{\mathbf{P}}_k(\widehat{K})$ for the inequality. In combination with [6, Theorem 3.1.4] this inequality becomes

$$\begin{aligned}|\widehat{E}(\widehat{\mathbf{v}}_K,\widehat{\mathbf{w}}_K)| &\le C\,|\widehat{\mathbf{v}}_K|_{p,\widehat{K}}|\widehat{\mathbf{w}}_K|_{q,\widehat{K}}\\ &\le C(\mathrm{mes}(K))^{-\frac{1}{2}}h^p|\mathbf{v}_K|_{p,K}\,(\mathrm{mes}(K))^{-\frac{1}{2}}h^q|\mathbf{w}_K|_{q,K}\\ &\le C\,(\mathrm{mes}(K))^{-1}\,h^{p+q}|\mathbf{v}_K|_{p,K}\,|\mathbf{w}_K|_{q,K}.\end{aligned} \tag{5.11}$$

The desired result is obtained by combining (5.9) with (5.11) and using the Cauchy–Schwartz inequality. ∎

If we apply Lemma 5.2 with $\mathbf{v}_h = \Pi_h\mathbf{u}$, with $p = k-1$, and with $q = 1$, and use (5.7) with $m = l = k-1$, we obtain

$$\begin{aligned}|(a-a_h)(s;\Pi_h\mathbf{u},\mathbf{w}_h)| &\le C\,|s|^3\,h^k\left(\sum_{K\in\mathcal{T}_h}|\Pi_h\mathbf{u}|^2_{k-1,K}\right)^{\frac{1}{2}}\left(\sum_{K\in\mathcal{T}_h}|\mathbf{w}_h|^2_{1,K}\right)^{\frac{1}{2}}\\ &\le C\,|s|^3\,h^k\left(\sum_{K\in\mathcal{T}_h}|\mathbf{u}|^2_{k-1,K}\right)^{\frac{1}{2}}\|\mathbf{w}_h\|_{1,s}.\end{aligned} \tag{5.12}$$

Now combining (5.6) with (5.8) (with $\lambda = k$) and (5.12) we obtain the following result.

LEMMA 5.3. *Suppose that $k \ge 3$ and that $f \in L^2(\Omega)$ is sufficiently regular that the solution $\mathbf{u}$ of (5.4) is in $H^{k+1}(\Omega)$. Then if $\mathbf{u}_h$ is the solution of (5.5), the following estimate holds:*

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,s} \le C\,h^k\left[(|\mathbf{u}|_{k+1,\Omega}+|s|\,|\mathbf{u}|_{k,\Omega})+\frac{|s|}{\eta}\left(|\mathbf{u}|_{k+1,\Omega}+|s|\,|\mathbf{u}|_{k,\Omega}+|s|^2|\mathbf{u}|_{k-1,\Omega}\right)\right],$$

with a constant C independent of h and of s.

REMARK 5.1. *One could, by this same technique, obtain estimates of order k for $k = 2$ but with greater regularity requirements: the requirement $k \ge 3$ is necessary because to obtain* (5.12) *we use $m = l = k-1$ in* (5.7)*. If we use $m = k-1$ but $l = k$ we may still use* (5.7) *but we obtain a term $\frac{|s|^3}{\eta}|u|_k$ in Lemma* 5.3.

5.2. Error estimates in the time domain. Recall that the weak form of the continuous problem (5.1) is

find $u : [0,T) \longrightarrow V = H^1(\Omega)$ such that

$$\frac{d^2}{dt^2}\int_\Omega u(x,t)\,v(x)\,dx + \int_\Omega \nabla u(x,t)\,\nabla v(x)\,dx = \int_\Omega f(x,t)\,v(x)\,dx \quad \forall\, v\in V, \tag{5.13}$$

$$\int_\Omega u(x,0)\,v(x)\,dx = 0 \quad \forall\, v \in V,$$

and the finite element problem is

(5.14)
find $u_h : [0,T) \longrightarrow \widetilde{\mathbf{V}}_h^k$ such that

$$\frac{d^2}{dt^2}\oint_\Omega u_h(x,t)\, v_h(x)\, dx + \int_\Omega \nabla u_h(x,t)\, \nabla v_h(x)\, dx = \int_\Omega f(x,t)\, v_h(x)\, dx \quad \forall\, v_h \in \widetilde{\mathbf{V}}_h^k,$$

$$\oint_\Omega u_h(x,t)\, v_h(x)\, dx = 0 \quad \forall\, v_h \in \widetilde{\mathbf{V}}_h^k.$$

Let $Q_T = \Omega \times (0,T)$ and denote Q_∞ simply by Q. Introduce the space

$$V_{T,0} = \{v \in H^1(Q_T) : v(x,0) = 0 \quad \forall\, x \in \Omega\}$$

and the following energy norm on $V_{T,0}$:

$$|||v|||_{1,T}^2 = \int_0^T \left[\left|\frac{\partial v}{\partial t}\right|_{0,\Omega}^2 + |\nabla v|_{0,\Omega}^2 \right] dt.$$

To exploit the results of the preceding section, we will use the Laplace transform

$$\mathbf{v}(x,y,s) = \frac{1}{2\pi}\int_0^\infty v(x,y,t)\exp(-st)dt \quad \text{with } s = \eta + i\omega \ (\eta > 0)$$

and Plancherel's theorem which guarantees that if $w \exp(-\eta\, t)$ is in $L^2(\mathbb{R})$ and w vanishes on $\mathbb{R}^-$, if $\mathbf{w}$ is the Laplace transform of w and $\eta > 0$, then

$$\int_{-\infty}^{\infty} \mathbf{w}^2(\eta + i\omega)d\omega = \int_0^\infty w^2(t)\exp(-2\eta t)dt.$$

This is of course immediately applicable when $T = \infty$.

