

## A NEW NONCONFORMING FINITE ELEMENT METHOD FOR THE COMPUTATION OF ELECTROMAGNETIC GUIDED WAVES I: MATHEMATICAL ANALYSIS\*

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**Abstract.** We are concerned with the problem of computing electromagnetic guided waves in a closed, inhomogeneous, cylindrical wave guide. These waves correspond to solutions of an eigenvalue problem, and classical methods produce, in addition to approximations to the solutions, spurious modes which are particularly troublesome because they correspond to nonzero approximations of the 0 eigenvalue. A nonconforming finite element method for the calculation of guided waves without spurious modes is introduced and analyzed.

**Key words.** electromagnetic guided waves, nonconforming finite elements, mixed finite elements, eigenvalue problems

**AMS subject classifications.** 65N15, 65N30, 65N25, 78A50

**1. Introduction.** This article is concerned with the numerical calculation of electromagnetic guided waves in a closed, inhomogeneous, cylindrical wave guide. The wave guide (see Fig. 1) will be identified with a right cylinder  $\tilde{\Omega}$  in  $\mathbb{R}^3$  of cross section  $\Omega$ ;  $\tilde{\Omega} = \Omega \times \mathbb{R}$ . As the wave guide is closed, we assume that  $\Omega$  is a bounded domain in  $\mathbb{R}^2$  with boundary  $\Gamma$  admitting a unit outward-pointing normal vector field  $\mathbf{n}$  defined almost everywhere. The boundary of  $\tilde{\Omega}$  is then  $\tilde{\Gamma} = \Gamma \times \mathbb{R}$ , and  $\tilde{\mathbf{n}} = (\mathbf{n}, 0)$  is a unit outward-pointing normal vector field on  $\tilde{\Omega}$ . In general, we shall denote a point in  $\mathbb{R}^2$  by  $\mathbf{x} = (x_1, x_2)$  and a point in  $\mathbb{R}^3$  by  $\tilde{\mathbf{x}} = (x_1, x_2, x_3) = (\mathbf{x}, x_3)$ .

Let  $\mathbf{E}$  denote the electric field and  $\mathbf{H}$  denote the magnetic field in  $\tilde{\Omega}$ . The conductive properties of  $\tilde{\Omega}$  are described by  $\epsilon$ , the dielectric permittivity, and  $\mu$ , the magnetic permeability. These are real scalar fields on  $\tilde{\Omega}$  that we assume to be independent of  $x_3$  and bounded above and away from 0. By an electromagnetic wave in  $\tilde{\Omega}$ , we shall mean a solution of Maxwell's equations in second-order form

$$(1.1) \quad \nu \frac{\partial^2 \mathbf{U}}{\partial t^2} + \text{Curl}(\eta \text{Curl} \mathbf{U}) = 0,$$

where either

$$(1.2) \quad \mathbf{U} = \mathbf{E}, \quad \nu = \epsilon \text{ and } \eta = 1/\mu,$$

or

$$(1.3) \quad \mathbf{U} = \mathbf{H}, \quad \nu = \mu \text{ and } \eta = 1/\epsilon.$$

We have

$$(1.4) \quad \begin{aligned} 0 < \nu_{\min} \leq \nu \leq \nu_{\max} < \infty, \\ 0 < \eta_{\min} \leq \eta \leq \eta_{\max} < \infty. \end{aligned}$$

An electromagnetic guided wave is a solution  $\mathbf{U}$  of (1.1) of the following form:

$$(1.5) \quad \mathbf{U}(x_1, x_2, x_3) = \tilde{\mathbf{u}}(x_1, x_2) e^{i(\omega t - \beta x_3)},$$

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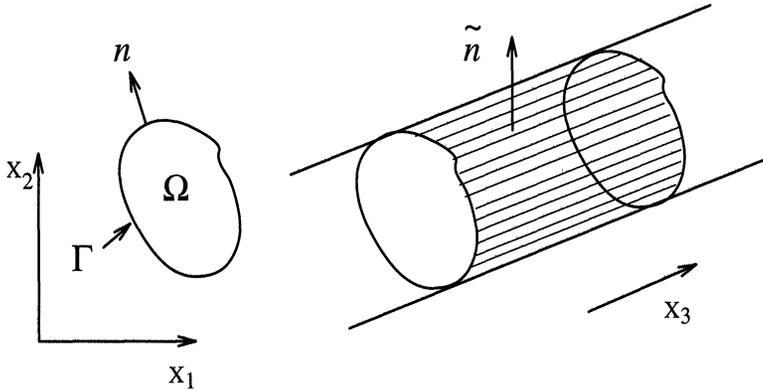


FIG. 1. The wave guide.

where  $\tilde{u}$ , the amplitude function, is a function of finite energy in  $\Omega$  in a sense to be made precise below, and where  $\omega$ , the frequency, and  $\beta$ , the wave number, are positive, real constants. Thus  $U$  is a harmonic plane wave propagating with no distortion in the positive  $x_3$ -direction with phase velocity  $\omega/\beta$ .

We impose the boundary condition

$$(1.6) \quad U \times \tilde{n} = 0 \quad \text{on } \tilde{\Gamma}.$$

Such a boundary condition is physically relevant since when  $U = E$ , for instance, this corresponds to a perfectly conducting boundary.

By introducing the operator  $c\tilde{u}r_l\beta$

$$c\tilde{u}r_l\beta\tilde{u} = \begin{bmatrix} \frac{\partial u_3}{\partial x_2} + i\beta u_2 \\ -i\beta u_1 - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{bmatrix},$$

we may write an equation on  $\Omega$  and a boundary condition on  $\Gamma$  that will be satisfied by the amplitude function  $\tilde{u} = (u, u_3)$  of a function  $U$  of the form (1.5) if and only if  $U$  is a solution of the equation (1.1) on  $\tilde{\Omega}$  and satisfies the boundary condition (1.6) on  $\tilde{\Gamma}$ :

$$(1.7) \quad c\tilde{u}r_l\beta(\eta c\tilde{u}r_l\beta\tilde{u}) = \nu\omega^2\tilde{u},$$

$$(1.8) \quad u \times n = 0, \quad u_3 = 0 \quad \text{on } \tilde{\Gamma}.$$

Thus the problem of finding a solution  $U$  of (1.1), (1.6) of the form (1.5) is reduced to that of finding a solution  $\tilde{u} = (u, u_3)$  of (1.7), (1.8). In terms of the operator  $c\tilde{u}r_l\beta$ , our finite energy condition can be made precise:

$$(1.9) \quad \int_{\Omega} |\tilde{u}(x)|^2 dx + \int_{\Omega} |c\tilde{u}r_l\beta\tilde{u}(x)|^2 dx < \infty.$$

However, for an arbitrary real pair  $(\omega, \beta)$ , there does not necessarily exist a solution of (1.7), (1.8). For a given wave number  $\beta$ , (1.7), (1.8) defines a selfadjoint eigenvalue problem. It is known, see §2, that an infinite number of solutions exists.

*Remark 1.1.* An alternative approach, perhaps physically more relevant, is, for a given frequency  $\omega$ , to find pairs  $(\beta, \tilde{\mathbf{u}})$  with  $\tilde{\mathbf{u}}$  satisfying (1.7), (1.8) for the wave number  $\beta$ . This approach yields a more difficult mathematical problem—a nonselfadjoint eigenvalue problem with a finite number of solutions. Other solutions corresponding to complex  $\beta$ 's are called evanescent modes. We hope to address this problem in a future work, but here we shall constrain our attention to the selfadjoint problem.

*Remark 1.2.* One may also consider the propagation of guided waves in an open wave guide, i.e., where one does not have a boundary condition on the boundary  $\tilde{\Gamma}$  of the wave guide that allows one to ignore the exterior domain  $\mathbb{R}^3 - \tilde{\Omega}$ . Again this is the physically more interesting situation as it is indeed the case with optic fibers. Then, however, one obtains a problem in all  $\mathbb{R}^3$ , instead of in  $\tilde{\Omega}$ , or by the introduction of the operator  $\tilde{curl}_\beta$  in all  $\mathbb{R}^2$ , the coefficients  $\nu$  and  $\eta$ , i.e.,  $\epsilon$  and  $\mu$ , considered to be constant outside  $\tilde{\Omega}$ . One must then introduce artificial boundary conditions at a distance not too far from  $\tilde{\Gamma}$  to obtain a tractable problem. Work on this extension of the ideas in this article is underway, cf. [14], [21], but shall not be further broached herein.

There have been several previous approaches to the eigenvalue problem (1.7), (1.8). First, for the case of constant coefficients (a homogeneous wave guide) a solution  $\mathbf{U}$  is divergence-free, and a solution of (1.1), (1.6) must be a solution of

$$(1.10) \quad \left(\frac{\nu}{\eta}\right) \frac{\partial^2 \mathbf{U}}{\partial t^2} - \Delta \mathbf{U} = 0 \quad \text{in } \tilde{\Omega},$$

$$(1.11) \quad \mathbf{U} \times \tilde{\mathbf{n}} = 0 \quad \text{and} \quad \text{div} \mathbf{U} = 0 \quad \text{on } \tilde{\Gamma}.$$

Thus a classical approach has been to solve the eigenvalue problem corresponding to (1.10), (1.11) which is an eigenvalue problem for the Laplacian. However we note that solutions of (1.10), (1.11) in the form (1.5) need not be divergence-free in  $\tilde{\Omega}$  and thus need not satisfy (1.1). Extraneous solutions are obtained.

For the nonhomogeneous wave guide,  $\nu \mathbf{U}$  is still divergence-free and  $\omega = 0$  is an eigenvalue with infinite-dimensional eigenspace. However, solutions in this eigenspace are not of interest physically as they do not propagate. Ideally one would like to base a finite element method on a variational formulation for which the trial functions are  $\nu$ -divergence-free in the sense that  $\text{div} \nu \tilde{\mathbf{u}} = 0$ . But it is not obvious how to construct a conforming approximation to the space of  $\nu$ -divergence-free trial functions.

Penalty methods have been introduced; cf. [4], [16], and [22]. These methods add a penalty term, dependent on a parameter  $\delta$ , to the bilinear form of the variational problem. One no longer has 0 as an eigenvalue, but other nonphysical eigenvalues are introduced. Just where in the spectrum these eigenvalues occur with respect to the smallest physical eigenvalues, those of primary interest to the engineer, can be manipulated by choice of the parameter  $\delta$ . See [5] for a nice explanation of this point.

If one simply ignores the  $\nu$ -divergence-free condition in the variational formulation and uses classical Lagrange finite elements, one obtains solutions corresponding to small non-physical real eigenvalues. These solutions, known in the literature as spurious solutions, may be thought of as approximations of  $\nu$ -divergence-free solutions, but they can be difficult to distinguish from solutions corresponding to small strictly positive, physical eigenvalues. Recently Bermúdez and Pedreira [5] proposed the use of the Nedelec finite elements [19] instead of Lagrange elements to approximate the amplitude function. They obtain a much better approximation in which spurious modes are eliminated. However, 0 is an eigenvalue of infinite multiplicity for the discrete problem and this can lead to some computational difficulties (as the Lanczos method can be very sensitive to the shift technique).

The approach we develop here is to define a nonconforming approximation to the space of  $\nu$ -divergence-free, trial functions. We are able to prove convergence results, and numerical experiments confirm the efficiency of the method; cf. [15].

In the remainder of this section we introduce a change of unknown function that allows us to work with real-valued functions instead of complex-valued functions. This represents a significant advantage with respect to computational efficiency. In §2 we give a mathematically rigorous formulation of the problem, and some results concerning the spectral theory for the continuous problem are recalled. We construct in §3 an approximation space in which we shall approximate the solution of the eigenvalue problem. The nonconforming method is defined in §4 and theoretical results concerning the accuracy of the approximation are given. An appendix detailing the demonstration of a generalization of a theorem of Kikuchi is included. Numerical results will be reported in Part II of this paper [15].

Our computations are greatly simplified by making the change of unknown function  $(u_1, u_2, u_3) \mapsto (u_1, u_2, -\imath u_3)$ , which we do with no change of notation, and by redefining the operator  $\text{curl}_\beta$ :

$$\text{curl}_\beta \tilde{\mathbf{u}} = \begin{bmatrix} \frac{\partial u_3}{\partial x_2} & -\beta u_2 \\ \beta u_1 & -\frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} & -\frac{\partial u_1}{\partial x_2} \end{bmatrix}.$$

If

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\imath \end{bmatrix},$$

then

$$\text{curl}_\beta = \imath \mathcal{T} \tilde{\text{curl}}_\beta \mathcal{T}.$$

Equation (1.7) becomes

$$(1.12) \quad \text{curl}_\beta^*(\eta \text{curl}_\beta \tilde{\mathbf{u}}) = \nu \omega^2 \tilde{\mathbf{u}} \quad \text{in } \Omega,$$

where  $\text{curl}_\beta^* = \text{curl}_{-\beta} = -\imath \mathcal{T}^* \tilde{\text{curl}}_\beta \mathcal{T}^*$  is the formal adjoint of  $\text{curl}_\beta$ . This formulation permits us to work with real vector-valued functions  $\tilde{\mathbf{u}}$ .

We shall also make use of the operators  $\text{div}_\beta$  and  $\text{grad}_\beta$ :

$$\begin{aligned} \text{div}_\beta \tilde{\mathbf{u}} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} - \beta u_3, \\ \text{grad}_\beta \varphi &= \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \beta \varphi \right). \end{aligned}$$

Further, in the remainder of this article,  $\text{curl}$  and  $\text{div}$  will denote operators on two-dimensional vector fields to scalar fields, and  $\text{curl}^*$  and  $\text{grad}$  will denote operators on scalar fields to

two-dimensional vector fields:

$$\begin{aligned} \operatorname{curl} \mathbf{u} &= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \\ \operatorname{div} \mathbf{u} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \\ \operatorname{grad} \varphi &= \left( \frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2} \right), \\ \operatorname{curl}^* \varphi &= \left( \frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right). \end{aligned}$$

Let us recall the definitions of the following function spaces:

$$\begin{aligned} \mathcal{H}(\operatorname{curl}, \Omega) &= \{\mathbf{u} \in \mathcal{L}^2(\Omega)^2 : \operatorname{curl} \mathbf{u} \in \mathcal{L}^2(\Omega)\}, \\ \mathcal{H}_0(\operatorname{curl}, \Omega) &= \{\mathbf{u} \in \mathcal{L}^2(\Omega)^2 : \operatorname{curl} \mathbf{u} \in \mathcal{L}^2(\Omega) \text{ and } \mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma\}, \end{aligned}$$

where  $\mathbf{u} \times \mathbf{n}$  is defined as an element of  $\mathcal{H}^{-1/2}(\Gamma)$  by Green’s formula

$$\langle \mathbf{u} \times \mathbf{n}, \gamma \rangle = \int_{\Omega} (\mathbf{u} \cdot \operatorname{curl}^* \varphi_{\gamma} - \operatorname{curl} \mathbf{u} \cdot \varphi_{\gamma}) dx \quad \text{for each } \gamma \in \mathcal{H}^{1/2}(\Gamma),$$

where  $\varphi_{\gamma}$  is an element of  $\mathcal{H}^1(\Omega)$  having trace  $\gamma$  on  $\Gamma$  and where  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $\mathcal{H}^{-1/2}(\Gamma)$  and  $\mathcal{H}^{1/2}(\Gamma)$ . Next, define

$$\begin{aligned} \mathcal{H}(\operatorname{curl}_{\beta}, \Omega) &= \{\tilde{\mathbf{u}} \in \mathcal{L}^2(\Omega)^3 : \operatorname{curl}_{\beta} \tilde{\mathbf{u}} \in \mathcal{L}^2(\Omega)^3\}, \\ \mathcal{H}_0(\operatorname{curl}_{\beta}, \Omega) &= \{\tilde{\mathbf{u}} \in \mathcal{L}^2(\Omega)^3 : \operatorname{curl}_{\beta} \tilde{\mathbf{u}} \in \mathcal{L}^2(\Omega)^3 \text{ and } \tilde{\mathbf{u}} \times \tilde{\mathbf{n}} = 0 \text{ on } \Gamma\}. \end{aligned}$$

Note that for  $\tilde{\mathbf{u}} = (\mathbf{u}, u_3) = (u_1, u_2, u_3)$  to be in the domain of  $\operatorname{curl}_{\beta}$ , more regularity is required of  $u_3$  than of  $u_1$  and  $u_2$ ; more precisely,  $\mathbf{u} \in \mathcal{H}(\operatorname{curl}, \Omega)$  and  $u_3 \in \mathcal{H}^1(\Omega)$ . We have the following equalities of vector spaces:

$$\mathcal{H}(\operatorname{curl}_{\beta}, \Omega) = \mathcal{H}(\operatorname{curl}, \Omega) \times \mathcal{H}^1(\Omega).$$

Thus we may identify  $\tilde{\mathbf{u}} \times \tilde{\mathbf{n}}$  with the vector  $(u_3, \mathbf{u} \times \mathbf{n}) \in \mathcal{H}^{1/2}(\Gamma) \times \mathcal{H}^{-1/2}(\Gamma)$  and write

$$\mathcal{H}_0(\operatorname{curl}_{\beta}, \Omega) = \mathcal{H}_0(\operatorname{curl}, \Omega) \times \mathcal{H}_0^1(\Omega).$$

The relevant Green formula is

$$\begin{aligned} (1.13) \quad & \langle \mathbf{u} \times \mathbf{n}, v_3 \rangle - \langle \mathbf{v} \times \mathbf{n}, u_3 \rangle \\ &= \int_{\Omega} (\tilde{\mathbf{u}} \cdot \operatorname{curl}_{\beta}^* \tilde{\mathbf{v}} - \operatorname{curl}_{\beta} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{v}}) dx \\ & \quad \text{for each } \tilde{\mathbf{v}} \in \mathcal{H}(\operatorname{curl}_{\beta}, \Omega). \end{aligned}$$

The following analogues of the corresponding relations for the classical operators  $\operatorname{curl}$ ,  $\operatorname{div}$ , and  $\operatorname{grad}$  hold:

$$\begin{aligned} \operatorname{div}_{\beta} \operatorname{curl}_{\beta}^* \tilde{\mathbf{u}} &= 0 \quad \text{for } \tilde{\mathbf{u}} \in \mathcal{H}(\operatorname{curl}_{\beta}, \Omega), \\ \operatorname{curl}_{\beta} \operatorname{grad}_{\beta} \varphi &= 0 \quad \text{for } \varphi \in \mathcal{H}^1(\Omega), \end{aligned}$$

and

$$\operatorname{div}_\beta^* = -\operatorname{grad}_\beta.$$

Finally, we point out that if  $\tilde{\mathbf{u}} \in \mathcal{H}(\operatorname{curl}_\beta, \Omega)$  and  $\operatorname{curl}_\beta \tilde{\mathbf{u}} = 0$ , then there is an element  $\varphi \in \mathcal{H}^1(\Omega)$  with  $\tilde{\mathbf{u}} = \operatorname{grad}_\beta \varphi$ , namely  $\varphi = \frac{1}{\beta} u_3$ , and that  $\varphi \in \mathcal{H}_0^1(\Omega)$  if and only if  $\tilde{\mathbf{u}} \in \mathcal{H}_0(\operatorname{curl}_\beta, \Omega)$ .

