Generalized cell-centered finite volume methods: application to two-phase flow in porous media

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1 Introduction

Finite volume methods and finite elements methods are generally opposed as competing approximation techniques. However there are ways to look at them which make these methods closer than what is usually thought, and this leads to the construction of generalized finite volume methods. Here we will restrict ourselves to cell-centered finite volumes.

On one hand, cell-centered finite volume methods are widely used by engineers and scientists who have to perform numerical simulations. When building such methods, one discretizes the domain, usually with rectangles, but not always, and equations are written inside each cell of the discretization. Then relations between cells are written. To do so, the solution of the problem is approximated by piecewise constants. Most often, before being eliminated, flux unknowns are also introduced to write the intercell relations which describe conservation. This procedure is very close to the physics and, in this fashion, even complicated physical laws can be implemented easily.

On the other hand, finite element methods have their advantages: they are based on a rigorous mathematical analysis, higher order methods can be obtained by increasing the degrees of the polynomials, and structured as well as unstructured meshes can be used depending on the application that is under study.

In this paper we show how to combine the advantages of these two approximation techniques into one numerical procedure. We retain the point of view of finite volume methods, but we call them generalized since, to approximate the unknowns, we use general discontinuous piecewise polynomial approximations instead of just piecewise constants. This will be achieved using mixed-hybrid finite element and Godunov's methods. This paper can be viewed as a sequence to a paper by Chavent-Roberts [CR91] where the method was described for linear elliptic and parabolic equations.

To illustrate our point we consider as an example a model for incompressible two-phase flow in a porous medium which results in a nonlinear system of two equations. One equation, the saturation equation, is parabolic and represents conservation of one of the phases which implies continuity of the normal components of the Darcy velocity of this phase. The other equation, the pressure equation, is elliptic and represents conservation of both phases which implies continuity of the normal components of the total Darcy velocity. Our numerical procedure will follow closely these physical requirements. Even in the homogeneous case the

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resulting schemes may be different from what has been already presented in [CJ86, CCJ+90, Daw91]. However the power of our numerical procedure will be shown in the case of a porous medium made of several subdomains with different rock properties.

Similar techniques have been used in nuclear engineering for the diffusion approximation and are called nodal methods [HJR88, Hen92]. In such a problem equations represent the conservation of neutrons, so the normal components of the current must be continuous. Efficient nodal methods achieve these physical constraints and can be obtained from mixed-hybrid finite elements.

2 Two-phase flow in porous media

As an example let us consider the model for incompressible two-phase flow in a porous medium which uses the global pressure formulation [CJ86].

The saturation equation expresses volume conservation for the wetting phase (which is equivalent to mass conservation since the flow is assumed to be incompressible):

$$\Phi \frac{\partial S}{\partial t} + \operatorname{div} \vec{\varphi}_w = 0 \tag{1}$$

where $S = S_w$ is the saturation of the wetting phase ($0 < S < 1$). The Darcy velocity of the wetting phase $\vec{\varphi}_w$ is given by the motion equation

$$\begin{cases} \vec{\varphi}_w = \vec{r} + \vec{f}, \\ \vec{r} = -K\vec{\nabla}\alpha(S), \quad \vec{f} = K(b_T(S)\vec{\varphi}_T + b_G(S)\vec{q}_G). \end{cases} \tag{2}$$

Here K denotes the tensor of absolute permeabilities, $\vec{\varphi}_T$ the total Darcy velocity which is the sum of the Darcy velocities of the wetting and the nonwetting phase and $\vec{q}_G$ denotes the gravity field :

$$\vec{\varphi}_T = \vec{\varphi}_w + \vec{\varphi}_{nw}, \quad \vec{q}_G = g\vec{\nabla} Z.$$

with g the gravity constant and Z the depth at the location.

Coefficients α, b_T, b_G depend on the mobilities k_w, k_{nw} and the capillary pressure p_c :

$$a = \frac{k_w k_{nw}}{k_w + k_{nw}} \frac{dp_c}{dS}, \quad \alpha = \int_0^S a(s)ds\,,$$

$$b_T = \frac{k_w}{k_w + k_{nw}}, \qquad b_G = \frac{k_w k_{nw}}{k_w + k_{nw}}\,(\rho_w - \rho_{nw}).$$

The mobilities are positive monotone functions of the saturation S; k_w is increasing with $k_w(0) = 0$ while k_{nw} is decreasing with $k_{nw}(1) = 0$. The capillary pressure $p_c = p_{nw} - p_w$ is a positive decreasing function of S if p_{nw}, p_w denote the pressures in the nonwetting and wetting phases.