5.2.1. Weighted $L^2(0,\infty;H^1(\Omega))$ and $H^1(0,\infty;L^2(\Omega))$ estimates. We introduce the weighted Sobolev spaces

$$H_\eta^l(\mathbb{R}^+;H^m(\Omega)) = \left\{ v \in L^2_{\text{loc}}(\mathbb{R}^+;H^m(\Omega)) : \frac{\partial^p v}{\partial t^p}\exp(-\eta t) \in L^2(\mathbb{R}^+;H^m(\Omega)),\ p = 0,\dots,l \right\}.$$

If u and u_h are the solutions of (5.13) and (5.14), respectively, and if $\mathbf{u}$, $\mathbf{u}_h$, and $\mathbf{f}$ are the Laplace transforms of u, u_h, and f, respectively, then $\mathbf{u}(\cdot,\cdot,s)$, respectively, $\mathbf{u}_h(\cdot,\cdot,s)$, is the solution of (5.4), respectively, (5.5). If $u \in H_\eta^l(0,\infty;H^{k+2-l}(\Omega))$, $l = 1,\dots,3$, then from Lemma 5.3 and Plancherel's theorem we derive that

$$\int_0^\infty \left(|u-u_h|_{1,\Omega}^2 + \left|\frac{\partial(u-u_h)}{\partial t}\right|_{0,\Omega}^2 \right)\exp(-2\eta t)dt$$

$$\leq C\, h^{2k} \left[\int_0^\infty \left(|u|_{k+1,\Omega}^2 + \left|\frac{\partial u}{\partial t}\right|_{k,\Omega}^2 \right)\exp(-2\eta t)dt + \frac{1}{\eta^2}\int_0^\infty \left(\left|\frac{\partial u}{\partial t}\right|_{k+1,\Omega}^2 + \left|\frac{\partial^2 u}{\partial t^2}\right|_{k,\Omega}^2 + \left|\frac{\partial^3 u}{\partial t^3}\right|_{k-1,\Omega}^2 \right)\exp(-2\eta t)dt \right]. \tag{5.15}$$

5.2.2. Estimates in $L^2(0,T;H^1(\Omega))$ and $H^1(0,T;L^2(\Omega))$. In addition to the norm $\|\cdot\|_{m,l,T} = \|\cdot\|_{H^l(0,T;H^m(\Omega))}$ we shall use the seminorm $|\cdot|_{m,l,T}$: for each

$$v \in H^l(0,T;H^m(\Omega)), \quad |v|^2_{m,l,T} = \int_0^T \left|\frac{\partial^l v}{\partial t^l}\right|^2_{m,\Omega} dt.$$

Let $T_0 > 0$. Suppose that u is in $H^l(0,T;H^m(\Omega))$ for some T, $T_0 < T$. We wish to extend u to a function $\tilde{u}$ in $H^l(0,\infty;H^m(\Omega))$ in such a way that

$$|\tilde{u}|_{m,l,\infty} \le C \sum_{j=0}^{l} |u|_{m,j,T}$$

with a constant C depending only on Ω and T_0. First let ψ be a cut-off function in $C^\infty(\mathbb{R})$ that vanishes on $(-\infty,0]$ and is constantly 1 on $[T_0,\infty)$. Then let $\xi \in H^l(-\infty,T;H^m(\Omega))$ be defined by

$$\xi(x,t) = \begin{cases} 0 & \text{if } t \le 0, \\ \psi(t)\,u(x,t) & \text{if } 0 \le t \le T. \end{cases}$$

Now use the extension operator $\mathcal{E} : H^l(-\infty,T;H^m(\Omega)) \longrightarrow H^l(-\infty,\infty;H^m(\Omega))$, defined by

$$\mathcal{E}(\xi)(x,t) = \begin{cases} \xi(x,t) & \text{for } t \le T, \\ \displaystyle\sum_{j=1}^{l} a_j \xi(x, T - j(t-T)) & \text{for } T \le t \end{cases}$$

for appropriate choice of the constants a_j (see [32, p. 284]). Then

$$|\mathcal{E}(\xi)|_{m,l,\infty} \le C|\xi|_{m,l,T} \le C\|\psi\|_{l,(0,T_0)} \sum_{j=0}^{l} |u|_{m,j,T}$$

as ψ does not depend on x. Finally the extension $\tilde{u}$ of u is obtained by piecing together u and $\mathcal{E}(\xi)$ at $t = T$:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } 0 \le t \le T, \\ \mathcal{E}(\xi)(t) & \text{if } T \le t. \end{cases}$$

Then $\tilde{u} \in H^l(0,\infty;H^m(\Omega))$ and we have

$$\begin{aligned} |\tilde{u}|_{m,l,\infty} &\le |u|_{m,l,T} + |\mathcal{E}(\xi)|_{m,l,\infty} \\ &\le (1 + C\,\|\psi\|_{l,(0,T_0)}) \sum_{j=0}^{l} |u|_{m,j,T} \\ &\le C_{T_0} \sum_{j=0}^{l} |u|_{m,j,T}. \end{aligned} \tag{5.16}$$