**2. Mathematical formulation of the problem.** Our objective here is to give a more mathematically precise formulation of the physical problem presented in §1 and to recall a few results that we shall need in the construction of our numerical method.

We assume that the cross section  $\Omega \subset \mathbb{R}^2$  has the local cone property; see [1]. We point out that this assumption is not restrictive from the point of view of physical applications since it does not exclude domains with corners, not even nonconvex ones.

To give a variational formulation of the problem we shall need to define some function spaces. Let  $\mathcal{H}$  denote the Hilbert space  $\mathcal{L}^2(\Omega)^3$  equipped with the weighted inner product

$$(2.1) \quad (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu = (\nu \tilde{\mathbf{u}}, \tilde{\mathbf{v}})_{\mathcal{L}^2(\Omega)^3} = \int_\Omega \nu \tilde{\mathbf{u}} \tilde{\mathbf{v}} \, dx,$$

and let  $\mathcal{H}_\beta$  be the following  $\beta$ -dependent closed subspace:

$$(2.2) \quad \mathcal{H}_\beta = \{\tilde{\mathbf{u}} \in \mathcal{H} : \operatorname{div}_\beta(\nu \tilde{\mathbf{u}}) = 0\}.$$

Now as  $\mathcal{H}_\beta$  is closed in  $\mathcal{H}$  it has an orthogonal complement  $\mathcal{H}_\beta^\perp$  and the decomposition

$$(2.3) \quad \mathcal{H} = \mathcal{H}_\beta \oplus \mathcal{H}_\beta^\perp,$$

where

$$\mathcal{H}_\beta^\perp = \{\tilde{\mathbf{u}} \in \mathcal{H}; (\tilde{\mathbf{v}}, \tilde{\mathbf{u}})_\nu = 0 \text{ for each } \tilde{\mathbf{v}} \in \mathcal{H}_\beta\}$$

defines a Helmholtz-type decomposition of a three-dimensional vector field. In fact we have the following lemma.

LEMMA 2.1. *If  $\tilde{\mathbf{v}} \in \mathcal{H}$ , then  $\tilde{\mathbf{v}} \in \mathcal{H}_\beta^\perp$  if and only if  $\operatorname{curl}_\beta \tilde{\mathbf{v}} = 0$  and  $\tilde{\mathbf{v}} \times \tilde{\mathbf{n}} = 0$ .*

*Proof.* Suppose that  $\tilde{\mathbf{v}} \in \mathcal{H}_\beta^\perp$ . To show that  $\operatorname{curl}_\beta \tilde{\mathbf{v}} = 0$  in the distributional sense, it is sufficient to show that

$$(\operatorname{curl}_\beta^* \tilde{\varphi}, \tilde{\mathbf{v}})_{\mathcal{L}^2(\Omega)^3} = 0 \quad \text{for each } \tilde{\varphi} \in \mathcal{D}(\Omega)^3.$$

So let  $\tilde{\varphi} \in \mathcal{D}(\Omega)^3$ . Then  $\frac{1}{\nu} \operatorname{curl}_\beta^* \tilde{\varphi} \in \mathcal{H}$ , and

$$\operatorname{div}_\beta \left( \nu \left( \frac{1}{\nu} \operatorname{curl}_\beta^* \tilde{\varphi} \right) \right) = 0;$$

i.e.,  $\frac{1}{\nu} \operatorname{curl}_\beta^* \tilde{\varphi} \in \mathcal{H}_\beta$ . Thus

$$\left( \frac{1}{\nu} \operatorname{curl}_\beta^* \tilde{\varphi}, \tilde{\mathbf{v}} \right)_\nu = 0,$$

or, in other words,

$$(\operatorname{curl}_\beta^* \tilde{\varphi}, \tilde{\mathbf{v}})_{\mathcal{L}^2(\Omega)^3} = 0.$$

To show that  $\tilde{\mathbf{v}} \times \tilde{\mathbf{n}} = 0$  on  $\Gamma$ , we test  $\text{curl}_\beta \tilde{\mathbf{v}}$  against a function that does not necessarily vanish on  $\Gamma$ . Let  $\tilde{\varphi} \in \mathcal{H}(\text{curl}_\beta, \Omega)$ . By the density of  $\mathcal{D}(\Omega)^3$  in  $\mathcal{H}(\text{curl}_\beta, \Omega)$  we have

$$(\text{curl}_\beta^* \tilde{\varphi}, \tilde{\mathbf{v}})_{\mathcal{L}^2(\Omega)^3} = 0,$$

and by (1.13) we have that

$$(\tilde{\varphi}, \text{curl}_\beta \tilde{\mathbf{v}})_{\mathcal{L}^2(\Omega)^3} + \langle \mathbf{v} \times \mathbf{n}, \varphi_3 \rangle - \langle \varphi \times \mathbf{n}, \mathbf{v}_3 \rangle = 0.$$

However, as we have just seen,  $\text{curl}_\beta \tilde{\mathbf{v}} = 0$  so

$$\langle \mathbf{v} \times \mathbf{n}, \varphi_3 \rangle - \langle \varphi \times \mathbf{n}, \mathbf{v}_3 \rangle = 0 \quad \text{for each } \tilde{\varphi} \in \mathcal{H}(\text{curl}_\beta, \Omega).$$

Since the map  $\tilde{\varphi} \mapsto (\varphi \times \mathbf{n}, \varphi_3)$  is surjective from  $\mathcal{H}(\text{curl}_\beta, \Omega)$  to  $\mathcal{H}^{-1/2}(\Gamma) \times \mathcal{H}^{1/2}(\Gamma)$ , we can conclude that  $\tilde{\mathbf{v}} \times \tilde{\mathbf{n}} = 0$ .

Now suppose that  $\tilde{\mathbf{v}} \in \mathcal{H}$  and that  $\text{curl}_\beta \tilde{\mathbf{v}} = 0$  and  $\tilde{\mathbf{v}} \times \tilde{\mathbf{n}} = 0$ . Then there exists  $\varphi \in \mathcal{H}_0^1(\Omega)$  with  $\tilde{\mathbf{v}} = \text{grad}_\beta \varphi$ , and if  $\tilde{\mathbf{w}} \in \mathcal{H}_\beta$ , we have

$$\begin{aligned} (\tilde{\mathbf{v}}, \tilde{\mathbf{w}})_\nu &= (\nu \text{grad}_\beta \varphi, \tilde{\mathbf{w}}) \\ &= -(\varphi, \text{div}_\beta(\nu \tilde{\mathbf{w}})) \\ &= 0, \end{aligned}$$

and  $\tilde{\mathbf{v}} \in \mathcal{H}_\beta^\perp$ .  $\square$

With this lemma and the definition of  $\mathcal{H}_\beta$ , we obtain the following characterizations of  $\mathcal{H}_\beta$  and  $\mathcal{H}_\beta^\perp$ :

$$\begin{aligned} \mathcal{H}_\beta &= \left\{ \tilde{\mathbf{u}} = (\mathbf{u}, u_3) \in \mathcal{H} : \mathbf{u} \in \mathcal{H}(\text{div}_\nu, \Omega) \text{ and } u_3 = \frac{1}{\nu\beta} \text{div}(\nu \mathbf{u}) \right\}, \\ \mathcal{H}_\beta^\perp &= \left\{ \tilde{\mathbf{u}} = (\mathbf{u}, u_3) \in \mathcal{H} : u_3 \in \mathcal{H}_0^1(\Omega) \text{ and } \mathbf{u} = \frac{1}{\beta} \text{grad } u_3 \right\}, \end{aligned}$$

where

$$\mathcal{H}(\text{div}_\nu, \Omega) = \{\mathbf{u} \in \mathcal{L}^2(\Omega)^2 : \text{div}(\nu \mathbf{u}) \in \mathcal{L}^2(\Omega)\}.$$

In other words,  $\mathcal{H}_\beta$  is isomorphic to the space of two-dimensional vector fields  $\mathcal{H}(\text{div}_\nu, \Omega)$  while  $\mathcal{H}_\beta^\perp$  is isomorphic to the space of scalar fields  $\mathcal{H}_0^1(\Omega)$  under the following isomorphisms:

$$(2.4) \quad \begin{aligned} \mathcal{J} : \mathcal{H}(\text{div}_\nu, \Omega) &\longrightarrow \mathcal{H}_\beta, \\ \mathbf{u} &\longmapsto \left( \mathbf{u}, \frac{1}{\nu\beta} \text{div}(\nu \mathbf{u}) \right) \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} \frac{1}{\beta} \text{grad}_\beta : \mathcal{H}_0^1(\Omega) &\longrightarrow \mathcal{H}_\beta^\perp, \\ \varphi &\longmapsto \left( \frac{1}{\beta} \text{grad } \varphi, \varphi \right). \end{aligned}$$

We shall also need the space

$$(2.6) \quad \mathcal{W} = \mathcal{H}_0(\text{curl}, \Omega) \times \mathcal{H}_0^1(\Omega) = \mathcal{H}_0(\text{curl}_\beta, \Omega)$$

endowed with the norm

$$(2.7) \quad \|\tilde{\mathbf{v}}\|_{\mathcal{W}}^2 = \|\mathbf{v}\|_{\mathcal{L}^2(\Omega)^2}^2 + \|\operatorname{curl} \mathbf{v}\|_{\mathcal{L}^2(\Omega)}^2 + \|v_3\|_{\mathcal{H}^1(\Omega)}^2.$$

One easily checks that an equivalent norm is given by

$$(2.8) \quad \|\tilde{\mathbf{v}}\|_{\mathcal{W}}^2 = \|\tilde{\mathbf{v}}\|_{\mathcal{L}^2(\Omega)^3}^2 + \|\operatorname{curl}_\beta \tilde{\mathbf{v}}\|_{\mathcal{L}^2(\Omega)^3}^2.$$

The space that we shall approximate numerically is the following  $\beta$ -dependent closed subspace of  $\mathcal{W}$ :

$$(2.9) \quad \mathcal{W}_\beta = \mathcal{W} \cap \mathcal{H}_\beta,$$

with the norm inherited from  $\mathcal{W}$ . Of course  $\mathcal{W}$  also admits the decomposition

$$\mathcal{W} = \mathcal{W}_\beta \oplus \mathcal{W}_\beta^\perp,$$

where again orthogonality is with respect to the inner product  $(\cdot, \cdot)_v$ . As  $\mathcal{H}_\beta^\perp \subset \mathcal{W}$ , we have  $\mathcal{W}_\beta^\perp = \mathcal{H}_\beta^\perp$ . Further, the isomorphism  $\mathcal{J}$  clearly restricts to an isomorphism

$$\mathcal{J} : \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}_0^1(\operatorname{div}_v, \Omega) \longrightarrow \mathcal{W}_\beta,$$

where the space  $\mathcal{H}_0^1(\operatorname{div}_v, \Omega)$  is defined as follows:

$$\mathcal{H}_0^1(\operatorname{div}_v, \Omega) = \{\mathbf{u} \in \mathcal{L}^2(\Omega)^2 : \operatorname{div}(v\mathbf{u}) \in \mathcal{H}_0^1(\Omega)\}.$$

The variational formulation of (1.7), (1.8) may be expressed in terms of the following  $\beta$ -dependent bilinear form defined on  $\mathcal{W}$ :

$$(2.10) \quad a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_\Omega \eta \operatorname{curl}_\beta \tilde{\mathbf{u}} \operatorname{curl}_\beta \tilde{\mathbf{v}} \, dx, \quad \tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathcal{W}.$$

Then (1.7), (1.8) may be expressed as

$$(2.11) \quad \begin{aligned} &\text{find } \omega \in \mathbb{R}^+, \tilde{\mathbf{u}} \in \mathcal{W} \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \omega^2 (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_v \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}. \end{aligned}$$

The form  $a(\beta, \cdot, \cdot)$  is symmetric, but it is not  $\mathcal{W}$ -elliptic as  $a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0$ , and thus  $(0, \tilde{\mathbf{u}})$  is a solution of (2.11), whenever  $\operatorname{curl}_\beta \tilde{\mathbf{u}} = 0$ . In fact it is clear that  $a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}) = 0$  if and only if  $\operatorname{curl}_\beta \tilde{\mathbf{u}} = 0$ . Thus we have as an immediate consequence of Lemma 2.1 that  $a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}) > 0$  for each nonzero  $\tilde{\mathbf{u}} \in \mathcal{W}_\beta$ . Furthermore, we are not interested in solutions of (2.11) with  $\omega = 0$ . Such solutions do not propagate, and their numerical approximation poses considerable difficulty; cf. [5]. Physical solutions associated with  $\omega > 0$  will thus belong to  $\mathcal{W}_\beta$ . This leads us to consider the variational problem

$$(2.12) \quad \begin{aligned} &\text{find } \omega \in \mathbb{R}^+, \tilde{\mathbf{u}} \in \mathcal{W}_\beta \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \omega^2 (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_v \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}_\beta \end{aligned}$$

(which is clearly equivalent to the problem

$$\begin{aligned} &\text{find } \omega \in \mathbb{R}^+, \tilde{\mathbf{u}} \in \mathcal{W}_\beta \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \omega^2 (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_v \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}, \end{aligned}$$

since  $a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$  and  $(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu$  both vanish when  $\tilde{\mathbf{v}} \in \mathcal{W}_\beta^\perp$  and  $\tilde{\mathbf{u}} \in \mathcal{W}_\beta$ ) and we can show the following result.

LEMMA 2.2. *The bilinear form  $a(\beta, \cdot, \cdot)$  is  $\mathcal{W}_\beta$ -elliptic.*

*Proof.* Let  $\tilde{\mathbf{u}}$  belong to  $\mathcal{W}_\beta$ . Then

$$\begin{aligned} a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}) &= \int_{\Omega} \eta \operatorname{curl}_\beta \tilde{\mathbf{u}} \operatorname{curl}_\beta \tilde{\mathbf{u}} \, dx \\ &= \int_{\Omega} \eta (\operatorname{grad} u_3 - \beta \mathbf{u}) \cdot (\operatorname{grad} u_3 - \beta \mathbf{u}) \, dx \\ &\quad + \int_{\Omega} \eta \operatorname{curl} \mathbf{u} \operatorname{curl} \mathbf{u} \, dx \\ &\geq \eta_{\min} \|\operatorname{curl} \mathbf{u}\|_{\mathcal{L}^2(\Omega)}^2 + \frac{\eta_{\min}}{\nu_{\max}} \|\operatorname{grad} u_3\|_{\mathcal{L}^2(\Omega)_\nu}^2 + \frac{\eta_{\min}}{\nu_{\max}} \beta^2 \|\mathbf{u}\|_{\mathcal{L}^2(\Omega)}^2 \\ &\quad - 2 \frac{\eta_{\min}}{\nu_{\max}} \beta \int_{\Omega} \nu \mathbf{u} \operatorname{grad} u_3 \, dx, \end{aligned}$$

but we recall

$$\begin{aligned} -\frac{\eta_{\min}}{\nu_{\max}} \beta \int_{\Omega} \nu \mathbf{u} \operatorname{grad} u_3 \, dx &= \frac{\eta_{\min}}{\nu_{\max}} \beta \int_{\Omega} u_3 \operatorname{div} \nu \mathbf{u} \, dx \\ (2.13) \qquad \qquad \qquad &= \frac{\eta_{\min}}{\nu_{\max}} \beta^2 \int_{\Omega} \nu u_3^2 \, dx \\ &= \frac{\eta_{\min}}{\nu_{\max}} \beta^2 \|u_3\|_{\mathcal{L}^2(\Omega)_\nu}^2, \end{aligned}$$

and the lemma follows.  $\square$

To describe the spectral properties of the problem we introduce the unbounded operator  $\mathcal{A}_\beta$  defined on  $\mathcal{H}_\beta$  to  $\mathcal{H}$  as follows:

$$\begin{aligned} (2.14) \qquad \mathcal{D}(\mathcal{A}_\beta) &= \left\{ \tilde{\mathbf{u}} \in \mathcal{W}_\beta; \frac{1}{\nu} \operatorname{curl}_\beta^*(\eta \operatorname{curl}_\beta \tilde{\mathbf{u}}) \in \mathcal{H} \right\}, \\ \mathcal{A}_\beta(\tilde{\mathbf{u}}) &= \frac{1}{\nu} \operatorname{curl}_\beta^*(\eta \operatorname{curl}_\beta \tilde{\mathbf{u}}) \quad \text{for each } \tilde{\mathbf{u}} \in \mathcal{D}(\mathcal{A}_\beta). \end{aligned}$$

We clearly have the following relationship between the operator  $\mathcal{A}_\beta$  and the bilinear form  $a(\beta, \cdot, \cdot)$ :

$$(2.15) \qquad (\mathcal{A}_\beta(\tilde{\mathbf{u}}), \tilde{\mathbf{v}})_\nu = a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \quad \text{for each } (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \in \mathcal{D}(\mathcal{A}_\beta) \times \mathcal{W}.$$

For Theorem 2.1 we shall need the following lemma.