Plugging equation (2) into (1) the saturation equations become a nonlinear parabolic equation of diffusion-convection type. The vector $\vec{r}$ is the diffusion contribution to $\vec{\varphi}_w$ due to capillary effects and $\vec{f}$ is the convection contribution to $\vec{\varphi}_w$. The latter is itself the sum of gravity effects and of the contribution of the total flow rate $\vec{\varphi_T}$ which is given by the pressure equation that we now describe.

The pressure equation models the conservation of the total volume of the two phases. Since the flow is assumed to be incompressible this takes the form

$$\operatorname{div} \vec{\varphi}_T = 0. \tag{3}$$

Using the global pressure formulation [CJ86] the total flow rate is given by the motion equation

$$\vec{\varphi}_T = -Kd(S)(\vec{\nabla} p - \rho(S)\vec{q}_G) \tag{4}$$

where the global pressure p is given by

$$p = \frac{1}{2}(p_w + p_{nw}) + \gamma(S). \tag{5}$$

The coefficients γ, d, ρ are functions of the saturation S :

$$\gamma = \int_0^S (b_T(s) - \frac{1}{2})\frac{dp_c}{dS},$$
$$d = k_w + k_{nw}, \qquad \rho = \frac{k_w\rho_w + k_{nw}\rho_{nw}}{k_w + k_{nw}}.$$

Continuity of the phase pressures p_w, p_{nw} imply that the capillary pressure p_c, and consequently the saturation S, is continuous, and that so is the global pressure p because of its definition (5). Also, because of phase conservation, the normal components of the phase Darcy velocities $\vec{\varphi}_w, \vec{\varphi}_{nw}$, and consequently of the total Darcy velocity $\vec{\varphi}_T$, are continuous across any hypersurface.

However, if there are several rock types, that is subdomains with different relative permeability and capillary pressure curves, the same physical assumptions hold, that is continuity of the phase pressures and phase conservation. But, across the interface between two rock types, continuity of the capillary pressure

$$p_c^l(S^l) = p_c^r(S^r). \tag{6}$$

implies now that the saturation is discontinuous. Here the superscripts l and r designate the left and right sides of the interface. Furthermore, from (5) and the continuity of the phase pressures, we obtain

$$\frac{1}{2}(p_w + p_{nw}) = p^l - \gamma^l(S^l) = p^r - \gamma^r(S^r). \tag{7}$$

which shows that the global pressure is also discontinuous across the interface between two rock types. Phase conservation still implies that the normal components of the Darcy velocities $\vec{\varphi}_w, \vec{\varphi}_T$ are continuous across the interface.

3 Approximation spaces

The domain Ω where the equations are defined is discretized with a structured or an unstructured mesh. Denote by $C \in \mathcal{T}$ the cells and by $E \in \mathcal{E}$ the faces (or edges in 2-D) of the mesh.

In our framework, the scalar unknowns, pressure and saturation, as well as the vector unknowns, Darcy velocities, are approximated locally, cell per cell, by polynomials. They

are discontinuous from one cell to the other. Inside each cell, for both the scalar and vector unknowns, there are cell degrees of freedom which are moments calculated over the cell, and edge degrees of freedom which are moments of the scalar function and moments of the normal components of the vector function over the faces. This is a generalization of standard finite volumes where scalars are approximated as piecewise constants, and where the vector unknowns have one degree of freedom per edge, the latter unknowns being usually eliminated.

However mathematical analysis as well as numerical experience show that the approximation spaces for the scalar and vector unknowns cannot be chosen arbitrarily. They must satisfy certain compatibility conditions which were studied in the analysis of mixed and mixed-hybrid finite elements. We need three approximation spaces.

1. The total Darcy velocity $\vec{\varphi_T}$ and the Darcy velocity of the phase wetting $\vec{\varphi}_w$ are calculated in an approximation space $\vec{X}$ such that

$$\vec{X} = \{\vec{v} \in (L^2(\Omega))^2 \mid \vec{v}_{|C} \in \vec{X}_C, C \in \mathcal{T}\}.$$

   Note that functions of X are discontinuous from one cell to the other and that the inclusion of X_C into $H(\mathrm{div}, C)$ ensures that, inside the cell, components of the functions normal to the faces (or edges) can be defined.