We obtain an extension $\tilde{f}$ of f by $\tilde{f} = \frac{\partial^2 \tilde{u}}{\partial t^2} - \nabla \tilde{u}$ on Q, and the extension $\tilde{u}_h$ of u_h is the solution to the finite element problem on Q with $\tilde{f}$ as the right-hand side. We note that causality guarantees that $\tilde{u}$, $\tilde{u}_h$, and $\tilde{f}$ agree with u, u_h, and f, respectively, on Q_T. Applying (5.15) to $\tilde{u}$, then noting, for the left-hand side, that for $\beta \in L^2(0,\infty)$, $\int_0^T \beta^2\,dt \le \int_0^\infty \beta^2\,dt$ and that u and $\tilde{u}$ as well as u_h and $\tilde{u}_h$ agree on the interval $(0,T)$ and, for the right-hand side, that $e^{-\eta t} \le 1$ on $(0,T)$ and that $|\tilde{u}|^2_{m,l,\infty} \le C\sum_{j=0}^{l} |u|^2_{m,j,T}$ ((5.16) above) we obtain

$$\int_0^T \left(|u-u_h|^2_{1,\Omega} + \left|\frac{\partial(u-u_h)}{\partial t}\right|^2_{1,\Omega} \right) e^{-2\eta t}\,dt$$
$$\le C\,h^{2k}\left[|u|^2_{k+1,0,T} + \sum_{j=0}^{1}|u|^2_{k,j,T} + \frac{1}{\eta^2}\left(\sum_{j=0}^{1}|u|^2_{k+1,j,T} + \sum_{j=0}^{2}|u|^2_{k,j,T} + \sum_{j=0}^{3}|u|^2_{k-1,j,T} \right)\right].$$

Now, since $e^{-2\eta t}$ is bounded below on $[0,T]$ by $e^{-2\eta T}$ we obtain

$$|||u-u_h|||^2_{1,T} \le C\,C_{T_0}\,h^{2k}\,e^{2\eta T}\left[|u|^2_{k+1,0,T} + \sum_{j=0}^{1}|u|^2_{k,j,T} \right.$$
$$\left. + \frac{1}{\eta^2}\left(\sum_{j=0}^{1}|u|^2_{k+1,j,T} + \sum_{j=0}^{2}|u|^2_{k,j,T} + \sum_{j=0}^{3}|u|^2_{k-1,j,T} \right)\right].$$

Setting $\eta = \frac{1}{T}$ in order to minimize $e^{2\eta T}\frac{1}{\eta^2}$, we obtain the following theorem.

THEOREM 5.1. *Suppose that $k \ge 3$. Let T_0 be positive and suppose that $T_0 \le T < \infty$. Suppose that $f \in L^2(Q_T)$ is sufficiently regular that the solution, u, of* (5.13) *is in $H^{k+2}(Q_T)$. Then if u_h is the solution of* (5.14)*, we obtain the following estimate:*

$$|||u-u_h|||^2_{1,T} \le C\,h^{2k}\left[|u|^2_{k+1,0,T} + \sum_{j=0}^{1}|u|^2_{k,j,T} \right.$$
$$\left. + T^2\left(\sum_{j=0}^{1}|u|^2_{k+1,j,T} + \sum_{j=0}^{2}|u|^2_{k,j,T} + \sum_{j=0}^{3}|u|^2_{k-1,j,T} \right)\right]$$

with the constant C depending only on Ω and T_0.

REMARK 5.2. *To obtain estimates of order k for the error $u - u_h$ in the energy norm $\int_0^T (|u-u_h|^2_{1,\Omega} + |\frac{\partial u}{\partial t} - \frac{\partial u_h}{\partial t}|^2_{0,\Omega})\,dt$ we have used only $H^{k+2}(Q_T)$-regularity. In* [3] *estimates of order k in this same norm for the error $u - w_h$, where w_h is an elliptic projection of u into the solution space, $H^{k+3}(Q_T)$-regularity is used. To obtain estimates of order k for the full error $u - u_h$ in the L^∞-energy norm $\sup_{0\le t\le T}(|u - u_h|_{1,\Omega} + |\frac{\partial u}{\partial t} - \frac{\partial u_h}{\partial t}|_{0,\Omega})$, $H^{k+5}(Q_T)$-regularity is required; cf. Theorem* 5.2.

5.2.3. An $L^\infty(0,T;H^1(\Omega))$ estimate. In this section we use interpolation to obtain an estimate in the $\|\cdot\|_{L^\infty(0,T;H^1(\Omega))}$-norm. As, for each θ, $\frac{1}{2} < \theta < 1$, there is a continuous injection of $H^\theta(0,T;H^1(\Omega))$ into $L^\infty(0,T;H^1(\Omega))$, it suffices to obtain estimates in the $H^\theta(0,T;H^1(\Omega))$ norm. Recall that

$$H^\theta(0,T;H^1(\Omega)) = [L^2(0,T;H^1(\Omega)), H^1(0,T;H^1(\Omega))]_\theta.$$

If, for $s > \frac{1}{2}$ and j a nonnegative integer with $j < s - \frac{1}{2}$, we introduce the notation

$$H^s_{0,j}(Q_T) = \left\{ u \in H^s(Q_T) : \frac{\partial^i u}{\partial t^i} = 0,\ i = 0,\dots,j \right\},$$

we may write, for $\theta \in (0,1)$ and for m an integer with $m > \frac{5}{2} - \theta$,

$$H^{m+\theta}_{0,2}(Q_T) = [H^m_{0,1}(Q_T), H^{m+1}_{0,2}(Q_T)]_\theta,$$

as $H^{m+1}_{0,2}(Q_T)$ is dense in $[H^m_{0,1}(Q_T), H^{m+1}_{0,2}(Q_T)]_\theta$.