LEMMA 2.3. *The operator  $\mathcal{A}_\beta + I$  is an isomorphism from  $\mathcal{D}(\mathcal{A}_\beta)$  onto  $\mathcal{H}_\beta$ .*

*Proof.* By Lemma 2.2, for  $f \in \mathcal{H}_\beta$ , the problem

$$(2.16) \qquad \begin{aligned} &\text{find } \tilde{\mathbf{u}} \in \mathcal{W}_\beta \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu = (f, \tilde{\mathbf{v}})_\nu \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}_\beta \end{aligned}$$

admits a unique solution  $\tilde{\mathbf{u}}_f$ . We cannot immediately conclude, however, that  $\tilde{\mathbf{u}}_f$  is the unique solution of

$$(2.17) \qquad \begin{aligned} &\tilde{\mathbf{u}} \in \mathcal{D}(\mathcal{A}_\beta), \\ &(\mathcal{A}_\beta + I)\tilde{\mathbf{u}} = f \end{aligned}$$

because  $\mathcal{D}(\Omega)^3$  is not contained in  $\mathcal{W}_\beta$ . However, we have only to remark that if  $\tilde{\mathbf{v}} \in \mathcal{W}_\beta^\perp$ , by Lemma 2.1

$$\begin{aligned} a(\beta, \tilde{\mathbf{u}}_f, \tilde{\mathbf{v}}) &= 0, \\ (f, \tilde{\mathbf{v}})_\nu &= 0 \end{aligned}$$

since  $f \in \mathcal{H}_\beta$ , and

$$(\tilde{\mathbf{u}}_f, \tilde{\mathbf{v}})_\nu = 0$$

since  $\tilde{\mathbf{u}}_f \in \mathcal{H}_\beta$ . Thus the equation in (2.16) holds also for  $\tilde{\mathbf{v}} \in \mathcal{W}_\beta^\perp$  and since  $\mathcal{W} = \mathcal{W}_\beta \oplus \mathcal{W}_\beta^\perp$  does contain  $\mathcal{D}(\Omega)^3$ , it now follows that  $\mathbf{u}_f$  is the solution of (2.17).  $\square$

**THEOREM 2.1.** *With the given hypothesis concerning  $\nu, \eta$ , and  $\Omega$ , the operator  $\mathcal{A}_\beta$  is a positive-definite, selfadjoint operator with compact resolvent.*

*Proof.* Positive definiteness is shown in Lemma 2.2. The symmetry of  $\mathcal{A}_\beta$  is implied by that of  $a(\beta, \cdot, \cdot)$ , so selfadjointness is a consequence of Lemma 2.3.

That  $\mathcal{A}_\beta$  has compact resolvent follows from the fundamental fact that the inclusion of  $\mathcal{W}_\beta$  into  $\mathcal{H}_\beta$  is compact. To see this we note that  $\mathcal{H}_0^1(\Omega)$  is compactly included in  $\mathcal{L}^2(\Omega)$  and that under the hypothesis that  $\Omega$  satisfies the restricted cone property,  $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div}_\nu, \Omega)$  is compactly embedded in  $\mathcal{L}^2(\Omega)_\nu^2$ ; cf. [26].  $\square$

*Remark 2.1.* An important part of the proof of Theorem 2.1 is the compactness of the imbedding of  $\mathcal{W}_\beta$  in  $\mathcal{H}_\beta$ . If we require greater smoothness of  $\nu$ , for instance  $\nu \in \mathcal{W}^{1,\infty}(\Omega)$ , we may replace the requirement that  $\Omega$  satisfy the restricted cone condition by any of the following conditions:

- (i)  $\Gamma$  is Lipschitz, for then  $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div}_\nu, \Omega)$  is continuously embedded in  $\mathcal{H}^{1/2}(\Omega)_\nu^2$ ; cf. [9].
- (ii)  $\Omega$  is convex, for then  $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div}_\nu, \Omega)$  is continuously embedded in  $\mathcal{H}^1(\Omega)_\nu^2$ ; cf. [13].
- (iii)  $\Gamma$  is  $\mathcal{C}^{1,1}$ , for again  $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div}_\nu, \Omega)$  is continuously embedded in  $\mathcal{H}^1(\Omega)_\nu^2$ ; cf. [13].

**THEOREM 2.2.** *With the given hypothesis concerning  $\nu, \eta$ , and  $\Omega$ , the spectrum of  $\mathcal{A}_\beta$ ,  $\sigma(\mathcal{A}_\beta)$ , is a pure point spectrum*

$$\sigma(\mathcal{A}_\beta) = \{\lambda_1(\beta) \leq \lambda_2(\beta) \leq \dots \leq \lambda_n(\beta) \leq \dots < +\infty\}$$

with

$$\lambda_1(\beta) > 0$$

and

$$\lim_{n \rightarrow \infty} \lambda_n(\beta) = +\infty.$$

The set of eigenfunctions  $\mathbf{u}_j(\beta)$  associated with  $\lambda_j(\beta)$  is characterized as the set of nonzero solutions of

$$(2.18) \quad \begin{aligned} \tilde{\mathbf{u}} &\in \mathcal{W}_\beta, \\ a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &= \lambda_j(\beta)(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu \quad \text{for each } \tilde{\mathbf{v}} \in \mathcal{W}_\beta. \end{aligned}$$

*Proof.* In light of Theorem 2.1, the results concerning the spectrum of  $\mathcal{A}_\beta$  are classical results from the spectral theory of compact selfadjoint operators [11].

As for the characterization of the eigenfunctions, it is clear that any eigenfunction satisfies (2.18). To see that any solution of (2.18) is an eigenfunction it is sufficient, as in the proof of Lemma 2.3, to remark that the second equation of (2.18) holds for  $\tilde{\mathbf{v}} \in \mathcal{W}_\beta^\perp$  for any  $\tilde{\mathbf{u}} \in \mathcal{W}_\beta$ .  $\square$

We conclude this section with the following theorem, a direct consequence of Theorem 2.2.

**THEOREM 2.3.** *For any electromagnetic wave guide whose cross section  $\Omega$  satisfies the restricted cone property, there exists a countable family of guided modes  $\{\mathbf{U}_j(x, t), 1 \leq j < +\infty\}$ . Each mode exists for any value of the wave number  $\beta$  and the  $j$ th mode obeys the dispersion relation*

$$(2.19) \quad \omega^2 = \lambda_j(\beta), \quad \text{where } \lambda_j(\beta) \text{ is the } j\text{th eigenvalue of } \mathcal{A}_\beta.$$

**3. The approximation space  $\mathcal{W}_{\beta,h}$ .** We have seen in the preceding section that looking for a guided mode with wave number  $\beta$  is equivalent to seeking a solution to the following problem:

$$(3.1) \quad \begin{aligned} &\text{find } (\omega, \tilde{\mathbf{u}}) \in \mathbb{R}^+ \times \mathcal{W}_\beta \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \omega^2(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}_\beta. \end{aligned}$$

The first step in the numerical approximation of the problem is thus to construct an approximation space  $\mathcal{W}_{\beta,h}$  for  $\mathcal{W}_\beta$ .

Recall that  $\mathcal{W}_\beta$  is the subspace of  $\mathcal{W} = \mathcal{H}_0(\text{curl}, \Omega) \times \mathcal{H}_0^1(\Omega)$  consisting of those elements  $\tilde{\mathbf{u}}$  of  $\mathcal{W}$  for which  $\text{div}_\beta(\nu \tilde{\mathbf{u}}) = 0$ . Thus it would be natural to try to define  $\mathcal{W}_{\beta,h}$  to be the subspace of  $\mathcal{W}_h = \mathcal{R}_h \times \mathcal{P}_h$  consisting of those elements  $\tilde{\mathbf{u}}_h$  of  $\mathcal{W}_h$  for which  $\text{div}_\beta(\nu \tilde{\mathbf{u}}_h) = 0$ , where  $\mathcal{R}_h$  is a finite-dimensional subspace of  $\mathcal{H}_0(\text{curl}, \Omega)$  and  $\mathcal{P}_h$  is a finite-dimensional subspace of  $\mathcal{H}_0^1(\Omega)$ . However we shall see below, in §3.5, that for classical spaces  $\mathcal{R}_h$  and  $\mathcal{P}_h$ ,  $\tilde{\mathbf{u}}_h \in (\mathcal{R}_h \times \mathcal{P}_h) \cap \mathcal{W}_\beta$  implies that  $\tilde{\mathbf{u}}_h = 0$ . This leads us to the idea of constructing an external approximation of  $\mathcal{W}_\beta$  in which the divergence-free condition

$$\text{div}_\beta(\nu \tilde{\mathbf{u}}) = 0$$

is enforced only in a weak form.

Let us recall that the divergence-free condition,  $\text{div}_\beta(\nu \tilde{\mathbf{u}}) = 0$ , defines the subspace  $\mathcal{W}_\beta$  of  $\mathcal{W}$  as the isomorphic image of the space  $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div}_\nu, \Omega)$  of two-dimensional vectors under the map  $\mathcal{J}$  which associates to a two-dimensional vector a third component in such a way that the divergence-free condition is satisfied:

$$\mathcal{J}(\mathbf{u} = (u_1, u_2)) = \left( u_1, u_2, \frac{1}{\beta} \text{div}_\nu(\mathbf{u}) \right),$$

where for notational convenience we have introduced the operator  $\text{div}_\nu$  defined by

$$\text{div}_\nu \mathbf{u} = \frac{1}{\nu} \text{div } \nu \mathbf{u}.$$

What we would like to do is to define a discrete analogue of  $\mathcal{J}$  that would map  $\mathcal{R}_h$  into  $\mathcal{R}_h \times \mathcal{P}_h$  by defining its third component in such a way that a weak divergence-free condition is satisfied. We could define  $\mathcal{J}_h$  from  $\mathcal{R}_h$  into  $\mathcal{R}_h \times \mathcal{P}_h$  by

$$(3.2) \quad \mathcal{J}_h(\mathbf{u}_h) = \left( u_{h,1}, u_{h,2}, \frac{1}{\beta} \text{div}_{\nu,h} \mathbf{u}_h \right),$$

where  $\operatorname{div}_{v,h} \mathbf{u}_h \in \mathcal{P}_h$  is determined by

$$(3.3) \quad \int_{\Omega} v \operatorname{div}_{v,h} \mathbf{u}_h \varphi_h \, dx = - \int_{\Omega} v \mathbf{u}_h \operatorname{grad} \varphi_h \, dx \quad \text{for each } \varphi_h \in \mathcal{P}_h.$$

We would then define  $\mathcal{W}_{\beta,h}$  to be the image of  $\mathcal{J}_h$ .

However, if  $\mathcal{W}_{\beta,h}$  is to be a finite element space we need a basis of elements with small support to preserve computational efficiency. If  $\operatorname{div}_{v,h}$  is defined by (3.3), the support of  $\operatorname{div}_{v,h} \mathbf{u}$  could be all of  $\Omega$  even if the support of  $\mathbf{u}_h$  is very small. We could not obtain a suitable basis for  $\mathcal{W}_{\beta,h}$  as the image of a basis for  $\mathcal{R}_h$ . This leads us to the idea of defining a discrete divergence operator  $\operatorname{Div}_{v,h}$  using numerical integration. For  $\mathbf{u}_h \in \mathcal{R}_h$ ,  $\operatorname{Div}_{v,h} \mathbf{u}_h$  is determined by

$$(3.4) \quad \oint_{\Omega} v \operatorname{Div}_{v,h} \mathbf{u}_h \varphi_h \, dx = - \int_{\Omega} v \mathbf{u}_h \operatorname{grad} \varphi_h \, dx \quad \text{for each } \varphi_h \in \mathcal{P}_h,$$

where  $\oint$  is a numerical integration operator.

Our finite element space is thus the space

$$(3.5) \quad \mathcal{W}_{\beta,h} = \mathcal{J}_h(\mathcal{R}_h) = \left\{ (\mathbf{u}_h, u_{h,3}) \in \mathcal{R}_h \times \mathcal{P}_h; u_{h,3} = \frac{1}{\beta} \operatorname{Div}_{v,h} \mathbf{u}_h \right\}.$$

But, of course, to complete our definition we need to define the spaces  $\mathcal{R}_h$  and  $\mathcal{P}_h$  and the numerical integration operator in (3.4).

*Remark 3.1.* Those who are familiar with fluid dynamics will see a certain analogy with the numerical approximation of Stokes or Navier–Stokes equations. For these problems, we have a vector-valued function  $\mathbf{u}$  (the velocity field) and a scalar one  $p$  (the pressure) as we have here  $\mathbf{u} = (u_1, u_2)$  and  $u_3$ . The difference is that for Stokes, the vector unknown (in  $\mathcal{H}^1(\Omega)$ ) is more regular than the scalar one (in  $\mathcal{L}^2(\Omega)$ ). Here, we encounter the opposite situation.

Henceforth, to avoid the technical difficulties linked to the approximation of the domain, we assume that  $\bar{\Omega}$  is a polygonal domain in  $\mathbb{R}^2$ . Let  $\mathcal{T}_h$  be a uniformly regular triangulation of  $\Omega$ ,  $\bar{\Omega} = \cup_{K \in \mathcal{T}_h} \bar{K}$  by a finite number of rectangles and/or triangles of mesh size  $h$ ;  $h = \max_{K \in \mathcal{T}_h} (\operatorname{diam} K)$ .

**3.1. The space  $\mathcal{P}_h$  and its approximation properties.** We shall consider the Lagrangian finite elements which are linear on each triangle of  $\mathcal{T}_h$  and bilinear on each rectangle of  $\mathcal{T}_h$ :

$$(3.6) \quad \begin{aligned} \mathcal{P}_h &= \{ \varphi \in \mathcal{H}_0^1(\Omega) \cap C(\bar{\Omega}); \\ &\varphi|_K \in P_1(K) \text{ if } K \in \mathcal{T}_h \text{ is a triangle, and} \\ &\varphi|_K \in Q_1(K) \text{ if } K \in \mathcal{T}_h \text{ is a rectangle} \}. \end{aligned}$$

An element of  $\mathcal{P}_h$  is continuous, vanishes on the boundary, and is uniquely determined by its values at the interior vertices of  $\mathcal{T}_h$ . Thus its dimension is the number of interior vertices of  $\mathcal{T}_h$ . Further, if for each interior vertex  $M$  of  $\mathcal{T}_h$ ,  $\varphi_M$  denotes the element of  $\mathcal{P}_h$  having value 1 at  $M$  and 0 at every other vertex of  $\mathcal{T}_h$ , then  $\{\varphi_M; M \text{ is an interior vertex of } \mathcal{T}_h\}$  is a basis of  $\mathcal{P}_h$ .

We denote by  $\Pi_{\mathcal{P}_h}$  the  $\mathcal{L}^2(\Omega)_v$  orthogonal projection operator from  $\mathcal{L}^2(\Omega)_v$  onto  $\mathcal{P}_h$ :

$$(3.7) \quad (\Pi_{\mathcal{P}_h} \varphi - \varphi, \psi_h)_v = 0 \quad \text{for each } \psi_h \in \mathcal{P}_h.$$

We have the following well-known estimate [7, (17.12)]:

$$(3.8) \quad \begin{aligned} \|\varphi - \Pi_{\mathcal{P}_h} \varphi\|_{\mathcal{L}^2(\Omega)_v} &\leq C \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} h^m |\varphi|_{m,\Omega}, \quad m = 1, 2, \\ \|\operatorname{grad}(\varphi - \Pi_{\mathcal{P}_h} \varphi)\|_{\mathcal{L}^2(\Omega)_v^2} &\leq C \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} h |\varphi|_{2,\Omega} \end{aligned}$$

whenever  $\varphi$  has sufficient regularity for the above norms and seminorms to be defined.