2. The pressure p and the saturation S are first approximated inside the cells in a space M which is a subset of $L^2(\Omega)$. Again functions of M are discontinuous from one cell to the other.

3. Then the pressure p and the saturation S are also approximated on the faces (or edges). We introduce TP and TS which approximate traces of the pressure and the saturation on the faces (or edges) in the space $N \subset \prod_{C\in\mathcal{T}} \prod_{E\subset\partial C} L^2(E)$. Note that there are two traces per face (or edge) that we will denote by $(TP)^l, (TP)^r$ and $(TS)^l, (TS)^r$.

As already mentionned spaces $\vec{X}, M$ and N cannot be chosen arbitrarily. Their choice must be based on the theory of mixed-finite elements [BF91, RT91] and there is now a large catalogue of spaces available in two or three dimensions, for structured or nonstructured meshes [RT77, BDM85, BDDF87, BDFM87, Ned80, Ned86]. Figure 1 gives examples in two dimensions.

4 Discretization of the pressure equation

Let us consider first the pressure equation that we have written as a system of two equations, a conservation equation (3) and a motion equation (4). To follow closely the physics we are going to write these two equations on each cell of the discretization and then equations on the edges expressing the intercells relations.

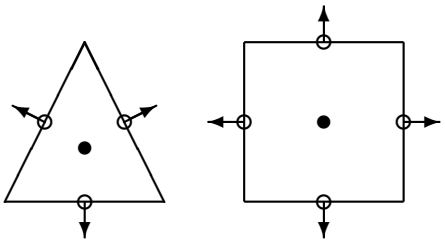
To simplify the notations we use the same notations for the approximate functions as for the continuous ones. So we like to find $\vec{\varphi_T} \in \vec{X}, p \in M, TP \in N$ satisfying the following equations.

Cell equations :

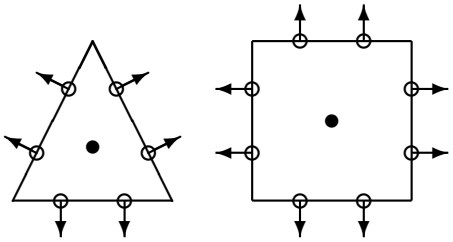
1. conservation equations

$$\int_C \mathrm{div}\vec{\varphi}_T \; q = -\int_C \vec{\varphi}_T \cdot \vec{\nabla} q \; + \int_{\partial C} (\vec{\varphi}_T \cdot \vec{n}_C) q = 0, \quad q \in M, \; C \in \mathcal{T}. \tag{8}$$

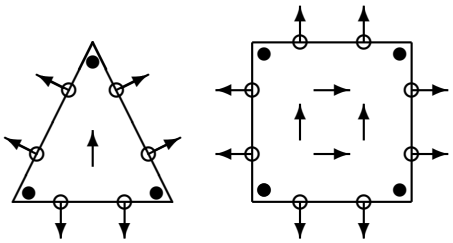
Raviart-Thomas elements of index 1



Brezzi-Douglas-Marini elements of index 1



Raviart-Thomas elements of index 2



↑ degrees of freedom for $\vec{X}$
● degrees of freedom for M ○ degrees of freedom for N

Figure 1: Examples of mixed finite element spaces in two dimensions.

This is equation (3), multiplied by test functions q and integrated over a cell C. It may or may not be integrated by parts. When it is, the degrees of freedom associated with the normal components appear explicitely in the approximation equation.

2. motion equations

$$\int_C (Kd)^{-1}\vec{\varphi}_T \cdot \vec{v} - \int_C p \,\mathrm{div}\vec{v} + \sum_{E\subset\partial C} \int_E TP\ \vec{v}\cdot\vec{n} = \int_C \rho\,\vec{q}_G \cdot \vec{v},$$
$$\vec{v} \in \vec{X}_C,\ C \in \mathcal{T}. \tag{9}$$

This equation is obtained by multiplying equation (4) by $(Kd)^{-1}$ and by test functions $\vec{v}$, and by integrating over the cell C.