Now, for $h > 0$, let $\mathcal{L}_h$ denote the linear operator that associates with $v \in H^m_{0,1}(Q_T)$ the error $v - v_h \in L^2(0,T;H^1(\Omega))$, where v_h is the solution to (5.14) for $f = \frac{\partial^2 v}{\partial t^2} - \Delta v$. From Theorem 5.1 we have that this operator is continuous:

$$\|\mathcal{L}_h v\|_{L^2(0,T;H^1(\Omega))} \le C_0\, T\, h^k \|v\|_{H^{k+2}_{0,1}(Q_T)}. \tag{5.17}$$

For an $H^1(0,T;H^1(\Omega))$-estimate, we note that the restriction of the operator $\mathcal{L}_h$ to the subspace $H^{m+1}_{0,2}(Q_T)$ commutes with the operator $\frac{\partial}{\partial t}$:

$$\begin{array}{ccc} H^{m+1}_{0,2}(Q_T) & \xrightarrow{\mathcal{L}_h} & H^1(0,T,H^1(\Omega)) \\ \frac{\partial}{\partial t}\downarrow & & \frac{\partial}{\partial t}\downarrow \\ H^m_{0,1}(Q_T) & \xrightarrow{\mathcal{L}_h} & L^2(0,T,H^1(\Omega)), \end{array}$$

so that we have, for $v \in H^{k+3}_{0,2}(Q_T)$,

$$\|\mathcal{L}_h v\|_{H^1(0,T,H^1(\Omega))} \le C_1\, T\, h^k \|v\|_{H^{k+3}_{0,2}(Q_T)}. \tag{5.18}$$

Now as the inclusions $H^{m+1}_{0,2}(Q_T) \hookrightarrow H^m_{0,1}(Q_T)$ and $H^1(0,T,H^1(\Omega)) \hookrightarrow L^2(0,T,H^1(\Omega))$ are dense and continuous, we can combine a standard interpolation result (see [29, Theorem 5.6]) with (5.17) and (5.18) to obtain

$$\|\mathcal{L}_h v\|_{H^\theta(0,T,H^1(\Omega))} \le C_0^{1-\theta}\, C_1^\theta\, T\, h^k \|v\|_{H^{k+2+\theta}_{0,2}(Q_T)}.$$

Thus we have demonstrated the following theorem.

THEOREM 5.2. *Suppose $k \ge 3$. Let T_0 be positive and suppose that $T_0 \le T < \infty$. Let θ be in the interval $(\frac{1}{2}, 1)$ and suppose that $f \in L^2(Q_T)$ is sufficiently regular that the solution, u, of* (5.13) *is in $H^{k+2+\theta}(Q_T)$. Then if u_h is the solution of* (5.14)*, we obtain the following estimate:*

$$\|u - u_h\|_{L^\infty(0,T,H^1(\Omega))} \le C_\theta\, T\, h^k \|u\|_{H^{k+2+\theta}_{0,2}(Q_T)}$$

with the constant C_θ depending only on θ, Ω, and T_0.

6. Time discretization.

6.1. Presentation of the schemes. The semidiscrete equation can be rewritten

$$\frac{d^2 u_h}{dt^2} + \widetilde{A}_h u_h = 0, \tag{6.1}$$

where $\widetilde{A}_h$ denotes the bounded operator in V_h associated with the bilinear form $a(u_h, v_h)$ and the inner product $(u_h, v_h)_h$:

$$\forall\, (u_h, v_h) \in V_h \times V_h, \quad (\widetilde{A}_h u_h, v_h)_h = a(u_h, v_h).$$

For the time discretization, the simplest scheme consists of using the classical leapfrog scheme with three time steps:

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h u_h^n = 0, \tag{6.2}$$

where u_h^n is the approximate solution at time $t^n = n\Delta t$ and Δt denotes the time step. Such a scheme yields second order accuracy with respect to Δt which is generally not sufficient for a higher order finite element method. For instance, the dispersion analysis (see [33]) on uniform meshes of the semidiscrete problem shows that one obtains errors in space on the phase velocity of plane waves which are $O(h^4)$ and $O(h^6)$, respectively, for the $\widetilde{P}_2$ and $\widetilde{P}_3$ elements. On the other hand, the stability analysis allows us to take a time step Δt which is proportional to the space step h. If one does not want to lose the accuracy provided by the space discretization, one must then also use higher order schemes with respect to time. It is the purpose of this section to propose a strategy for reaching this goal. Of course, as we have to approximate an ordinary differential equation we have at our disposal all the machinery of Runge–Kutta or multistep methods. We shall restrict our investigation to schemes respecting the following criteria:

(i) The scheme uses three time steps.
(ii) The scheme is explicit.
(iii) The scheme is centered.

Criterion (ii) is obvious in our context and motivated by efficiency considerations only. Criterion (i) prevents a priori the use of any start-up procedure but, in fact, also appears necessary to combine higher order accuracy, explicitness, and stability. Criterion (iii) aims at preserving in the discrete model the reversible nature and the conservative character of the continuous model. It ensures in particular that no numerical dissipation is introduced by the time discretization. A convenient strategy consists in applying the so-called modified equation technique to (6.1). Such schemes can be seen as appropriate modifications of the leapfrog scheme (6.2) constructed by looking at the truncation error associated with the leapfrog scheme. More precisely, if $U_h^n = u_h(t^n)$, where $u_h(t)$ denotes the exact solution of (6.1), we have

$$\frac{U_h^{n+1} - 2U_h^n + U_h^{n-1}}{\Delta t^2} = \frac{\partial^2 u_h}{\partial t^2}(t_n) + 2\sum_{l=1}^{k-1}(-1)^{l+1}\frac{\Delta t^{2l}}{(2l+2)!}\widetilde{A}_h^{l+1} U_h^n + O(\Delta t)^{2k}. \tag{6.3}$$

Therefore a $2k$th order scheme in Δt is given by

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h u_h^n - 2\sum_{l=1}^{k-1}(-1)^{l+1}\frac{\Delta t^{2l}}{(2l+2)!}\widetilde{A}_h^{l+1} u_h^n = 0. \tag{6.4}$$