*Remark 3.2.* If  $\varphi$  is not in  $\mathcal{H}^2(\Omega)$  but we have a partition of  $\Omega$ ,  $\bar{\Omega} = \bigcup_i \bar{\Omega}_i$ ,  $\Omega_i \cap \Omega_j = \emptyset$  if  $i \neq j$ , such that  $\varphi \in \mathcal{H}^2(\Omega_i)$  for each  $i$ , and such that each  $T \in \mathcal{T}_h$  is contained in  $\Omega_i$  for some  $i$ , the seminorm  $|\varphi|_{\mathcal{H}^2(\Omega)}$  may be replaced by  $\sum_i |\varphi|_{\mathcal{H}^2(\Omega_i)}$  in both equations of (3.8).

**3.2. The space  $\mathcal{R}_h$  and its approximation properties.** We use the lowest-order mixed elements of Nedelec [19] for the approximation of the space  $H_0(\text{curl}, \Omega)$ . In the two-dimensional case, these elements are easily deduced from those of Raviart–Thomas [23] for the approximation of the space  $H(\text{div}, \Omega)$  (simply apply a rotation of angle  $\frac{\pi}{2}$ ):

$$(3.9) \quad \begin{aligned} \mathcal{R}_h &= \{v \in \mathcal{H}_0(\text{curl}, \Omega); \\ v|_K &\in \{\alpha + \gamma(-x_2, x_1)^t, \alpha \in P_0(K)^2, \gamma \in P_0(K)\} \text{ if } K \in \mathcal{T}_h \text{ is a triangle, and} \\ v|_K &\in Q_{0,1}(K) \times Q_{1,0}(K) \text{ if } K \in \mathcal{T}_h \text{ is a rectangle.} \end{aligned}$$

An element of  $\mathcal{R}_h$  has tangential components which are continuous across the interior edges of  $\mathcal{T}_h$  and 0 on the boundary edges. It is uniquely determined by the constant values of its tangential components on the interior edges. Thus the dimension of  $\mathcal{R}_h$  is equal to the number of interior edges of  $\mathcal{T}_h$ , and if for each interior edge  $a$ ,  $w_a$  denotes the element of  $\mathcal{R}_h$  having tangential component equal to 1 on  $a$  and equal to 0 on every other edge, then  $\{w_a; a \text{ an interior edge of } \mathcal{T}_h\}$  is a basis of  $\mathcal{R}_h$ .

We shall make use of the operator  $\Pi_{\mathcal{R}_h} : \mathcal{H}_0(\text{curl}, \Omega) \longrightarrow \mathcal{R}_h$  which is just the projection in the  $\mathcal{H}(\text{curl}, \Omega)$ -norm:

$$(3.10) \quad (\Pi_{\mathcal{R}_h}(v) - v, w_h)_v + (\text{curl}(\Pi_{\mathcal{R}_h}(v) - v), \text{curl}(w_h))_v = 0 \quad \text{for each } w_h \in \mathcal{R}_h.$$

We have the estimate

$$(3.11) \quad \begin{aligned} \|u - \Pi_{\mathcal{R}_h} u\|_{\mathcal{L}^2(\Omega)^2} &\leq C \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} h |u|_{1,\Omega}, \\ \|\text{curl}(u - \Pi_{\mathcal{R}_h} u)\|_{\mathcal{L}^2(\Omega)} &\leq C \sqrt{\frac{\nu_{\max}}{\nu_{\min}}} h (|u|_{1,\Omega} + |\text{curl}u|_{1,\Omega}) \end{aligned}$$

whenever  $u$  has sufficient regularity for the above norms and seminorms to be defined. This estimate follows from [24, Thm. 6.3].

*Remark 3.3.* Again, if  $u$  is not in  $\mathcal{H}^1(\Omega)$  or if  $\text{curl} u$  is not in  $\mathcal{H}^1(\Omega)$  but we have a partition of  $\bar{\Omega} = \bigcup_i a\bar{\Omega}_i$  such that  $u \in \mathcal{H}^1(\Omega_i)$  and  $\text{curl} u \in \mathcal{H}^1(\Omega_i)$  for each  $i$ , the seminorms  $|u|_{1,\Omega}$  and  $|\text{curl}u|_{1,\Omega}$  may be replaced by  $\sum_i |u|_{1,\Omega_i}$  and  $\sum_i |\text{curl}u|_{1,\Omega_i}$ , respectively.

Here we point out one more property of the pair of spaces  $\mathcal{R}_h$  and  $\mathcal{P}_h$  which we shall exploit later:

$$(3.12) \quad \text{if } \varphi_h \in \mathcal{P}_h, \text{ then } \text{grad}(\varphi_h) \in \mathcal{R}_h.$$

This appears as a fundamental compatibility relation between the spaces  $\mathcal{R}_h$  and  $\mathcal{P}_h$ .

**3.3. The numerical integration scheme and its approximation properties.** The quadrature scheme used here is the trapezoidal rule; the integral of a function over a cell is approximated by the average of the function values at its vertices multiplied by the area of the cell:

$$\oint_{\Omega} f \, dx = \sum_{K \in \mathcal{T}_h} \frac{\text{meas}(K)}{nv(K)} \sum_{M \text{ a vertex of } K} f(M),$$

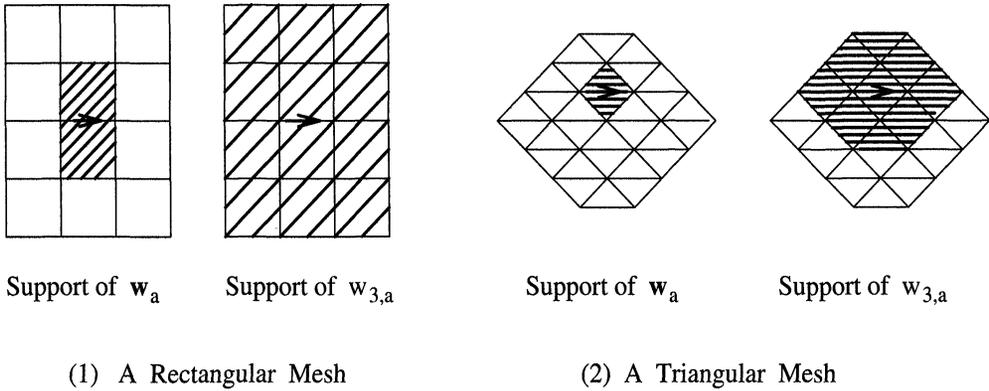


FIG. 2. Support of  $w_{3,a} = \text{support of } \text{Div}_{v,h} w_a$ .

where  $nv(K)$  denotes the number of vertices of  $K$ . This scheme is exact for functions in  $\mathcal{P}_h$ , and we have that if  $\varphi_h$  and  $\psi_h$  are in  $\mathcal{P}_h$ , and if  $\alpha$  is regular ( $\mathcal{W}^{1,\infty}$ ) on each cell of  $\mathcal{T}_h$ , then, [8, Chap. 4.1],

$$(3.13) \quad \left| \oint_{\Omega} \alpha \varphi_h \psi_h dx - \int_{\Omega} \alpha \varphi_h \psi_h dx \right| \leq C h \alpha_{\max} \|\varphi_h\|_{\mathcal{H}^1(\Omega)} \|\psi_h\|_{\mathcal{L}^2(\Omega)}.$$

We also introduce here the numerical integration operator  $\Upsilon_h$  from  $\mathcal{P}_h$  to  $\mathcal{P}_h$  defined by

$$(3.14) \quad \oint_{\Omega} \Upsilon_h(\varphi_h) \psi_h dx = \int_{\Omega} \varphi_h \psi_h dx \quad \text{for each } \psi_h \in \mathcal{P}_h.$$

We point out that  $\text{Div}_{v,h} = \Upsilon_h \circ \text{div}_{v,h}$ .

**3.4. Basis elements and dimension.** Note that  $\mathcal{W}_{\beta,h}$  is isomorphic to the space  $\mathcal{R}_h$  and has by construction the same degrees of freedom as  $\mathcal{R}_h$ . It is then easy to construct a basis of  $\mathcal{W}_{\beta,h}$  by considering the set  $\{\tilde{w}_a; a \text{ an interior edge of } \mathcal{T}_h\}$ , where  $\tilde{w}_a = (w_a, w_{3,a} = \frac{1}{\beta} \text{Div}_{v,h} w_a)$ . The support of  $w_{3,a}$  is simply the union of all the elements of  $\mathcal{T}_h$  which admit at least one common vertex with one of the two elements of the support of  $w_a$  (see Fig. 2).

*Remark 3.4.* Let us again emphasize the importance of using a quadrature formula for the evaluation of the integral  $\int_{\Omega} v \psi \varphi dx$  in (3.3) for the definition of the discrete divergence operator. If we had evaluated this integral exactly, we would have lost the local character of the discrete operator  $\text{Div}_{v,h}$  since its evaluation would have involved the inversion of the mass matrix associated with the bilinear form  $(\varphi, \psi) \mapsto \int_{\Omega} v \varphi \psi dx$  on  $P_h \times P_h$ . Then, for a basis function  $\tilde{w}_a = (w_a, w_{3,a}) = (w_a, \frac{1}{\beta} \text{div}_{v,h} w_a)$ , the support of  $w_{3,a}$  would have coincided with all  $\Omega$ , leading to prohibitively expensive calculations.

It is interesting to compare the dimension of the space  $\mathcal{W}_{\beta,h}$  (which is the same as the dimension of  $\mathcal{R}_h$ ) with that of the space  $\mathcal{R}_h \times \mathcal{P}_h$  in which we would naturally work if we did not take into account the divergence-free condition. We consider the case where  $\Omega$  is a square and examine the two following examples.

*Example 1.* We consider a mesh of squares. Denoting by  $N$  the number of small squares in one direction, then  $\text{Dim } \mathcal{R}_h = 2N(N - 1)$  and  $\text{Dim } \mathcal{P}_h = (N - 1)^2$ .

*Example 2.* We consider the previous mesh, and we split each small square into two triangles. Then  $\text{Dim } \mathcal{R}_h = 3N^2 - 2N$  and  $\text{Dim } \mathcal{P}_h = (N - 1)^2$ .

We see that we gain at least a factor 3/2 for Example 1 and 4/3 for Example 2, on the dimension of the matrices we have to deal with, when  $N$  is large. However this benefit is counterbalanced by the fact that the matrices are not as sparse.

**3.5.  $(\mathcal{R}_h \times \mathcal{P}_h) \cap \mathcal{H}_\beta = \{0\}$ .** We assume here that  $\nu = 1$ . Suppose that  $\tilde{v}_h = (v_h, v_{h,3})$  belongs to  $\mathcal{R}_h \times \mathcal{P}_h$  and that  $\operatorname{div}_\beta(\nu \tilde{v}_h) = 0$ . Then  $\frac{1}{\beta} \operatorname{div} v_h = v_{h,3}$ . However, for each element  $K \in \mathcal{T}_h$ , whether  $K$  is a rectangle or a triangle,  $\operatorname{div} v_h|_K = 0$ . Thus  $v_{3h}$  is identically 0 since elements of  $\mathcal{P}_h$  are continuous and vanish on the boundary. We now have imposed on  $v_h$  an additional continuity requirement, namely continuity of the divergence, at each interior edge of  $\mathcal{T}_h$ , and it is not difficult to check that this together with the boundary requirement for elements of  $\mathcal{R}_h$  implies that  $v_h$  is also identically 0.

**3.6. Approximation properties of the space  $\mathcal{W}_{\beta,h}$ .** Here we derive some approximation properties of the space  $\mathcal{W}_{\beta,h}$ . Indeed we shall see that because of the numerical integration used in the definition of  $\mathcal{W}_{\beta,h}$ ; i.e., in the definition of the discrete divergence operator  $\operatorname{Div}_{v,h}$ , we have been able to obtain a useful approximation result only in the case that the family of triangulations  $\{\mathcal{T}_h; h \in \Lambda\}$  is asymptotically uniform. (See definition below.) The results obtained are thus not used in the remainder of this article. We have nonetheless thought it of some interest to include them and to point out an analogy with finite volume methods [12] or cell-centered finite difference methods.

We would like to be able to show that any element of  $\mathcal{W}_\beta$  can be approximated sufficiently well by an element of the space  $\mathcal{W}_{\beta,h}$ . That is we would like to be able to define an operator  $\tilde{\Pi} : \mathcal{W}_\beta \rightarrow \mathcal{W}_{\beta,h}$  such that we have at the very least that

$$(3.15) \quad \|\tilde{u} - \tilde{\Pi}\tilde{u}\|_{\mathcal{W}} \rightarrow 0 \quad \text{as } h \rightarrow 0 \quad \text{for each } \tilde{u} \in \mathcal{W}_\beta.$$

Given that any map  $\tilde{\Pi} : \mathcal{W}_\beta \rightarrow \mathcal{W}_{\beta,h}$  uniquely defines a map  $\Pi : \mathcal{H}_0(\operatorname{curl}, \Omega) \rightarrow \mathcal{R}_h$  and conversely, we need a projection operator  $\Pi$  from  $\mathcal{H}_0(\operatorname{curl}, \Omega)$  onto  $\mathcal{R}_h$  such that if  $u \in \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}(\operatorname{div}_\nu, \Omega)$  then  $\mathcal{J}_h(\Pi u)$  is a good approximation of  $\mathcal{J}(u)$ ; i.e.,  $\Pi u$  is a good approximation of  $u$  in the  $\mathcal{H}_0(\operatorname{curl}, \Omega)$ -norm and  $\operatorname{Div}_{v,h}(\Pi u)$  is a good approximation of  $\operatorname{div}_\nu u$  in the  $\mathcal{H}^1(\Omega)$ -norm.

We take the map  $\Pi$  to be  $\Pi_{\mathcal{R}_h}$ , the projection in the  $\mathcal{H}(\operatorname{curl}, \Omega)$ -norm of  $\mathcal{H}_0(\operatorname{curl}, \Omega)$  onto  $\mathcal{R}_h$ , as defined by (3.10), and we obtain the following lemma.

LEMMA 3.1. *The following diagram commutes:*

$$\begin{array}{ccc}
 \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}(\operatorname{div}_\nu, \Omega) & \xrightarrow{\operatorname{div}_\nu} & \mathcal{L}^2(\Omega) \\
 \Pi_{\mathcal{R}_h} \downarrow & & \downarrow \Pi_{\mathcal{P}_h} \\
 \mathcal{R}_h & \xrightarrow{\operatorname{div}_{v,h}} & \mathcal{P}_h
 \end{array}$$

That is,

$$\Pi_{\mathcal{P}_h} \circ \operatorname{div}_\nu = \operatorname{div}_{v,h} \circ \Pi_{\mathcal{R}_h}.$$

*Proof.* Let  $u \in \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \mathcal{H}(\operatorname{div}_\nu, \Omega)$ . It is sufficient to show that

$$(\Pi_{\mathcal{P}_h} \operatorname{div}_\nu u, \varphi_h)_\nu = (\operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} u, \varphi_h)_\nu \quad \text{for each } \varphi_h \in \mathcal{P}_h.$$

First note that for  $\varphi_h \in \mathcal{P}_h$ ,

$$\begin{aligned}
 (\Pi_{\mathcal{P}_h} \operatorname{div}_\nu u, \varphi_h)_\nu &= (\operatorname{div}_\nu u, \varphi_h)_\nu \\
 &= -(u, \operatorname{grad} \varphi_h)_\nu
 \end{aligned}$$

by definition of  $\Pi_{\mathcal{P}_h}$ , (3.7). For any  $\varphi_h \in \mathcal{P}_h$ ,

$$(\operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u}, \varphi_h)_v = -(\Pi_{\mathcal{R}_h} \mathbf{u}, \operatorname{grad} \varphi_h)_v$$

by definition of  $\operatorname{div}_{v,h}$ . But  $(\mathbf{u} - \Pi_{\mathcal{R}_h} \mathbf{u}, \operatorname{grad} \varphi_h)_v = 0$  by definition of  $\Pi_{\mathcal{R}_h}$  since  $\operatorname{grad} \varphi_h \in \mathcal{R}_h$  and  $\operatorname{curl} \operatorname{grad} \varphi_h = 0$ .  $\square$

LEMMA 3.2. Suppose that  $\mathbf{u} \in \mathcal{H}_0(\operatorname{curl}, \Omega)$  is such that  $\operatorname{div}_v \mathbf{u} \in \mathcal{H}_0^1(\Omega) \cap \mathcal{H}^2(\Omega)$ . Then

$$(3.16) \quad \|\operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u} - \operatorname{div}_v \mathbf{u}\|_{\mathcal{L}^2(\Omega)} \leq C \sqrt{\frac{v_{\max}}{v_{\min}}} h^2 |\operatorname{div}_v \mathbf{u}|_{2,\Omega}$$

and

$$(3.17) \quad \|\operatorname{grad}(\operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u} - \operatorname{div}_v \mathbf{u})\|_{\mathcal{L}^2(\Omega)^2} \leq C \sqrt{\frac{v_{\max}}{v_{\min}}} h |\operatorname{div}_v \mathbf{u}|_{2,\Omega},$$

where the positive constant  $C$  depends only on the open set  $\Omega$ .