Edge equations, for edges within a given rock type:

3. pressure continuity

Since the pressure must be continuous across the edges, the two traces of the pressure must be equal, so $(TP)^l = (TP)^r$ on all interior edges. On the boundary, if we denote by $\mathcal{E}_{PD}$ the set of boundary edges where Dirichlet conditions are imposed on the pressure, we have also

$$TP = p_d \,\text{on}\, E \in \mathcal{E}_{PD}.$$

4. conservation equations

To preserve conservation the normal components of the total Darcy velocity must be continuous across the interior edges. If we assume also that we have homogenous Neumann conditions on the part of the boundary where we do not have Dirichlet conditions, all these conditions can be written as

$$\int_E \vec{\varphi}_T^l \cdot \vec{n}\,\tau = \int_E \vec{\varphi}_T^r \cdot \vec{n}\,\tau, \quad \tau \in N, \tag{10}$$

where $\vec{n}$ denotes one of the normals to E.

Edge equations, for edges between two rock types:

3bis. pressure discontinuity

Since traces on the edges are available in our approximation framework, equation (7) can be approximated in a straightforward manner:

$$(TP)^l - \gamma^l((TS)^l) = (TP)^r - \gamma^r((TS)^r). \tag{11}$$

4bis. conservation equations

Total conservation still implies equation (10).

We claim that the above method is a finite volume method for the following reasons.

– Equations have been written cell by cell and intercell relations are expressed by the edge equations.

– The scheme is cellwise conservative. Indeed since the approximation space M contains piecewise constants we have

$$\int_C \mathrm{div}\vec{\varphi}_T = \sum_{E\subset\partial C} \int_E \vec{\varphi}_T \cdot \vec{n}_C = 0,\ C \in \mathcal{T}.$$

– Let us consider the case when the domain is discretized with rectangular meshes with faces (or sides) parallel to the coordinate axes and when the Raviart-Thomas-Nedelec elements of index 1 are used. If one uses the trapezoidal rule in all the coordinate directions to calculate the integral $\int_C (Kd)^{-1}\vec{\varphi}_T \cdot \vec{v}$ one can eliminate the total velocity $\vec{\varphi}_T$ and obtains the standard cell-centered finite volume method with a five-point stencil in two dimensions or a seven-point stencil in three dimensions [Jaf84, WW88]. On triangular meshes the relation between mixed finite elements and cell-centered finite volumes is investigated in [ABMO95].

One should also note that, for a given rock type, the above method is equivalent to the mixed or mixed-hybrid finite element method in the sense that the numbers for p and $\vec{\varphi}_T$ obtained from the machine are the same. The difference in the formulation lies in the fact that in our finite volume method all continuity requirements are enforced explicitly.

Once the approximate formulation has been written, the resulting system of equations is very large and the next step is to choose the most efficient way to implement it. However this is beyond the scope of this paper.

5 Discretization of the saturation equation

We shall proceed for the saturation equation in a similar way to that for the pressure equation. However the saturation equation is of diffusion-convection type and convection is often dominant. Therefore a special treatment of the convection terms will be introduced using Godunov techniques. Since we are emphasizing space discretization we will not discretize time and leave to the reader the choice of his favorite time discretization.

Again using for simplicity the same notations for continuous and approximate functions we like to find $\vec{r}(t) \in \vec{X}, S(t) \in M, TS(t) \in N$ satisfying the following equations.

Cell equations :

1. conservation equations

$$\int_C \frac{\partial S}{\partial t} - \int_C \vec{\varphi}_w \cdot \vec{\nabla} q \ + \int_{\partial C} F_w q = 0, \quad q \in M,\ C \in \mathcal{T} \tag{12}$$

This equation is the discretized analogue of equation (1). We multiplied by test functions, integrated over the cell and integrated by parts the divergence term. The quantities $\vec{\varphi}_w$ and F_w are defined below.

2. motion equations

Inside each cell we write the discrete analogue of equation (2) :

$$\vec{\varphi}_w = \vec{r} + \vec{f}(S) \tag{13}$$

Inside the cell all quantities in the definition (2) of $\vec{f}$ are well defined and so is $\vec{f}(S)$. However for the capillary contribution $\vec{r}$, which is a gradient, we introduce a weak formulation. We multiply definition (2) of $\vec{r}$ by K^{-1} and by test functions, we integrate over the cell and integrate by parts. We obtain :

$$\int_C K^{-1}\vec{r}\cdot\vec{v} - \int_C \alpha(S)\ \mathrm{div}\vec{v} + \sum_{E\subset\partial C}\int_E \alpha(TS)\ \vec{v}\cdot\vec{n}_C = 0,$$
$$\vec{v}\in\vec{X}_C,\ C\in\mathcal{T}. \qquad (14)$$

Edge equations for edges within a given rock type:

3. saturation continuity

As for the pressure, the saturation must be continuous across the edges, so the two traces of the saturation must be equal: $(TS)^l = (TS)^r$ on all interior edges. On the boundary, if we denote by $\mathcal{E}_{SD}$ the set of boundary edges where Dirichlet conditions are imposed on the saturation, we have also $TS = S_d$ on $E\in\mathcal{E}_{SD}$.