From the computational point of view, note that the cost of one time step for (6.4) is k times the cost of one time step for (6.2). To see that, one evaluates the polynomial in $\widetilde{A}_h$ appearing in (6.4) using Horner's rule. For instance, for $k = 2$ (order 4), the scheme is

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h u_h^n - \frac{\Delta t^2}{12}\widetilde{A}_h^2 u_h^n = 0 \tag{6.5}$$

and is implemented as

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h \left[u_h^n - \frac{\Delta t^2}{12}\widetilde{A}_h u_h^n\right] = 0, \tag{6.6}$$

where the quantity between brackets appears as the result of an intermediate time step. For $k = 3$ (order 6), the scheme is

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h u_h^n - \frac{\Delta t^2}{12}\widetilde{A}_h^2 u_h^n + \frac{\Delta t^4}{360}\widetilde{A}_h^3 u_h^n = 0, \tag{6.7}$$

and we use the expression

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h\left[u_h^n - \frac{\Delta t^2}{12}\widetilde{A}_h\left[u_h^n - \frac{\Delta t^2}{30}\widetilde{A}_h u_h^n\right]\right] = 0. \tag{6.8}$$

6.2. Stability analysis. An important feature relative to the analysis and the practical implementation of such schemes is their stability: explicit schemes are constrained by a CFL stability condition. For the leapfrog scheme (6.2), this condition is (see [28])

$$\frac{\Delta t^2\|\widetilde{A}_h\|}{4} \le 1, \tag{6.9}$$

where the norm of operator $\|\widetilde{A}_h\|$ is defined by

$$\|\widetilde{A}_h\| = \sup_{v_h\in V_h}\frac{a(u_h, v_h)}{|v_h|_h^2}. \tag{6.10}$$

It is also possible to obtain a sufficient stability condition for the general $2k$th order scheme (6.4). More precisely, we have the following condition.

LEMMA 6.1. *Scheme* (6.4) *is stable if*

$$\frac{\Delta t^2\|\widetilde{A}_h\|}{4} \le \alpha_k, \tag{6.11}$$

where the coefficient α_k *is characterized by*

$$\alpha_k = \sup\{\alpha > 0 \;/\; \forall x \in [0, \alpha],\ 0 \le xQ_k(x) \le 4\}, \tag{6.12}$$

where Q_k *is the polynomial defined by*

$$Q_k(x) = 1 + 2\sum_{l=1}^{k-1}\frac{(-1)^l x^l}{(2l+2)!}.$$

In particular, one has

$$\alpha_1 = 1, \quad \alpha_2 = 3, \quad \alpha_3 = -\frac{5^{2/3}}{2} + \frac{\sqrt[3]{5}}{2} + 5/2 \simeq 1.8930,$$

$$\alpha_4 = \frac{\left(\sqrt[3]{27}^{2/3}\left(109 + 27\sqrt{41}\sqrt{5}\right)^{2/3} - 238 + 14\,2^{2/3}\sqrt[3]{7}\sqrt[3]{109 + 27\sqrt{41}\sqrt{5}}\right)\sqrt[3]{27}^{2/3}}{42\sqrt[3]{109 + 27\sqrt{41}\sqrt{5}}}$$

$$\simeq 5.3703.$$

Proof. Let us denote by $\sigma(\widetilde{A}_h)$ the spectrum of $\widetilde{A}_h$, made up of positive real eigenvalues. Let us remark that scheme (6.4) can be rewritten

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \widetilde{A}_h(\Delta t)u_h^n = 0, \tag{6.13}$$

where $\widetilde{A}_h(\Delta t)$ is defined by

$$\widetilde{A}_h(\Delta t) = \widetilde{A}_h\, Q_k(\Delta t^2 \widetilde{A}_h).$$

The von Neuman analysis shows that the scheme is stable if and only if the eigenvalues of the symmetric operator $\widetilde{A}_h(\Delta t)$ are positive and smaller than $4/\Delta t^2$. This leads to the requirement

$$\forall\, \lambda \in \sigma(\widetilde{A}_h), \quad 0 \le \lambda \Delta t^2\, Q_k(\lambda \Delta t^2) \le 4.$$

Let us introduce the set

$$E_k = \{x \ge 0 \ / \ 0 \le x\, Q_k(x) \le 4\}.$$

Looking at the behavior of Q_k at infinity, it is easy to see that E_k is a compact set whose first connected component is the segment $[0, \alpha_k]$. As we also have the inclusion

$$\sigma(\widetilde{A}_h) \subset [0, \|\widetilde{A}_h\|],$$

it is very easy to conclude. □

Note that, for instance, the time step allowed for the fourth order scheme is $\sqrt{3}$ (1.732) times greater than that for the second order scheme, which compensates in large part for the fact that the computational time per time step is twice as large. Also note that $\|\widetilde{A}_h\|$ is of the order $1/h^2$ for small h. An explicit expression for $\|\widetilde{A}_h\|$ can be obtained in the case of a regular mesh using a Fourier method (see [33]).

7. Numerical results.

We conclude with a presentation of two types of numerical experiments, the first to show the accuracy and the efficiency of the method, the second one to show that it can be used with unstructured meshes.