*Proof.* Since by Lemma 3.1 we have  $\operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u} = \Pi_{\mathcal{P}_h} \operatorname{div}_v \mathbf{u}$ , (3.16) and (3.17) follow from (3.8).  $\square$

THEOREM 3.1. Let  $\tilde{\mathbf{u}} = (\mathbf{u}, u_3) \in \mathcal{W}_\beta$  be such that  $\mathbf{u} \in \mathcal{H}^1(\Omega)^2$ ,  $\operatorname{curl} \mathbf{u} \in \mathcal{H}^1(\Omega)$ , and  $u_3 \in \mathcal{H}^2(\Omega)$ . Let

$$\tilde{\mathbf{u}}_h = \left( \Pi_{\mathcal{R}_h} \mathbf{u}, \frac{1}{\beta} \Pi_{\mathcal{P}_h} u_3 \right) = \left( \Pi_{\mathcal{R}_h} \mathbf{u}, \frac{1}{\beta} \operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u} \right).$$

Then

$$(3.18) \quad \begin{aligned} \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathcal{H}(\operatorname{curl}, \Omega) \times \mathcal{H}^1(\Omega)} \\ \leq C(\beta, \nu) h \{ |\mathbf{u}|_1 + |\operatorname{curl} \mathbf{u}|_1 + |\operatorname{div}_v \mathbf{u}|_2 \}. \end{aligned}$$

*Proof.* Theorem 3.1 follows immediately from (3.8), (3.11), and Lemmas 3.1 and 3.2.  $\square$

Remark 3.5. As in Remarks 3.2 and 3.3, the estimate (3.18) can be extended to the case where the function  $\tilde{\mathbf{u}}$  is only piecewise regular in the sense defined in these two remarks.

Theorem 3.1 would give us not only the desired pointwise convergence in  $\mathcal{W}$ , but also the stronger  $\mathcal{W}$ -norm convergence of  $\tilde{\mathbf{u}}_h = (\Pi_{\mathcal{R}_h} \tilde{\mathbf{u}}, \frac{1}{\beta} \operatorname{div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u})$  toward  $\tilde{\mathbf{u}}$ . However  $\tilde{\mathbf{u}}_h$  is not in  $\mathcal{W}_{\beta,h}$  since we had to resort to the use of a quadrature rule to define  $\mathcal{W}_{\beta,h}$ . Of course we would like to be able to replace  $\operatorname{div}_{v,h}$  in Theorem 3.1 by  $\Upsilon_h \circ \operatorname{div}_{v,h} = \operatorname{Div}_{v,h}$ , and we clearly have the analogue of Lemma 3.1:

$$(3.19) \quad \Upsilon_h \circ \Pi_{\mathcal{P}_h} \circ \operatorname{div}_v = \Upsilon_h \circ \operatorname{div}_{v,h} \circ \Pi_{\mathcal{R}_h} = \operatorname{Div}_{v,h} \circ \Pi_{\mathcal{R}_h}.$$

But we would have the analogue of Lemma 3.2 only if we had the estimates for  $\Upsilon_h \circ \Pi_{\mathcal{P}_h}$  that (3.8) gives for  $\Pi_{\mathcal{P}_h}$ . These estimates we have been able to obtain only under the hypothesis that the family  $\{\mathcal{T}_h; h \in \Lambda\}$  of triangulations is *asymptotically uniform*.

DEFINITION 3.1. A family of triangulations  $\{\mathcal{T}_h; h \in \Lambda\}$  is said to be *asymptotically uniform* if

$$\max_{M \in N_h^0} \int_{\Omega} \varphi_M(x) (x - M) dx = \mathcal{O}(h^4),$$

where  $N_h^0$  is the set of interior nodes of  $\mathcal{T}_h$  and, for  $M \in N_h^0$ ,  $\varphi_M$  is the basis function of  $\mathcal{P}_h$  associated with the vertex  $M$ .

LEMMA 3.3. *If the family of triangulations  $\{\mathcal{T}_h; h \in \Lambda\}$  is asymptotically uniform, we have for each  $\varphi \in \mathcal{H}^2(\Omega)$  that*

$$(3.20) \quad \begin{aligned} \|\Upsilon_h \circ \Pi_{\mathcal{P}_h} \varphi - \varphi\|_{\mathcal{L}^2(\Omega)} &\leq C h^2 \|\varphi\|_{\mathcal{H}^2(\Omega)}, \\ \|\text{grad}(\Upsilon_h \circ \Pi_{\mathcal{P}_h} \varphi - \varphi)\|_{\mathcal{L}^2(\Omega)^2} &\leq C h \|\varphi\|_{\mathcal{H}^2(\Omega)}. \end{aligned}$$

*Proof.* For a complete proof of Lemma 3.3, see [15].  $\square$

THEOREM 3.2. *Assume that the family of triangulations  $\{\mathcal{T}_h; h \in \Lambda\}$  is asymptotically uniform. Let  $\tilde{\mathbf{u}} = (\mathbf{u}, u_3) \in \mathcal{W}_\beta$  be such that  $\mathbf{u} \in \mathcal{H}^1(\Omega)^2$ ,  $\text{curl} \mathbf{u} \in \mathcal{H}^1(\Omega)$ , and  $u_3 \in \mathcal{H}^2(\Omega)$ . Let*

$$\tilde{\mathbf{u}}_h = \left( \Pi_{\mathcal{R}_h} \mathbf{u}, \frac{1}{\beta} \Upsilon_h \circ \Pi_{\mathcal{P}_h} u_3 \right) = \left( \Pi_{\mathcal{R}_h} \mathbf{u}, \frac{1}{\beta} \text{Div}_{v,h} \Pi_{\mathcal{R}_h} \mathbf{u} \right).$$

Then

$$(3.21) \quad \begin{aligned} \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathcal{H}(\text{curl}, \Omega) \times \mathcal{H}^1(\Omega)} \\ \leq C(\beta, v) h \{ \|\mathbf{u}\|_1 + \|\text{curl} \mathbf{u}\|_1 + \|\text{div}_v \mathbf{u}\|_2 \}. \end{aligned}$$

*Proof.* Theorem 3.2 follows from (3.8), (3.11), (3.19), and Lemma 3.3.  $\square$

Remark 3.6. The same comment concerning the regularity of  $\mathbf{u}$  as in Remark 3.5 applies to the estimate (3.21).

**4. Analysis of the finite element method.** The nonconforming finite element method is as follows:

$$(4.1) \quad \begin{aligned} \text{find } (\omega, \tilde{\mathbf{u}}_h) \in \mathbb{R}^+ \times \mathcal{W}_{\beta,h} \text{ such that} \\ a(\beta, \tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) = \omega^2 (\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h)_v \text{ for each } \tilde{\mathbf{v}}_h \in \mathcal{W}_{\beta,h}. \end{aligned}$$

We point out that even though  $\mathcal{W}_{\beta,h}$  is not contained in  $\mathcal{W}_\beta$ , it is contained in  $\mathcal{W}$  and  $a(\beta, \cdot, \cdot)$  is defined on all of  $\mathcal{W}$ . So (4.1) is meaningful. Further we can show the following lemma.

LEMMA 4.1. *The bilinear form  $a(\beta, \cdot, \cdot)$  is  $\mathcal{W}_{\beta,h}$ -elliptic, uniformly in  $h$ .*

*Proof.* With  $\tilde{\mathbf{u}} \in \mathcal{W}_\beta$  replaced by  $\tilde{\mathbf{u}}_h \in \mathcal{W}_{\beta,h}$  in the proof of the continuous analogue, Lemma 2.2, the proof is the same up to (2.13) where we obtain instead

$$(4.2) \quad \begin{aligned} -\frac{\eta_{\min}}{\nu_{\max}} \beta \int_{\Omega} \nu \mathbf{u}_h \text{grad} u_{h,3} \, dx &= \frac{\eta_{\min}}{\nu_{\max}} \beta \oint_{\Omega} u_{h,3} \nu \text{Div}_{v,h} \mathbf{u}_h \, dx \\ &= \frac{\eta_{\min}}{\nu_{\max}} \beta^2 \oint_{\Omega} \nu u_{h,3}^2 \, dx \\ &\geq 0, \end{aligned}$$

which completes the proof since, by the Poincaré lemma,

$$\|u_{h,3}\|_{\mathcal{L}^2(\Omega)_v}^2 \leq C \|\text{grad} u_{h,3}\|_{\mathcal{L}^2(\Omega)_v}^2. \quad \square$$

Just as solutions of (3.1) are eigenpairs for the operator  $\mathcal{A}_\beta$ , those of (4.1) are eigenpairs of the operator  $\mathcal{A}_{\beta,h}$ , which is defined on  $\mathcal{W}$  as follows:

$$\begin{aligned} \mathcal{A}_{\beta,h}(\tilde{\mathbf{u}}) &\in \mathcal{W}_{\beta,h}, \\ (\mathcal{A}_{\beta,h}(\tilde{\mathbf{u}}), \tilde{\mathbf{v}})_v &= a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \quad \text{for each } \tilde{\mathbf{v}} \in \mathcal{W}_{\beta,h}. \end{aligned}$$

Thus as the image of  $\mathcal{A}_{\beta,h}$  has finite dimension, we clearly have the following analogue of Theorems 2.1 and 2.2.

**THEOREM 4.1.** *With the given hypothesis concerning  $\nu$ ,  $\eta$ , and  $\Omega$ , the operator  $\mathcal{A}_{\beta,h}$  is a symmetric, positive-definite operator of finite rank.*

**THEOREM 4.2.** *With the given hypothesis concerning  $\nu$ ,  $\eta$ , and  $\Omega$ , the spectrum of  $\mathcal{A}_{\beta,h}$ ,  $\sigma(\mathcal{A}_{\beta,h})$ , is a pure point spectrum*

$$\sigma(\mathcal{A}_{\beta,h}) = \{\lambda_{h,1}(\beta) \leq \lambda_{h,2}(\beta) \leq \cdots \leq \lambda_{h,n_h}(\beta)\}$$

with

$$\lambda_{h,1}(\beta) > 0.$$

The set of eigenfunctions,  $\tilde{\mathbf{u}}_{hj}(\beta)$ , associated with  $\lambda_{h,j}(\beta)$ , is the set of nonzero solutions of

$$(4.3) \quad \begin{aligned} \tilde{\mathbf{u}}_{hj} &\in \mathcal{W}_{\beta,h}, \\ a(\beta, \tilde{\mathbf{u}}_{hj}, \tilde{\mathbf{v}}_h) &= \lambda_{h,j}(\beta) (\tilde{\mathbf{u}}_{hj}, \tilde{\mathbf{v}}_h)_\nu \quad \text{for each } \tilde{\mathbf{v}}_h \in \mathcal{W}_{\beta,h}. \end{aligned}$$

To see that the eigenpairs  $(\omega, \tilde{\mathbf{u}})$  of  $\mathcal{A}_\beta$  are well approximated by those of  $\mathcal{A}_{\beta,h}$ , we shall recall some results from the theory of spectral approximation. First, however, as this theory applies to compact operators, we define the operators  $\mathcal{T}_\beta$  and  $\mathcal{T}_{\beta,h}$ .

Let  $\mathcal{T}_\beta$  be the operator from  $\mathcal{H}$  into  $\mathcal{H}$ , defined by

$$(4.4) \quad \begin{aligned} \mathcal{T}_\beta \tilde{\mathbf{u}} &\in \mathcal{W}_\beta, \\ a(\beta, \mathcal{T}_\beta \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) &= (\tilde{\mathbf{u}}, \tilde{\mathbf{v}})_\nu \quad \text{for each } \tilde{\mathbf{v}} \in \mathcal{W}_\beta \end{aligned}$$

and  $\mathcal{T}_{\beta,h}$  be the operator from  $\mathcal{H}$  into  $\mathcal{H}$  defined by

$$(4.5) \quad \begin{aligned} \mathcal{T}_{\beta,h} \tilde{\mathbf{u}} &\in \mathcal{W}_{\beta,h}, \\ a(\beta, \mathcal{T}_{\beta,h} \tilde{\mathbf{u}}, \tilde{\mathbf{v}}_h) &= (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}_h)_\nu \quad \text{for each } \tilde{\mathbf{v}}_h \in \mathcal{W}_{\beta,h}. \end{aligned}$$

We have that  $\mu$  is an eigenvalue of  $\mathcal{T}_\beta$  if and only if  $\frac{1}{\mu}$  is an eigenvalue of  $\mathcal{A}_\beta$ , and  $\mu_h$  is an eigenvalue for  $\mathcal{T}_{\beta,h}$  if and only if  $\frac{1}{\mu_h}$  is an eigenvalue for  $\mathcal{A}_{\beta,h}$ . The operators  $\mathcal{T}_\beta$  and  $\mathcal{T}_{\beta,h}$  are compact, selfadjoint operators on  $\mathcal{H}$ . For each eigenvalue  $\mu$  of  $\mathcal{T}_\beta$  we let  $E(\mu)$  be the eigenspace associated with  $\mu$  and let  $m(\mu)$  denote its multiplicity. Similarly, let  $E_h(\mu_h)$  denote the eigenspace associated with the eigenvalue  $\mu_h$  of  $\mathcal{T}_{\beta,h}$ .

**4.1. Convergence analysis.** Here we would like to make the hypothesis that the coefficient functions  $\nu$  and the triangulations  $\mathcal{T}_h$  that we consider are such that for each  $K \in \mathcal{T}_h$ ,  $\nu \in \mathcal{W}^{1,\infty}(K)$ . (Thus  $\nu$  is piecewise Lipschitzian.)

For our convergence analysis we have not used the classical min–max approach described by Babuška and Osborn [3] and by Strang and Fix [25] for variationally formulated eigenvalue problems as our finite element method is nonconforming. Nor have we succeeded in adapting this method to the nonconforming case. Indeed this theory would require the convergence in the operator norm of  $\mathcal{L}(\mathcal{H})$  that we did not succeed in proving. (In fact, we conjecture that this is not the case.)

Also in [18], the authors obtain convergence results for a mixed formulation of the eigenvalue problem, which we mention here since we shall indeed be led to use the equivalence of our problem with its mixed formulation. But here also norm convergence is required.

We are thus led to return to the earlier works of Anselone [2, Chap. 4] and Osborn [20]. To use their results we must establish that for the compact operators  $\mathcal{T}_\beta$  and  $\{\mathcal{T}_{\beta,h}; h \in \Lambda\}$ , the following two hypotheses are met:

- (H1) *The uniform discrete compactness hypothesis.* If  $\mathcal{B} \subset \cup_{h \in \Lambda} \mathcal{W}_{\beta,h}$  is bounded in  $\mathcal{W}$ , then it is relatively compact in  $\mathcal{H}$ .
- (H2) *The consistency hypothesis.* For each  $f \in \mathcal{H}$ ,  $\mathcal{T}_{\beta,h}(f)$  converges to  $\mathcal{T}_{\beta}f$  in the  $\mathcal{H}$ -norm, i.e.,

$$\lim_{h \rightarrow 0} \|\mathcal{T}_{\beta} f - \mathcal{T}_{\beta,h} f\|_{\mathcal{H}} = 0.$$

Now from Anselone [2] we know that for  $\mu \neq 0$  an eigenvalue of  $\mathcal{T}_{\beta}$  of multiplicity  $m$ , and for  $\epsilon > 0$  but less than the distance from  $\mu$  to any neighboring eigenvalue, the sphere of radius  $\epsilon$  around  $\mu$  contains, for  $h$  sufficiently small, exactly  $m$  eigenvalues,  $\mu_{h,i}$ ,  $i = 1, \dots, m$  (counted with multiplicity) of  $\mathcal{T}_{\beta,h}$ , and the dimension of  $E(\mu)$  is equal to that of  $E_h(\mu) = \bigoplus_{i=1}^m E_h(\mu_{h,i})$ .

Before stating Osborn’s results estimating the distance between the eigenvalues of  $\mathcal{T}_{\beta}$  and those of  $\mathcal{T}_{\beta,h}$  and the distance between the eigenvectors of  $\mathcal{T}_{\beta}$  and those of  $\mathcal{T}_{\beta,h}$ , we recall the classical notion of the distance between two subspaces  $M$  and  $N$  of a Hilbert space  $X$ :  $\delta(M, N)$ . The gap between  $M$  and  $N$  is defined by

$$\delta(M, N) = \max \{ \hat{\delta}(M, N), \hat{\delta}(N, M) \},$$

where

$$\hat{\delta}(M, N) = \sup_{\{x \in M; \|x\|_X=1\}} \text{dist}(x, N).$$

Now Osborn has shown [20] (see his Theorem 4 together with the remarks following Theorem 6) that if (H1) and (H2) are satisfied, then for  $\mu \neq 0$  an eigenvalue of  $\mathcal{T}_{\beta}$  of multiplicity  $m$  and  $\mu_{h,i}$  an eigenvalue of  $\mathcal{T}_{\beta,h}$  approximating  $\mu$  guaranteed by the theory of Anselone, we have

$$(4.6) \quad |\mu - \mu_{h,i}| \leq C \left\{ \sum_{i,j=1}^m |((\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h}) \tilde{\varphi}_i, \tilde{\varphi}_j)| + \|(\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h})|_{E(\mu)}\|^2 \right\},$$

where  $\{\tilde{\varphi}_i\}_{i=1}^m$  is a basis for  $E(\mu)$  and where  $(\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h})|_{E(\mu)}$  denotes the restriction of  $\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h}$  to  $E(\mu)$ .