4. conservation equations

We need to define the flow rate of the wetting phase F_w on both sides of each interior edge and we will write that these flow rates must be equal in order to preserve conservation of the wetting phase. The flow rate F_w is the sum of the capillary and convective contributions :

$$F_w = \vec{r}\cdot\vec{n}_C + F.$$

To approximate convection we use Godunov type methods, so F, which approximates $\vec{f}\cdot\vec{n}_C$ is a numerical flux, calculated along the normal to the edge with an exact or approximate Riemann solver [God59, BJ91]. However there are two ways to do so, depending on whether one uses or not the traces TS on the saturation in order to calculate F.

First variant to approximate convection: without using TS
This is the standard way: $F = F(S^l, S^r)$ where S^l and S^r are the traces of the cell values of the saturation in the cells having E in common. Since F is uniquely defined (modulo the sign) on the edge, the necessary continuity of F_w to preserve conservation of the wetting phase, implies that the normal components of $\vec{r}$ must be continuous across the interior edges :

$$F_w^l = F_w^r \Longleftrightarrow \int_E \vec{r}\cdot\vec{n}\,\tau = \int_E \vec{r}\cdot\vec{n}\,\tau,\quad \tau\in N,\ E\in\overset{\circ}{\mathcal{E}}$$

where $\overset{\circ}{\mathcal{E}}$ denotes the set of interior edges. Thus, in this variant, the numerical scheme enforces separate conservation of the diffusion and convection contributions to the Darcy velocity of the wetting phase, while the physics requires that only their sum should be conservative. This will not be the case in the second variant for approximating convection that we present now.

Second variant to approximate convection: using TS
Now we write $F^l = F(S^l, TS^l), F^r = F(TS^r, S^r)$ where S^l, S^r and TS^l, TS^r are the cell values and edge values of the saturation in the two neighbouring cells. Then we write continuity of the water flow rates:

$$F_w^l = F_w^r, \text{ with } F_w^l = \vec{r}^l \cdot \vec{n} + F^l, \quad F_w^r = \vec{r}^r \cdot \vec{n} + F^r.$$

As we see, the numerical scheme enforces only conservation of the wetting phase which is the true physical assumption, and not separate continuity of the normal components of the diffusion and convection contributions to the Darcy velocity $\vec{\varphi}_w$. However such a procedure collapses when there is no capillarity since edge values of the saturation are not then calculated.

For boundary edges, if we are given a flow rate for the wetting phase, say equal to 0 for a closed boundary, on the boundary edges which do not belong to $\mathcal{E}_{SD}$ we simply plug

$$F_w = 0 \,\text{on}\, E \in \mathcal{E} \setminus \mathcal{E}_{SD}, E \subset \partial\Omega$$

into equation (12) and the normal components of $\vec{r}$ must also vanish.

Remark : When $S \in M$ is piecewise linear (or bilinear) the method must be stabilized with limiters [CJ86, VJ93].

Edge equations for edges between two rock types:

3. saturation discontinuity

Using the saturation edge values it is easy to write the discrete analogue of equation (6) for continuity of the capillary pressure:

$$p_c^l(TS^l) = p_c^r(TS^r).$$

4. conservation equations

We proceed in a similar way as we did for edges within a given rock type. The flow rate of the wetting phase F_w is defined on both sides of each interior edge and we write that these flow rates must be equal in order to preserve conservation of the wetting phase. Again when calculating the convective part of the flow rate, two variants may be proposed depending on whether one uses or not the edge values of the saturation.

In the first variant, when edge values are not used, the only difficulty lies in the Riemann solver which is now associated with a Riemann problem with a flux function which changes across the interface between the two rock types. Such Riemann solvers have been studied in [GR91, Jaf96].

6 Conclusion

For structured as well as nonstructured meshes, we presented a general framework for finite volume methods which allows for the use of nonconstant approximation inside the discretization cells. These methods use the approximation spaces provided by mixed-hybrid finite

element methods. Using incompressible two-phase flow as an example, it was shown that this approximation technique is very convenient for problems with heterogeneities, and that even in the homogeneous case, it may lead to new schemes.

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