7.1. A test in a homogeneous medium.

We first consider the very simple test case: the propagation of a wave in a homogeneous two-dimensional (2D) medium of velocity 1, with a Dirichlet boundary condition. The source is quasi-punctual and located at point x_S.

$$\text{(7.1)} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = g(x) f(t), \quad x \in \Omega, \quad t \in \,]0,T[, \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in \,]0,T[, \end{cases}$$

where

$$\text{(7.2)} \qquad \begin{cases} f(t) = \begin{cases} 2a(2a(t-b)^2 - 1)\exp(-a(t-b)^2) & \forall\, t \in [0, 3.5], \\ 0 \quad \text{otherwise}, \end{cases} \\ a = \left(\dfrac{\pi}{1.31}\right)^2, \quad b = 1.35, \\ g(x) = \exp(-7|x - x_S|). \end{cases}$$

We consider the case in which Ω is a square and the source is located at its center, i.e.,

$$\text{(7.3)} \qquad \begin{cases} \Omega = \,]0,12[\, \times \,]0,12[, \\ x_S = (6,6). \end{cases}$$

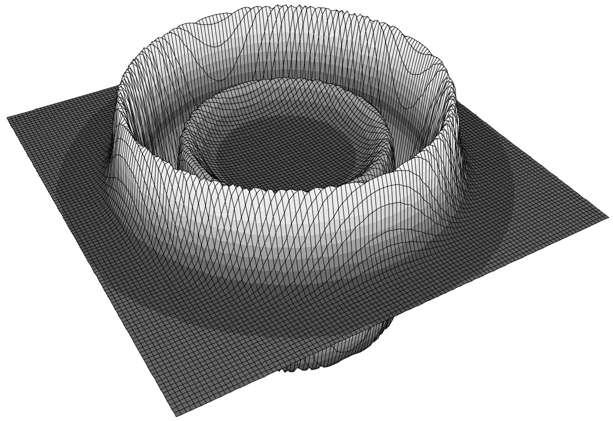

FIG. 7.1. *Snapshot of the solution on a regular mesh after five seconds.*

The source (7.2) generates a wave whose central wavelength is about 0.5. We take $T = 50$ so that the wave propagates along more than 100 wavelengths (a hard test case for second order methods). For the computation, we use a uniform mesh. A snapshot of the solution after a short time (before any reflexion on the boundary) is presented in Figure 7.1.

This gives an idea of the variation in space of the solution that has to be compared with the choice of the space step Δx. Our goal is to compare the accuracy of our new $\widetilde{P}_2$ and $\widetilde{P}_3$ elements with respect to the usual P_1 method. For this comparison, we adapt, in each case, the step Δx of the mesh in order that the number of degrees of freedom should be approximately the same in each experiment:

$$\begin{cases} \bullet \text{ For } P_1,\ \Delta x = \dfrac{2}{15}. \\[2ex] \bullet \text{ For } \widetilde{P}_2,\ \Delta x = \dfrac{4}{15}. \\[2ex] \bullet \text{ For } \widetilde{P}_3,\ \Delta x = \dfrac{2}{5}. \end{cases}$$

The ratio $\alpha = \Delta t/\Delta x$ is taken so that time step Δt is equal to 0.04 for second order methods in time and to 0.08 for fourth order in time (we recall that the stability condition is roughly twice as large for schemes, fourth order in time, which balances the cost of the corrective term).

In Figures 7.2–7.6, we represent the computed solution at point (9,3) as a function of time. The top pictures correspond to short times ($0 \le t \le 25$) while the bottom pictures correspond to later times ($25 \le t \le 50$). In fact, in each picture, we have two curves: the dotted line corresponds to the exact solution, which can be computed, for instance, using the principle of images and the continuous line corresponds to the approximate solution.

In Figure 7.2, we represent the P_1 solution, computed with a centered scheme second order in time. We see clearly that an unacceptable error appears quite early on. This points out the dispersive nature of the waves as well as the importance of using higher order schemes.

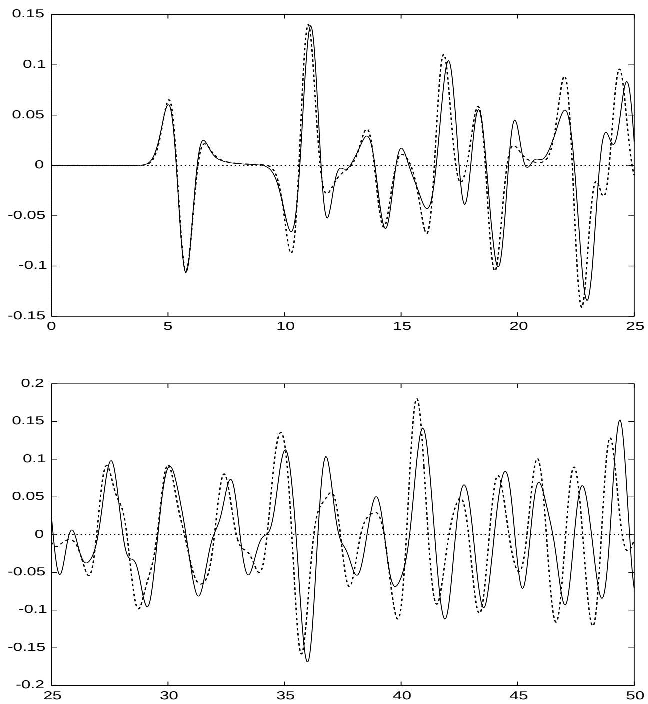

FIG. 7.2. *Seismogram P_1, second order in time and space, 3.75 elements per wavelength, $\alpha = 0.3$, $t \in [0, 25]$ (top), $t \in [25, 50]$ (bottom).*