Equation (4.6) gives us an estimate of the error in approximating the eigenvalues. The error in approximating the eigenvectors is given by [20, Thm. 1]

$$(4.7) \quad \delta(E(\mu), E_h(\mu)) \leq C \|(\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h})|_{E(\mu)}\|$$

for sufficiently small  $h$ .

To obtain thus the desired estimates, we must show that in our case (H1) and (H2) are satisfied, and we must estimate the two terms

$$\|(\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h})|_{E(\mu)}\| \quad \text{and} \quad \sum_{i,j=1}^m |((\mathcal{T}_{\beta} - \mathcal{T}_{\beta,h}) \tilde{\varphi}_i, \tilde{\varphi}_j)|.$$

**4.2. (H1): The uniform discrete compactness result.** The compactness of the operator  $\mathcal{T}_{\beta} : \mathcal{H} \rightarrow \mathcal{H}$  is due to the compactness of the inclusion of  $\mathcal{W}_{\beta}$  into  $\mathcal{H}_{\beta}$ ; cf. Remark 2.1. The compactness of the operators  $\mathcal{T}_{\beta,h} : \mathcal{H} \rightarrow \mathcal{H}$  is guaranteed by the fact that they are of finite rank;  $\mathcal{W}_{\beta,h}$  is obviously compactly included in  $\mathcal{H}_{\beta}$ . However, for the convergence results that we wish to obtain, some uniformity in  $h$  of the compactness of the operators  $\mathcal{T}_{\beta,h}$  is needed. This uniformity is guaranteed by the fact that the spaces  $\mathcal{W}_{\beta,h}$  are uniformly

compactly imbedded in  $\mathcal{H}$  in the sense stated in Lemma 4.2 below. Lemma 4.2 is thus a discrete analogue of the compactness of the embedding of  $\mathcal{W}_\beta$  into  $\mathcal{H}$ . This proof strongly depends on an analogous discrete compactness result of Kikuchi [17, Thm. 1] for the spaces  $\mathcal{R}_h$ . This theorem however has been proved only for the case in which the triangulation is a triangulation by triangles (and only for the Nedelec spaces of lowest order). Thus we are able to demonstrate (H1) only in this case. Further, for Kikuchi's result, the coefficient  $\nu$  is constant. Though his proof is easily extended to include the case with our hypotheses concerning  $\nu$ , we have nevertheless included an appendix to show this.

What we would like to show now is that  $\cup_{h \in \Lambda} \mathcal{W}_{\beta,h}$ , is compactly included in  $\mathcal{H}$  in the following sense.

LEMMA 4.2. *Assume that the boundary  $\Gamma$  of  $\Omega$  is Lipschitz continuous. If  $\mathcal{B} \subset \cup_{h \in \Lambda} \mathcal{W}_{\beta,h}$  is bounded in  $\mathcal{W}$ , then it is relatively compact in  $\mathcal{H}$ .*

*Proof.* The proof is an elaboration and slight generalization of one given in [5]. We include a rather detailed argument here because in that of [5] the coefficient  $\nu$  is constant and no numerical integration scheme is used. Let  $\{\tilde{w}_n \in \mathcal{W}_{\beta,h_n}\}_{n=1}^\infty$  be a sequence in  $\cup_{h \in \Lambda} \mathcal{W}_{\beta,h}$  which is bounded in  $\mathcal{W}$ . We must show that there is a subsequence, convergent in  $\mathcal{H} = \mathcal{L}^2(\Omega)^3$ . (We may assume that the sequence  $h_n$  converges toward 0 for otherwise, given the regularity assumption for the family  $\{\mathcal{T}_h; h \in \Lambda\}$ , the dimension of the spaces  $\mathcal{W}_{\beta,h_n}$  is bounded and we would show convergence in  $\mathcal{H}$  of a subsequence of a sequence in  $\cup_{n=1}^\infty \mathcal{W}_{\beta,h_n}$  bounded in  $\mathcal{W}$  just as we would for the union of a finite number of spaces  $\mathcal{W}_{\beta,h_n}$ .)

Now  $\tilde{w}_n = (w_n, w_{n,3})$ , and the sequence  $\{w_{n,3}\}_{n=1}^\infty$  is clearly bounded in  $\mathcal{H}_0^1(\Omega)$  so we can extract a subsequence converging, weakly in  $\mathcal{H}_0^1(\Omega)$ , strongly in  $\mathcal{L}^2(\Omega)$ , to some  $w_3 \in \mathcal{H}_0^1(\Omega)$ . Similarly  $\{w_n\}_{n=1}^\infty$  is bounded in  $\mathcal{H}_0(\text{curl}, \Omega)$ , but this is not enough to ensure strong convergence of a subsequence in  $\mathcal{L}^2(\Omega)^2$ . The idea is to use a kind of discrete Helmholtz decomposition, writing  $w_n$  as follows:

$$(4.8) \quad w_n = \beta \text{grad } p_n + u_n,$$

where  $p_n$  is defined by

$$(4.9) \quad \begin{aligned} p_n &\in \mathcal{P}_{h_n}, \\ \oint_{\Omega} \nu w_{n,3} q_h dx &= - \int_{\Omega} \nu \text{grad } p_n \text{grad} q_h dx \quad \text{for each } q_h \in \mathcal{P}_{h_n}. \end{aligned}$$

The element  $p_n$  is well defined since  $\mathcal{P}_{h_n} \subset \mathcal{H}_0^1(\Omega)$  (and  $(p, q) \mapsto \int_{\Omega} \nu \text{grad} p \text{grad} q dx$  is elliptic on  $\mathcal{H}_0^1(\Omega)$ ). Further  $\beta \text{grad } p_n$  is in  $\mathcal{R}_{h_n}$  as is  $w_n$ . We have thus written  $w_n$  as a sum of a gradient of an element of  $\mathcal{P}_{h_n}$  and an element of  $\mathcal{R}_{h_n}$ . To see that this element  $u_n$  of  $\mathcal{R}_{h_n}$  is in fact  $\text{Div}_{\nu,h}$ -free, we need only to show that  $\text{Div}_{\nu,h_n} \text{grad } p_n = \frac{1}{\beta} \text{Div}_{\nu,h_n} w_n$ . But this is true because both sides of the equation are equal to  $w_{n,3}$ ,—the left-hand side by definition of  $\text{Div}_{\nu,h_n}$  and the right-hand side because  $\tilde{w}_n \in \mathcal{W}_{\beta,h_n}$ .

We shall first show that the sequence  $\{p_n\}_{n=1}^\infty$  has a subsequence, still denoted  $\{p_n\}_{n=1}^\infty$ , converging strongly in  $\mathcal{H}_0^1(\Omega)$ . We then see that the corresponding subsequence  $\{u_n\}_{n=1}^\infty$  is bounded in  $\mathcal{H}(\text{curl}, \Omega)$  as both  $\{w_n\}_{n=1}^\infty$  and  $\{\text{grad } p_n\}_{n=1}^\infty$  are ( $\text{curl grad } p_n = 0$ ). We now have a bounded sequence in  $\mathcal{H}(\text{curl}, \Omega)$  on which  $\text{Div}_{\nu,h_n}$  vanishes. To obtain a subsequence converging strongly in  $\mathcal{L}^2(\Omega)^2$  and thereby completing the proof, we shall apply an extension of a theorem of Kikuchi [17, Thm. 1] which is demonstrated in Appendix A.

Now to show that  $\{\text{grad } p_n\}_{n=1}^\infty$  is a bounded sequence in  $\mathcal{L}^2(\Omega)^2$ , we calculate, using (4.9), (3.13), and the Poincaré inequality,

$$\begin{aligned} \|\text{grad } p_n\|_{\mathcal{L}^2(\Omega)^2}^2 &= \oint_{\Omega} \nu w_{n,3} p_n dx \\ &\leq \left| \oint_{\Omega} \nu w_{n,3} p_n dx - \int_{\Omega} \nu w_{n,3} p_n dx \right| + |(w_{n,3}, p_n)_\nu| \end{aligned}$$

$$\begin{aligned} &\leq C \{h_n \|w_{n,3}\|_{\mathcal{L}^2(\Omega)} |p_n|_{\mathcal{H}^1(\Omega)} + \|w_{n,3}\|_{\mathcal{L}^2(\Omega)_v} \|p_n\|_{\mathcal{L}^2(\Omega)_v}\} \\ &\leq C \|w_{n,3}\|_{\mathcal{L}^2(\Omega)_v} |p_n|_{\mathcal{H}^1(\Omega)}, \end{aligned}$$

and since  $\{w_{n,3}\}_{n=1}^\infty$  is bounded in  $\mathcal{L}^2(\Omega)$ ,

$$\|\text{grad } p_n\|_{\mathcal{L}^2(\Omega)_v^2} \leq C \|w_{n,3}\|_{\mathcal{L}^2(\Omega)_v} \leq C.$$

Thus  $\{p_n\}_{n=1}^\infty$  is bounded in  $\mathcal{H}_0^1(\Omega)$  and we can extract a subsequence converging, weakly in  $\mathcal{H}_0^1(\Omega)$ , strongly in  $\mathcal{L}^2(\Omega)$ , to  $p \in \mathcal{H}_0^1(\Omega)$ . Now we would like to show that  $\{\text{grad } p_n\}_{n=1}^\infty$  in fact converges strongly in  $\mathcal{L}^2(\Omega)^2$  to  $\text{grad } p$  in  $\mathcal{L}^2(\Omega)^2$ . We must show that

$$\lim_{n \rightarrow \infty} \|\text{grad } p_n\|_{\mathcal{L}^2(\Omega)_v^2}^2 = \|\text{grad } p\|_{\mathcal{L}^2(\Omega)_v^2}^2.$$

We have

$$\|\text{grad } p_n\|_{\mathcal{L}^2(\Omega)_v^2}^2 = \oint_{\Omega} \nu w_{n,3} p_n \, dx,$$

and by (3.13)

$$\lim_{n \rightarrow \infty} \oint_{\Omega} \nu w_{n,3} p_n \, dx = \int_{\Omega} \nu w_3 p \, dx$$

since  $\{w_{n,3}\}_{n=1}^\infty$  and  $\{p_n\}_{n=1}^\infty$  are bounded in  $\mathcal{H}_0^1(\Omega)$  and  $\mathcal{H}_0^1(\Omega)^2$ , respectively, and converge strongly in  $\mathcal{L}^2(\Omega)$  and in  $\mathcal{L}^2(\Omega)^2$ , respectively, to  $w_3$  and to  $p$ , respectively. Thus to conclude that  $\{\text{grad } p_n\}_{n=1}^\infty$  in fact converges strongly in  $\mathcal{L}^2(\Omega)^2$  to  $\text{grad } p$ , we only have to show that

$$\|\text{grad } p\|_{\mathcal{L}^2(\Omega)_v^2}^2 = \int_{\Omega} \nu w_3 p \, dx.$$

Toward this end we recall that  $\text{grad } p_n$  converges weakly to  $\text{grad } p$  in  $\mathcal{L}^2(\Omega)$  and we let  $q_n$  be a sequence with  $q_n \in \mathcal{P}_{h_n}$  converging strongly to  $p$  in  $\mathcal{H}_0^1(\Omega)$ . (For example let  $q_n = \Pi_{\mathcal{P}_{h_n}}(p)$ .) Then  $\text{grad } q_n$  converges strongly to  $\text{grad } p$  in  $\mathcal{L}^2(\Omega)$ , and

$$\|\text{grad } p\|^2 = \lim_{n \rightarrow \infty} (\text{grad } p_n, \text{grad } q_n)_\nu.$$

But,

$$(\text{grad } p_n, \text{grad } q_n)_\nu = \oint_{\Omega} \nu w_{n,3} q_n \, dx$$

and

$$\lim_{n \rightarrow \infty} \oint_{\Omega} \nu w_{n,3} q_n \, dx = \int_{\Omega} \nu w_3 p \, dx.$$

To show the strong convergence of  $\{\mathbf{u}_n\}_{n=1}^\infty$  in  $\mathcal{L}^2(\Omega)^2$  we use an extension of a theorem of Kikuchi [17, Thm. 1] which is demonstrated in Appendix A. In our context, we state this theorem as follows.

**THEOREM A.1.** *Suppose that*

- $\{\mathbf{u}_n\}_{n=1}^\infty$  is a bounded sequence in  $\mathcal{H}_0(\text{curl}, \Omega)$ ,
- $\mathbf{u}_n \in \mathcal{R}_{h_n}$  for each  $n$ ,
- $\text{Div}_{\nu, h_n} \mathbf{u}_n = 0$  for each  $n$ .

*Then there is a subsequence of  $\{\mathbf{u}_n\}_{n=1}^\infty$  converging strongly in  $\mathcal{L}^2(\Omega)^2$  and weakly in  $\mathcal{H}(\text{curl}, \Omega)$  to an element  $\mathbf{u}$  in  $\mathcal{H}(\text{curl}, \Omega)$  provided that the following hypothesis holds:*

(HR) *The regularity hypothesis. There is a constant  $\sigma$ ,  $0 < \sigma \leq 1$ , and a positive constant  $C$  such that if  $\mathbf{u} \in \mathcal{H}_0(\text{curl}, \Omega)$  then we may write*

$$\mathbf{u} = \mathbf{w} + \text{grad } q \quad \text{with } \mathbf{w} \in \mathcal{H}^\sigma(\Omega)^2 \quad \text{and } q \in \mathcal{H}_0^1(\Omega)$$

and with the estimate

$$\|\mathbf{w}\|_{\mathcal{H}^\sigma(\Omega)} + \|\text{grad } q\|_{\mathcal{L}^\infty(\Omega)} \leq C \|\mathbf{u}\|_{\mathcal{H}(\text{curl}, \Omega)}.$$

Lemma 4.2 will have been demonstrated once we show the following lemma.

LEMMA 4.3. *If  $\Omega$  is Lipschitz, then the regularity hypothesis (HR) holds.*

*Proof.* We suppose that  $\mathbf{u} \in \mathcal{H}_0(\text{curl}, \Omega)$ . Let  $\psi$  be the unique solution of

$$\begin{aligned} \Delta \psi &= \text{curl } \mathbf{u}, \\ \frac{\partial \psi}{\partial n} &= 0, \end{aligned}$$

and let

$$\mathbf{w} = \text{Curl}^* \psi.$$

By regularity for Lipschitz domains,  $\psi \in \mathcal{H}^1(\Omega)$  and

$$\|\psi\|_{\mathcal{H}^1(\Omega)} \leq \|\text{curl } \mathbf{u}\|_{\mathcal{L}^2(\Omega)}.$$

Further  $\mathbf{w} = \text{Curl}^* \psi \in \mathcal{L}^2(\Omega)^2$ ,  $\text{div } \mathbf{w} = 0$ , and  $\text{curl } \mathbf{w} = \Delta \psi = \text{curl } \mathbf{u} \in \mathcal{L}^2(\Omega)$ . Also,  $\mathbf{w} \times \mathbf{n} = \frac{\partial \psi}{\partial n} = 0$ . Thus by a result of Costabel [9],  $\mathbf{w} \in \mathcal{H}^{1/2}(\Omega)$ , and

$$\begin{aligned} \|\mathbf{w}\|_{\mathcal{H}^{1/2}(\Omega)} &\leq C (\|\mathbf{w}\|_{\mathcal{L}^2(\Omega)^2} + \|\text{div } \mathbf{w}\|_{\mathcal{L}^2(\Omega)} + \|\text{curl } \mathbf{w}\|_{\mathcal{L}^2(\Omega)} + \|\mathbf{w} \times \mathbf{n}\|_{\mathcal{L}^2(\Gamma)^2}) \\ &\leq C (\|\text{Curl}^* \psi\|_{\mathcal{L}^2(\Omega)^2} + \|\text{curl } \mathbf{u}\|_{\mathcal{L}^2(\Omega)}) \\ &\leq 2C \|\text{curl } \mathbf{u}\|_{\mathcal{L}^2(\Omega)}. \end{aligned}$$

To conclude we need to show that the function  $\mathbf{u} - \mathbf{w}$  is the gradient of a function in  $\mathcal{H}_0^1(\Omega)$ . We clearly have  $(\mathbf{u} - \mathbf{w}) \in \mathcal{H}_0(\text{curl}^0, \Omega)$ ; i.e.,

$$\text{curl}(\mathbf{u} - \mathbf{w}) = 0 \quad \text{and} \quad (\mathbf{u} - \mathbf{w}) \wedge \mathbf{n}|_\Gamma = 0.$$

Thus, if  $\Omega$  is simply connected we are done. Otherwise we recall that  $\mathcal{H}_0(\text{curl}^0, \Omega)$  is the space of gradients of functions in  $\mathcal{H}_c^1(\Omega)$ , the space of functions in  $\mathcal{H}^1(\Omega)$  which are constant on each component  $\Gamma_i$  of  $\Gamma$ . Then if  $\Gamma$  has, say,  $N$  components,  $\Gamma_1, \Gamma_2, \dots, \Gamma_N$ ,  $(\mathbf{u} - \mathbf{w}) = \text{grad } q + \sum_i \alpha_i \text{grad } \phi_i$ , where  $q \in \mathcal{H}_0^1(\Omega)$  and  $\phi_i \in \mathcal{H}^1(\Omega)$  is defined by

$$\begin{aligned} \Delta \phi_i &= 0, \\ \phi_i|_{\Gamma_j} &= \delta_{i,j}. \end{aligned}$$

In this case we modify  $\mathbf{w}$ , putting  $\hat{\mathbf{w}} = \mathbf{w} + \sum_i \alpha_i \text{grad } \phi_i$ ; see [10] for details. By elliptic regularity results for the Dirichlet problem, we know that  $\phi_i \in \mathcal{H}^{3/2}(\Omega)$  for each  $i$ ; see [9], and the lemma follows.  $\square$

**4.3. (H2): The consistency result.** The object of this section is to show the following result.

LEMMA 4.4. *For each  $f \in \mathcal{H}$ ,  $\mathcal{T}_{\beta,h} f$  converges to  $\mathcal{T}_\beta f$  in the  $\mathcal{W}$ -norm, i.e.,*

$$\lim_{h \rightarrow 0} \|\mathcal{T}_\beta f - \mathcal{T}_{\beta,h} f\|_{\mathcal{W}} = 0.$$

(The hypothesis (H2) merely requires convergence in the  $\mathcal{H}$ -norm but this is certainly implied by convergence in the  $\mathcal{W}$ -norm.) Before giving a proof of Lemma 4.4 showing pointwise

convergence of  $\mathcal{T}_{\beta,h}$  to  $\mathcal{T}_\beta$ , we would like to make a comment concerning the difficulty for showing this convergence.