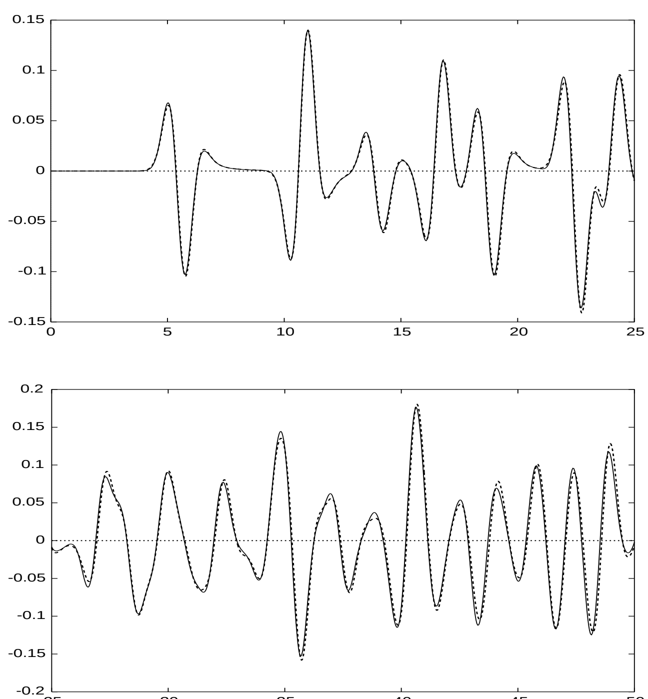

FIG. 7.3. *Seismogram P_2, fourth order in space and second order in time and $\alpha = 0.15$, 1.875 elements per wavelength, $t \in [0, 25]$ (top), $t \in [25, 50]$ (bottom).*

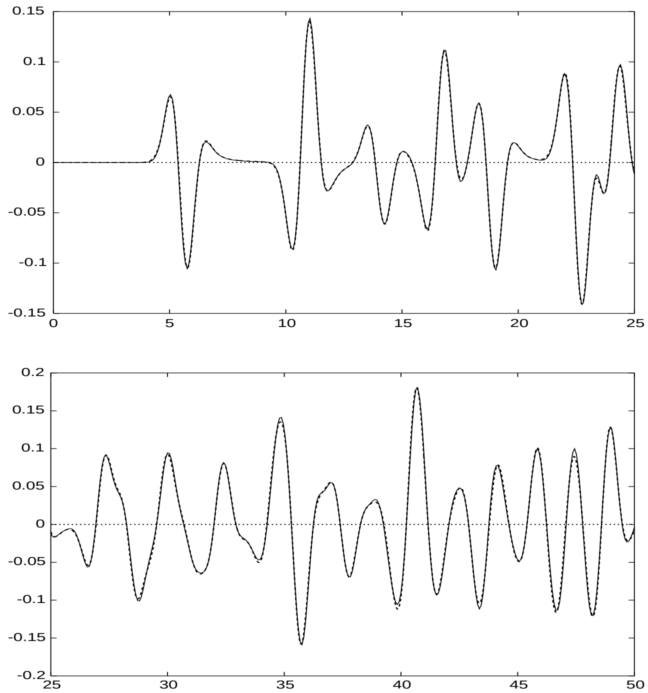

FIG. 7.4. *Seismogram P_2, fourth order in space and time and $\alpha = 0.3$, 1.875 elements per wavelength, $t \in [0, 25]$ (top), $t \in [25, 50]$ (bottom).*

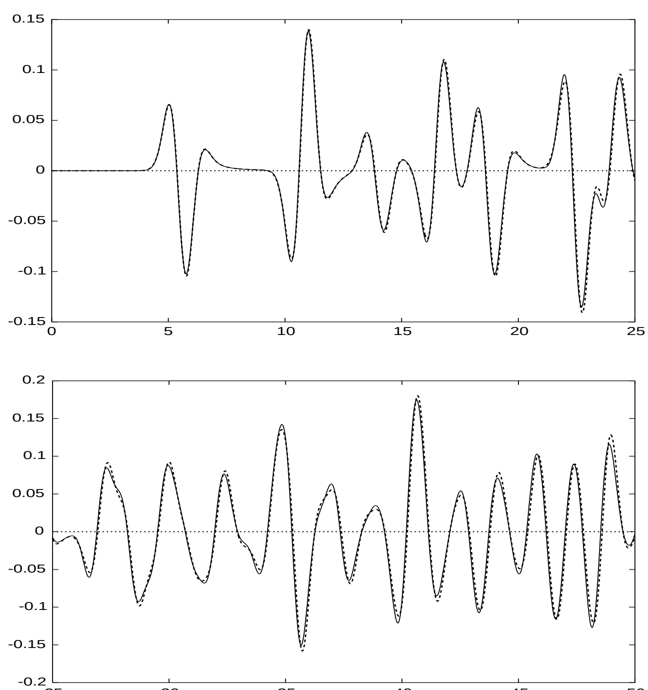

FIG. 7.5. *Seismogram P_3, fourth order in space and second order in time and $\alpha = 0.1$, 1.25 elements per wavelength, $t \in [0, 25]$ (top), $t \in [25, 50]$ (bottom).*

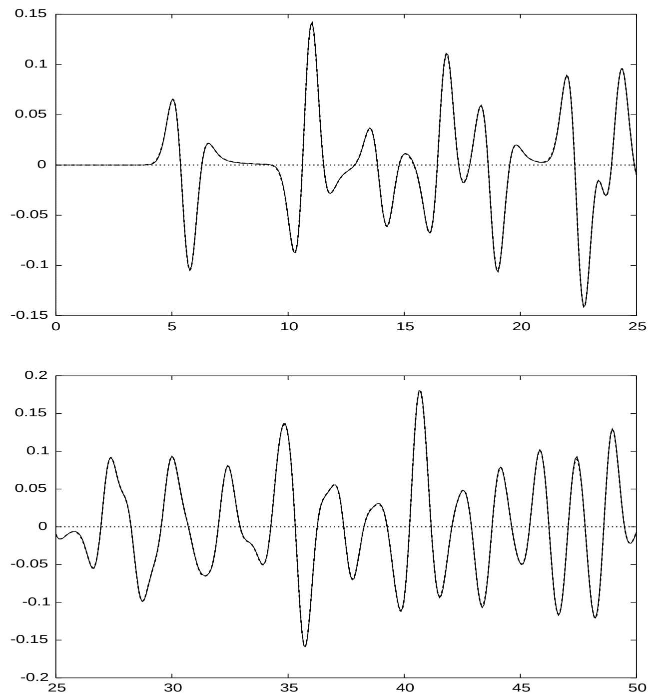

FIG. 7.6. *Seismogram* P_3, *fourth order in space and time and* $\alpha = 0.2$, 1.25 *elements per wavelength,* $t \in [0, 25]$ *(top),* $t \in [25, 50]$ *(bottom).*

In Figure 7.3, we represent the solution computed with the $\widetilde{P}_2$ space, still using a second order scheme in time. In comparison with Figure 7.2, we observe a major improvement of the quality of the solution due to the gain in spatial accuracy. However, a small phase-shift appears at later times. The result shown in Figure 7.4, for which we have used here a scheme fourth order in time, then clearly shows the advantage of increasing the time accuracy: the phase-shift observed in Figure 7.3 has disappeared.

The results seen in Figures 7.5 and 7.6 have been computed with the space $\widetilde{P}_3$. The comparison of these results seems to indicate that it is of no use to continue to increase the time accuracy: with the $\widetilde{P}_3$ element and a fourth order scheme in time (Figure 7.6), we get a quasi-perfect solution, despite the fact that the mesh is rather coarse in this experiment. Moreover, it can be shown that the method sixth order in time, which requires a significant increase in computation time, is less stable than the method fourth order in time.

7.2. An example of a computation in a complex geometry. As an illustration of the ability of our code to handle complex geometries, we treat here a problem of wave propagation in an exterior domain. The domain, the complement of a "cone-sphere"-shaped obstacle, can be seen in Figure 7.7, where we also show the computational mesh, which is in fact nonregular only in a neighborhood of the obstacle. We observe the diffraction of an incident wave of the same nature as that considered in section 7.1. This incident wave is emitted by a point source located at point (7.5,12). We show, in Figure 7.7, a snapshot of the total field. We see in particular the diffraction phenomenon due to the summit of the cone.

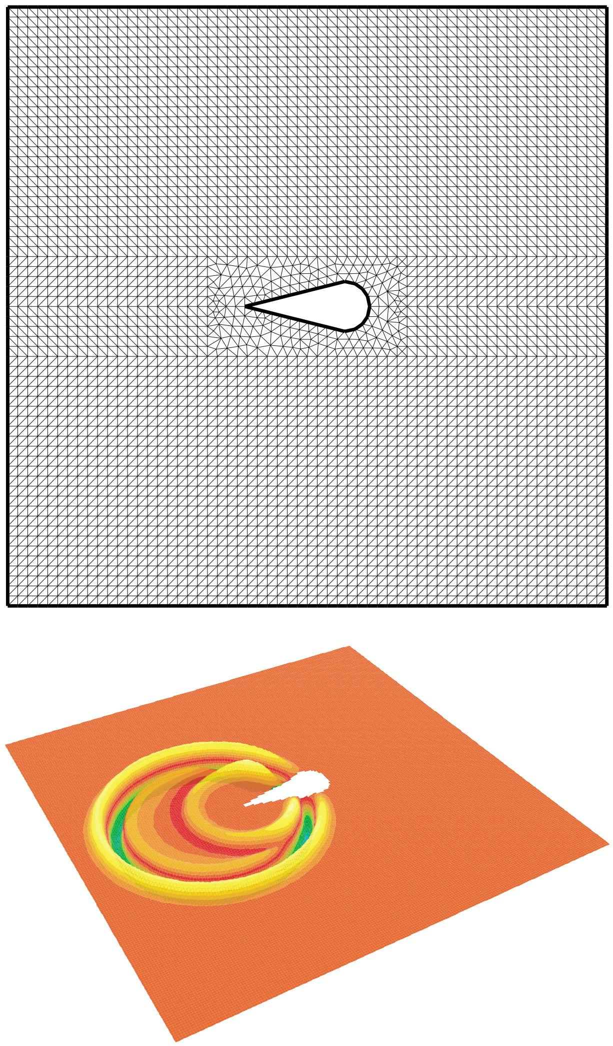

FIG. 7.7. *Mesh for a cone-sphere (top) and the solution after* 5.3*s (bottom).*

8. Conclusion. In this paper, we have constructed triangular finite elements of degrees 2 and 3 which lead to stable, explicit, third, and fourth order accurate methods for the approximation of the 2D wave equation on arbitrary meshes. These methods are obtained by using quadrature rules permitting mass lumping together with higher order finite difference time stepping. The present work might be generalized in several directions. We conclude here by presenting the actual state of our research in these directions:

- **Generalization to higher order elements in two dimensions.** We have been able to construct the $\widetilde{P}_4$ space, which gives fifth order accuracy, along the same lines as those of section 4. However, it would be nice to have a general strategy for constructing the $\widetilde{P}_k$ spaces for any k. This does not seem to be at all easy. The work is in progress.

- **Generalization to dimension 3.** This work is underway; we have already constructed the $\widetilde{P}_2$ space. This example shows that extension to the three-dimensional case is not as straightforward as one might think. (See also the work by Mulder [26]).
- **Generalization to other models.** The methodology presented here can be extended to edge elements for Maxwell's equations [12], [17], [18] and also to mixed formulations of elastodynamics equations [4], [5]. The essential additional difficulty lies in the fact that the quantities to be approximated are no longer scalar, so that obtaining mass lumping is more difficult even in the case of lower order elements.

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