*Remark 4.1.* The second lemma of Strang (see [8, Lem. 4.2.2] or [7, Thm. 31.1]) gives us

$$(4.10) \quad \begin{aligned} & \|\mathcal{T}_\beta f - \mathcal{T}_{\beta,h} f\|_{\mathcal{W}} \\ & \leq C \left( \inf_{\tilde{w}_h \in \mathcal{W}_{\beta,h}} \|\mathcal{T}_\beta f - \tilde{w}_h\|_{\mathcal{W}} + \sup_{\{\tilde{w}_h \in \mathcal{W}_{\beta,h}; \|\tilde{w}_h\|_{\mathcal{W}}=1\}} |a(\beta, \mathcal{T}_\beta f, \tilde{w}_h) - (f, \tilde{w}_h)_v| \right). \end{aligned}$$

Thus the error is made up of two terms: the interpolation error,

$$E_{Int} = \inf_{\tilde{w}_h \in \mathcal{W}_{\beta,h}} \|\mathcal{T}_\beta f - \tilde{w}_h\|_{\mathcal{W}},$$

and an error due to the fact that the method is nonconforming,

$$E_{NC} = \sup_{\{\tilde{w}_h \in \mathcal{W}_{\beta,h}; \|\tilde{w}_h\|_{\mathcal{W}}=1\}} |a(\beta, \mathcal{T}_\beta f, \tilde{w}_h) - (f, \tilde{w}_h)_v|.$$

Let us first consider  $E_{NC}$ . We have seen in the proof of Lemma 2.2 that for  $f \in \mathcal{H}_\beta$ , i.e.,  $\text{div}_\beta(\nu f) = 0$ ,

$$a(\beta, \mathcal{T}_\beta f, \tilde{w}) = (f, \tilde{w})_v \quad \text{for each } \tilde{w} \in \mathcal{W}$$

so that  $E_{NC} = 0$ . If  $f \in \mathcal{H}_\beta^\perp$ ; i.e., if  $f = \text{grad}_\beta p$  for some  $p \in \mathcal{H}_0^1(\Omega)$ , then one checks from the definition of  $\mathcal{T}_\beta$  that  $\mathcal{T}_\beta f = 0$  so that

$$E_{NC} = \sup_{\{\tilde{w}_h \in \mathcal{W}_{\beta,h}; \|\tilde{w}_h\|_{\mathcal{W}}=1\}} |(f, \tilde{w}_h)_v|.$$

But

$$(f, \tilde{w}_h)_v = (\text{grad}_\beta p_h, \tilde{w}_h)_v + (\text{grad}_\beta (p - p_h), \tilde{w}_h)$$

for any  $p_h \in \mathcal{P}_h$ ,

$$\begin{aligned} (\text{grad}_\beta p_h, \tilde{w}_h)_v & \leq \left| \oint_{\Omega} \nu \text{Div}_{\nu,h} \mathbf{w}_h p_h \, dx - \int_{\Omega} \nu \text{Div}_{\nu,h} \mathbf{w}_h p_h \, dx \right| \\ & \leq C h \|\text{Div}_{\nu,h} \mathbf{w}_h\|_{\mathcal{L}^2(\Omega)} \|p_h\|_{\mathcal{H}^1(\Omega)} \\ & \leq C h \|\tilde{w}_h\|_{\mathcal{W}} \|p_h\|_{\mathcal{H}^1(\Omega)}, \end{aligned}$$

and

$$(\text{grad}_\beta (p - p_h), \tilde{w}_h) \leq C \|p - p_h\|_{\mathcal{H}^1(\Omega)} \|\tilde{w}_h\|_{\mathcal{L}^2(\Omega)^3}.$$

Taking  $p_h = \Pi_{\mathcal{P}_h} p$ , we see that  $E_{NC}$  tends to 0 with  $h$ . (Moreover, we have  $E_{NC} \leq C h |p|_2$  whenever  $p \in \mathcal{H}^2(\Omega)$ .)

We turn now to  $E_{Int}$ . If  $f \in \mathcal{H}_\beta^\perp$ , then  $E_{Int} = 0$  since  $\mathcal{T}_\beta f = 0$ . However, for  $f \in \mathcal{H}_\beta$  we need an interpolation result. In §3.1 we were able to show that for the case where the family of meshes  $\{\mathcal{T}_h; h \in \Lambda\}$  is asymptotically uniform we have the following interpolation result:

$$\lim_{h \rightarrow 0} \inf_{\tilde{w}_h \in \mathcal{W}_{\beta,h}} \|\mathcal{T}_\beta f - \tilde{w}_h\|_{\mathcal{W}} = 0.$$

Thus  $E_{Int}$  tends to 0 with  $h$  and from Theorem 3.2 we see that it is bounded by  $Ch (|\mathcal{T}_\beta f|_1 + |\text{curl} \mathcal{T}_\beta f|_1 + |\text{div}_v \mathcal{T}_\beta f|_2)$  if  $\mathcal{T}_\beta f$  is sufficiently regular.

Thus if the mesh is asymptotically uniform, we obtain pointwise convergence of  $\mathcal{T}_{\beta,h} f$  toward  $\mathcal{T}_\beta f$  in  $\mathcal{W}$ .

*Remark 4.2.* Note that even with an asymptotically uniform mesh, Strang's lemma does not give strong convergence since, if  $f \in \mathcal{H}_\beta^\perp$  for instance, we obtain ( $f = \text{grad} p$ )

$$\|\mathcal{T}_\beta f - \mathcal{T}_{\beta,h} f\| \leq Ch |p|_{2,\Omega},$$

while we would like to have

$$\|\mathcal{T}_\beta f - \mathcal{T}_{\beta,h} f\| \leq o(h) |\text{grad} p|_{\mathcal{L}^2(\Omega)^2}.$$

Although Strang's lemma provides the desired convergence only for asymptotically uniform meshes, our numerical results (cf. [15]) indicate that this restriction on the meshes is not necessary. The difficulty in the case of the nonasymptotically uniform mesh is caused by the approximate, divergence-free constraint on the approximation space  $\mathcal{W}_{\beta,h}$ . We do not encounter this difficulty in approximating  $\mathcal{W}$  by  $\mathcal{W}_h$ . One might hope to control this error more easily by imposing the constraint via Lagrange multipliers.

This leads us to introduce a mixed formulation of the problem. Consider the problem

$$(4.11) \quad \begin{aligned} & \text{find } (\tilde{\mathbf{u}}, p) \in \mathcal{W} \times \mathcal{H}_0^1(\Omega) \text{ such that} \\ & a(\beta, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + b(\tilde{\mathbf{v}}, p) = (f, \tilde{\mathbf{v}})_v \text{ for each } \tilde{\mathbf{v}} \in \mathcal{W}, \\ & b(\tilde{\mathbf{u}}, q) = 0 \text{ for each } q \in \mathcal{H}_0^1(\Omega), \end{aligned}$$

where  $b(\cdot, \cdot)$  is the bilinear form on  $\mathcal{W} \times \mathcal{H}_0^1(\Omega)$  defined by

$$(4.12) \quad b(\tilde{\mathbf{v}}, q) = (\tilde{\mathbf{v}}, \text{grad}_\beta q)_v \quad \text{for each } (\tilde{\mathbf{v}}, q) \in \mathcal{W} \times \mathcal{H}_0^1(\Omega).$$

Problem (4.11) has a unique solution [6] since  $a(\beta, \cdot, \cdot)$  is elliptic on  $\mathcal{W}_\beta$  (Lemma 2.2), and  $b(\cdot, \cdot)$  satisfies the inf-sup condition

$$\inf_{\{q \in \mathcal{H}_0^1(\Omega); \|\mathcal{H}_0^1(q)\| = 1\}} \sup_{\{\tilde{\mathbf{v}} \in \mathcal{W}; \|\tilde{\mathbf{v}}\|_{\mathcal{W}} = 1\}} b(\tilde{\mathbf{v}}, q) > 0.$$

(Simply take  $\tilde{\mathbf{v}} = \text{grad}_\beta q / \|\text{grad}_\beta q\|_{\mathcal{W}}$ .)

If we denote by  $(\tilde{\mathbf{u}}_f, p_f)$  the solution  $(\tilde{\mathbf{u}}, p)$  of (4.11), it is easy to check that

$$(4.13) \quad \tilde{\mathbf{u}}_f = \mathcal{T}_\beta f$$

and that  $p_f$  is the solution  $p$  of

$$(4.14) \quad \begin{aligned} & \text{find } p \in \mathcal{H}_0^1(\Omega) \text{ such that} \\ & b(\text{grad}_\beta \varphi, p) = (f, \text{grad}_\beta \varphi)_v \text{ for each } \varphi \in \mathcal{H}_0^1(\Omega). \end{aligned}$$

Note that (4.14) has a unique solution for  $f \in \mathcal{H}$  since  $b(\text{grad}_\beta \cdot, \cdot)$  is elliptic on  $\mathcal{H}_0^1(\Omega)$ .

We point out here that if  $f \in \mathcal{H}_\beta$ , i.e., if  $\text{div}_\beta(\nu f) = 0$ , then  $p_f = 0$ . So, with (4.11) and (4.13) we obtain that if  $f \in \mathcal{H}_\beta$  then

$$(4.15) \quad a(\beta, \mathcal{T}_\beta f, \tilde{\mathbf{v}}) = (f, \tilde{\mathbf{v}})_v \quad \text{for each } \tilde{\mathbf{v}} \in \mathcal{W}.$$

We also introduce the mixed version of the discretized problem:

$$(4.16) \quad \begin{aligned} &\text{Find } (\tilde{\mathbf{u}}_h, p_h) \in \mathcal{W}_h \times \mathcal{P}_h \text{ such that} \\ &a(\beta, \tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}_h) + b_h(\tilde{\mathbf{v}}_h, p_h) = (f, \tilde{\mathbf{v}}_h)_v \text{ for each } \tilde{\mathbf{v}}_h \in \mathcal{W}_h, \\ &b_h(\tilde{\mathbf{u}}_h, q_h) = 0 \text{ for each } q_h \in \mathcal{P}_h, \end{aligned}$$

where

$$\mathcal{W}_h = \mathcal{R}_h \times \mathcal{P}_h$$

and  $b_h(\cdot, \cdot)$  is the bilinear form on  $\mathcal{W}_h \times \mathcal{P}_h$  defined by

$$(4.17) \quad b_h(\tilde{\mathbf{v}}_h, q_h) = \int_{\Omega} v \mathbf{v}_h \operatorname{grad} q_h dx + \beta \oint_{\Omega} v v_{h,3} q_h dx$$

for each  $(\tilde{\mathbf{v}}_h, q_h) \in \mathcal{W}_h \times \mathcal{P}_h$ .

We have seen that  $a(\beta, \cdot, \cdot)$  is elliptic on  $\mathcal{W}_{\beta,h}$  with a constant of ellipticity independent of  $h$ , and it is not difficult to check that the bilinear forms  $b_h(\cdot, \cdot)$ ,  $h \in \Lambda$ , satisfy the uniform inf-sup condition

$$\inf_{h \in \Lambda} \inf_{\{q_h \in \mathcal{P}_h; \|q_h\|_{\mathcal{T}^1(\Omega)}=1\}} \sup_{\{\tilde{\mathbf{v}}_h \in \mathcal{W}_h; \|\tilde{\mathbf{v}}_h\|_{\mathcal{W}}=1\}} b_h(\tilde{\mathbf{v}}_h, q_h) > 0.$$

(Take  $\tilde{\mathbf{v}}_h = \operatorname{grad}_{\beta} q_h / \|\operatorname{grad}_{\beta} q_h\|_{\mathcal{W}}$ .) Thus we know that there is a unique solution  $(\tilde{\mathbf{u}}_{h,f}, p_{h,f})$  of (4.16), and as before one easily checks that

$$(4.18) \quad \tilde{\mathbf{u}}_{h,f} = \mathcal{T}_{\beta,h} f$$

and  $p_{h,f}$  is the unique solution  $p_h$  of

$$(4.19) \quad \begin{aligned} &\text{find } p_h \in \mathcal{P}_h \text{ such that} \\ &b_h(\operatorname{grad}_{\beta} \varphi_h, p_h) = (f, \operatorname{grad}_{\beta} \varphi_h)_v \text{ for each } \varphi_h \in \mathcal{P}_h. \end{aligned}$$

Just as in the continuous case we see that if  $f \in \mathcal{H}_{\beta}$ , then  $p_{h,f} = 0$  so

$$(4.20) \quad a(\beta, \mathcal{T}_{\beta,h} f, \tilde{\mathbf{v}}_h) = (f, \tilde{\mathbf{v}}_h)_v \quad \text{for each } \tilde{\mathbf{v}}_h \in \mathcal{W}_h.$$

The following estimate is the simplification to our case of an analogue for mixed methods of Strang's second lemma (see [24, Thms. 11.1, 11.2])<sup>1</sup>:

$$(4.21) \quad \begin{aligned} &\|\tilde{\mathbf{u}}_f - \tilde{\mathbf{u}}_{h,f}\|_{\mathcal{W}} + \|p_f - p_{h,f}\|_{\mathcal{T}^1(\Omega)} \\ &\leq C \left( \inf_{\tilde{\mathbf{v}}_h \in \mathcal{W}_h} \left\{ \|\tilde{\mathbf{u}}_f - \tilde{\mathbf{v}}_h\|_{\mathcal{W}} + \sup_{\{q_h \in \mathcal{P}_h; \|q_h\|_{\mathcal{T}^1(\Omega)}=1\}} |(b - b_h)(\tilde{\mathbf{v}}_h, q_h)| \right\} \right. \\ &\quad \left. + \inf_{q_h \in \mathcal{P}_h} \left\{ \|p_f - q_h\|_{\mathcal{T}^1(\Omega)} + \sup_{\{\tilde{\mathbf{v}}_h \in \mathcal{W}_h; \|\tilde{\mathbf{v}}_h\|_{\mathcal{W}}=1\}} |(b - b_h)(\tilde{\mathbf{v}}_h, q_h)| \right\} \right), \end{aligned}$$

where  $C$  is independent of  $h$  because the uniform inf-sup condition is satisfied. Then in the case where  $f \in \mathcal{H}_{\beta}$ , this estimate becomes

$$(4.22) \quad \begin{aligned} &\|\mathcal{T}_{\beta} f - \mathcal{T}_{\beta,h} f\|_{\mathcal{W}} \\ &\leq C \inf_{\tilde{\mathbf{v}}_h \in \mathcal{W}_h} \left\{ \|\mathcal{T}_{\beta} f - \tilde{\mathbf{v}}_h\|_{\mathcal{W}} + \sup_{\{q_h \in \mathcal{P}_h; \|q_h\|_{\mathcal{P}_h}=1\}} |(b - b_h)(\tilde{\mathbf{v}}_h, q_h)| \right\} \end{aligned}$$

since in this case  $p_f = 0$ .

<sup>1</sup>The first sup term in the right-hand side of (4.21) is missing in [24, Thm. 11.2]. To check this simply apply the instructions in [24] for obtaining Theorem 11.2 from Theorem 11.1.

*Proof of Lemma 4.4.* It follows from (3.11) and (3.8) that

$$\lim_{h \rightarrow 0} \inf_{\tilde{v}_h \in \mathcal{W}_h} \|\tilde{u}_f - \tilde{v}_h\|_{\mathcal{W}} = 0$$

and

$$\lim_{h \rightarrow 0} \inf_{q_h \in \mathcal{P}_h} \|p_f - q_h\|_{\mathcal{T}^1(\Omega)} = 0.$$

Since, for  $\tilde{v}_h = (v_h, v_{h,3}) \in \mathcal{W}_h$  and  $q_h \in \mathcal{P}_h$ ,

$$(b - b_h)(\tilde{v}_h, q_h) = \beta \left( \int_{\Omega} v v_{h,3} q_h dx - \oint_{\Omega} v v_{h,3} q_h dx \right),$$

we have by (3.13) that

$$\sup_{\{\tilde{v}_h \in \mathcal{W}_h; \|\tilde{v}_h\|_{\mathcal{W}}=1\}} |(b - b_h)(\tilde{v}_h, q_h)| \leq C h \|q_h\|_{\mathcal{L}^2(\Omega)}$$

and

$$\sup_{\{q_h \in \mathcal{P}_h; \|q_h\|_{\mathcal{T}^1(\Omega)}=1\}} |(b - b_h)(\tilde{v}_h, q_h)| \leq C h \|v_{h,3}\|_{\mathcal{L}^2(\Omega)}.$$

The lemma now follows.  $\square$

**4.4. Error estimates.** Now that (H1) and (H2) have been established we can use (4.6) and (4.7) to obtain estimates of the error made in the approximation of the eigenvalues and the eigenvectors of  $\mathcal{T}_\beta$  by calculating those of  $\mathcal{T}_{\beta,h}$ . Thus we need to estimate  $\|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})|_{E(\mu)}\|$  and  $\sum_{i,j=1}^m |((\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{\varphi}_i, \tilde{\varphi}_j)|$ . First, we enounce our main result.

**THEOREM 4.3.** *For  $\mu$  an eigenvalue of  $\mathcal{T}_\beta$  of multiplicity  $m$  and  $\mu_{h,i}$  an eigenvalue of  $\mathcal{T}_{\beta,h}$  approximating  $\mu$  guaranteed by the theory of Anselone, we have*

$$(4.23) \quad \lim_{h \rightarrow 0} |\mu - \mu_{h,i}| = 0$$

and

$$(4.24) \quad \lim_{h \rightarrow 0} \delta(E(\mu), E_h(\mu)) = 0$$

for sufficiently small  $h$ . Further whenever we also have  $E(\mu)$  contained in  $\mathcal{H}^1(\text{curl}, \Omega) \times \mathcal{H}^2(\Omega)$ , then

$$(4.25) \quad |\mu - \mu_{h,i}| \leq C h^2$$

and

$$(4.26) \quad \delta(E(\mu), E_h(\mu)) \leq C h.$$

*Proof.* First we note that since  $E(\mu)$  is of finite dimension, (H2), i.e., Lemma 4.4, implies that

$$\lim_{h \rightarrow 0} \|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})|_{E(\mu)}\| = 0.$$

Then to estimate  $\|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})|_{E(\mu)}\|$ , we observe that  $\tilde{\varphi} \in E(\mu)$  implies that  $\tilde{\varphi} \in \mathcal{W}_\beta \subset \mathcal{H}_\beta$ , and we have from (4.22) and the fact that  $\mathcal{T}_\beta \tilde{\varphi} = \mu \tilde{\varphi}$  that

$$\|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{\varphi}\|_{\mathcal{W}} \leq C \mu \inf_{\tilde{v}_h \in \mathcal{W}_h} \left\{ \|\tilde{\varphi} - \tilde{v}_h\|_{\mathcal{W}} + \sup_{q_h \in \mathcal{P}_h; \|q_h\|_{1,\Omega}=1} |(b - b_h)(\tilde{v}_h, q_h)| \right\}.$$

We suppose that  $\tilde{\varphi} \in \mathcal{H}^1(\text{curl}, \Omega) \times \mathcal{H}^2(\Omega)$ . Using (3.13), the estimation of the error due to the numerical integration scheme, we obtain

$$\|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{\varphi}\|_{\mathcal{W}} \leq C \mu \inf_{\tilde{v}_h \in \mathcal{W}_h} \{ \|\tilde{\varphi} - \tilde{v}_h\|_{\mathcal{W}} + h \nu_{\max} |v_{3h}|_{\mathcal{L}^\infty(\otimes)} \}.$$

With  $\tilde{v}_h = (\Pi_{\mathcal{R}_h} \varphi, \Pi_{\mathcal{P}_h} \varphi_3)$ , the interpolation estimates (3.8) and (3.11) and the continuity of the operator  $\Pi_{\mathcal{P}_h}$  from  $\mathcal{L}^2(\Omega)$  to  $\mathcal{L}^2(\Omega)$  give

$$(4.27) \quad \|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{\varphi}\|_{\mathcal{W}} \leq C \mu h (|\varphi|_{1,\Omega} + |\text{curl } \varphi|_{1,\Omega} + \|\varphi_3\|_{2,\Omega}).$$

Now since  $E(\mu)$  is finite dimensional, we have

$$(4.28) \quad \|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})|_{E(\mu)}\|_{\mathcal{W}} \leq C_\mu h,$$

which gives the result (4.26).

To estimate  $\sum_{i,j=1}^m |((\mathcal{T}_\beta - \mathcal{T}_{\beta,h}) \tilde{\varphi}_i, \tilde{\varphi}_j)|$ , where  $\{\tilde{\varphi}_i\}_{i=1}^m$  is a basis for  $E(\mu)$ , we remark that for  $\tilde{u} \in \mathcal{W}_\beta$  and  $\tilde{v} \in \mathcal{W}$ , the symmetry of the bilinear forms  $(\cdot, \cdot)_\nu$  and  $a(\beta, \cdot, \cdot)$  together with (4.15) and (4.20) implies that

$$\begin{aligned} |((\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{u}, \tilde{v})_\nu| &= |a((\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{u}, \mathcal{T}_\beta \tilde{v})| \\ &= |a((\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{u}, \mathcal{T}_\beta \tilde{v} - \tilde{v}_h)| \quad \text{for each } \tilde{v}_h \in \mathcal{W}_h \\ &\leq C \|(\mathcal{T}_\beta - \mathcal{T}_{\beta,h})\tilde{u}\|_{\mathcal{W}} \inf_{\tilde{v}_h \in \mathcal{W}_h} \|\mathcal{T}_\beta \tilde{v} - \tilde{v}_h\|_{\mathcal{W}}. \end{aligned}$$

Combining once again (3.8) and (3.11) and using (4.27), we get

$$\begin{aligned} &|((\mathcal{T}_\beta - \mathcal{T}_{\beta,h}) \tilde{\varphi}_i, \tilde{\varphi}_j)| \\ &\leq C \mu^2 h^2 (|\varphi_i|_{1,\Omega} + |\text{curl } \varphi_i|_{1,\Omega} + \|\varphi_{i3}\|_{2,\Omega}) (|\varphi_j|_{1,\Omega} + |\text{curl } \varphi_j|_{1,\Omega} + \|\varphi_{j3}\|_{2,\Omega}); \end{aligned}$$

that is,

$$\sum_{i,j=1}^m |((\mathcal{T}_\beta - \mathcal{T}_{\beta,h}) \tilde{\varphi}_i, \tilde{\varphi}_j)| \leq C_\mu h^2,$$

which concludes the proof of (4.25).  $\square$

*Remark 4.3.* Indeed  $E(\mu)$  is contained in  $\mathcal{H}^1(\text{curl}, \Omega) \times \mathcal{H}^2(\Omega)$  if  $\mathcal{T}_\beta$  maps  $\mathcal{L}^2(\Omega)$  into  $\mathcal{H}^1(\text{curl}, \Omega) \times \mathcal{H}^2(\Omega)$  which corresponds to a regularity result for Maxwell's equations; cf. [26].

*Remark 4.4.* Of course the estimates (4.25) and (4.26) still hold if the eigenfunctions  $\tilde{\varphi}_i$  are only piecewise regular in the sense made precise in Remarks 3.1 and 3.2. This remark is not purely academic since  $\tilde{\varphi}_i$  will be only piecewise regular if  $\epsilon$  and  $\mu$  are only piecewise regular.

*Remark 4.5.* Numerical results (cf. [15]) show that our estimates are optimal.

**Appendix A: An extension of a theorem of Kikuchi.** In this appendix we give a proof of the extension of the theorem of Kikuchi [17] used in the demonstration of the compactness result in §3.2. The extension to the case of variable coefficients is actually straightforward, but because of the conciseness of the proof in [17] we thought it useful to give some of the details.

We recall that  $\Omega$  is a bounded domain in  $\mathbb{R}^2$  and we suppose further that  $\Omega$  satisfies the following regularity hypothesis.

(HR) *The regularity hypothesis.* There is a constant  $\sigma$ ,  $0 < \sigma \leq 1$ , and a positive constant  $C$  such that if  $\mathbf{u} \in \mathcal{H}_0(\text{curl}, \Omega)$  then we may write

$$\mathbf{u} = \mathbf{w} + \text{grad } q \quad \text{with } \mathbf{w} \in \mathcal{H}^\sigma(\Omega)^2 \quad \text{and } q \in \mathcal{H}_0^1(\Omega)$$

and with the estimate

$$\|\mathbf{w}\|_{\mathcal{H}^\sigma(\Omega)^2} + \|\text{grad } q\|_{\mathcal{L}^2(\Omega)^2} \leq C \|\mathbf{u}\|_{\mathcal{H}(\text{curl}, \Omega)}.$$

**THEOREM A.1.** *Suppose that  $\Omega$  satisfies the regularity hypothesis (HR). Suppose that*

- $\{\mathbf{u}_n\}_{n=1}^\infty$  is a bounded sequence in  $\mathcal{H}_0(\text{curl}, \Omega)$ ,
- $\mathbf{u}_n \in \mathcal{R}_{h_n}$  for each  $n$ ,
- $\text{Div}_{\nu, h_n} \mathbf{u}_n = 0$  for each  $n$ .

*Then there is a subsequence of  $\{\mathbf{u}_n\}_{n=1}^\infty$  converging strongly in  $\mathcal{L}^2(\Omega)^2$  and weakly in  $\mathcal{H}(\text{curl}, \Omega)$  to an element  $\mathbf{u}$  in  $\mathcal{H}(\text{curl}, \Omega)$ .*

*Proof.* The classical Helmholtz decomposition theorem guarantees that  $\mathbf{u}_n$  may be decomposed as follows:

$$\mathbf{u}_n = \mathbf{v}^n \oplus \text{grad } p^n,^2$$

where  $\mathbf{v}^n \in \mathcal{H}(\text{div}_\nu 0, \Omega)$ ,  $p^n \in \mathcal{H}_0^1(\Omega)$ , and

$$\mathcal{H}(\text{div}_\nu 0, \Omega) = \{\mathbf{w} \in \mathcal{H}(\text{div}_\nu, \Omega) : \text{div}(\nu \mathbf{w}) = 0\}.$$

We have immediately that

$$\begin{aligned} \|\mathbf{u}_n\|_{\mathcal{L}^2(\Omega)_\nu^2}^2 &= \|\mathbf{v}^n\|_{\mathcal{L}^2(\Omega)_\nu^2}^2 + \|\text{grad } p^n\|_{\mathcal{L}^2(\Omega)_\nu^2}^2, \\ \text{curl } \mathbf{u}_n &= \text{curl } \mathbf{v}^n, \\ \mathbf{v}^n \wedge \mathbf{n} |_\Gamma &= 0. \end{aligned}$$

That  $\{\mathbf{v}^n\}_{n=1}^\infty$  has a subsequence converging strongly in  $\mathcal{L}^2(\Omega)$  now follows from the compactness result of Weber [26] used in §2. That the corresponding subsequence of  $\{p^n\}_{n=1}^\infty$  has a further subsequence converging strongly in  $\mathcal{L}^2(\Omega)$ , weakly in  $\mathcal{H}^1(\Omega)$ , follows from Rellich's lemma. The remainder of the demonstration is thus to show that this subsequence converges strongly in  $\mathcal{H}^1(\Omega)$ .

The first thing we show is that the limit  $p$  of  $\{p^n\}_{n=1}^\infty$  is in fact 0. We started with  $\mathbf{u}_n$  of 0 discrete divergence  $\nu$ . We subtract the part  $\mathbf{v}^n$  of 0 divergence  $\nu$ . Since as  $n$  increases, i.e., as  $h_n \rightarrow 0$ , the discrete divergence  $\nu$  should approach the divergence  $\nu$ , the remaining part of  $\mathbf{u}_n$ , i.e.,  $\text{grad } p^n$ , should approach 0. Thus we claim that  $\text{grad } p$ , and hence of course  $p$ , is in fact 0.

<sup>2</sup>Subscripts indicate that the element belongs to the corresponding finite-dimensional subspace; superscripts do not.

To show this we first remark that because  $\text{Div}_{v,h_n} \mathbf{u}_n = 0$  and  $\text{Div}_{v,h_n} \mathbf{v}^n = 0$ ,<sup>3</sup> we also have  $\text{Div}_{v,h_n} \text{grad } p^n = 0$ . Then we take a sequence  $\{\xi_n\}_{n=1}^\infty$ , with  $\xi_n \in \mathcal{P}_{h_n}$ , converging strongly in  $\mathcal{H}^1(\Omega)$  to  $p$ . Now we have

$$(\text{grad } p^n, \text{grad } \xi_n)_v \rightarrow (\text{grad } p, \text{grad } p)_v$$

since  $\{\xi_n\}_{n=1}^\infty$  converges strongly in  $\mathcal{H}^1(\Omega)$  to  $p$ . We also have

$$(\text{grad } p^n, \text{grad } \xi_n)_v = 0$$

since  $\text{Div}_{v,h_n} \text{grad } p^n = 0$  for each  $n$ . Thus  $\text{grad } p = 0$  and also  $p = 0$  since  $p \in \mathcal{H}_0^1(\Omega)$ .

We still must show, and this is the most delicate part of the demonstration, that  $\|\text{grad } p^n\|_{\mathcal{L}^2(\Omega)_v^2}$  converges to 0. This is where we shall make use of the regularity hypothesis as well as the as-yet-unused hypothesis that  $\mathbf{u}_n \in \mathcal{R}_{h_n}$ .

We have

$$\begin{aligned} \|\text{grad } p^n\|_{\mathcal{L}^2(\Omega)_v^2}^2 &= (\text{grad } p^n, \mathbf{u}_n - \mathbf{v}^n)_v \\ &= (\text{grad } p^n, \mathbf{u}_n)_v \\ &= (\text{grad } p^n - \text{grad } \phi_n, \mathbf{u}_n)_v \\ &\leq C \|\text{grad } p^n - \text{grad } \phi_n\|_{\mathcal{L}^2(\Omega)_v^2} \end{aligned}$$

for any  $\phi_n \in \mathcal{P}_{h_n}$ . We now must choose  $\phi_n$ . (There is, of course, a sequence  $\{p_n\}_{n=1}^\infty$  with  $p_n \in \mathcal{P}_{h_n}$  converging strongly to  $p$  in  $\mathcal{H}^1(\Omega)$ . But, we do not have that  $\|p^n - p_n\|$  converges to 0 since we do not know that  $p^n$  converges strongly to  $p$  in  $\mathcal{H}^1(\Omega)$ . This is precisely what we are trying to show.)

By the regularity hypothesis we may write

$$\mathbf{v}^n = \mathbf{z}^n + \text{grad } q^n$$

with  $\mathbf{z}^n \in \mathcal{H}^\sigma(\Omega)^2$ ,  $q^n \in \mathcal{H}_0^1(\Omega)$ ,

$$\|\mathbf{z}^n\|_{\mathcal{H}^\sigma(\Omega)^2} \leq C \|\mathbf{v}^n\|_{\mathcal{H}(\text{curl}, \Omega)} \leq C \|\mathbf{u}_n\|_{\mathcal{H}(\text{curl}, \Omega)},$$

and

$$\|\text{grad } q^n\|_{\mathcal{L}^2(\Omega)_v^2} \leq C \|\mathbf{v}^n\|_{\mathcal{H}(\text{curl}, \Omega)} \leq C \|\mathbf{u}_n\|_{\mathcal{H}(\text{curl}, \Omega)},$$

which implies that  $\mathbf{z}^n$ , up to extractions of subsequences, converges weakly in  $\mathcal{H}^\sigma(\Omega)$  and thus strongly in  $\mathcal{L}^2(\Omega)^2$  to some  $\mathbf{z} \in \mathcal{H}^\sigma(\Omega)$  and thus  $q^n$  converges weakly in  $\mathcal{H}^1(\Omega)$  and thus strongly in  $\mathcal{L}^2(\Omega)$  to some  $q \in \mathcal{H}_0^1(\Omega)$ . But, as  $\mathbf{v}^n$  and  $\mathbf{z}^n$  converge strongly in  $\mathcal{L}^2(\Omega)$ , so does  $\text{grad } q^n$ . Thus  $q^n$  converges strongly in  $\mathcal{H}^1(\Omega)$ .

This permits us to write

$$\text{grad } p^n = \mathbf{u}_n - \mathbf{z}^n - \text{grad } q^n$$

and to obtain

$$\|\text{grad } p^n\|_{\mathcal{L}^2(\Omega)_v^2}^2 \leq C \|\mathbf{u}_n - \mathbf{z}^n - \text{grad } q^n - \text{grad } \phi_n\|_{\mathcal{L}^2(\Omega)_v^2}.$$

What have we gained over the previous situation

$$\begin{aligned} \text{grad } p^n &= \mathbf{u}_n - \mathbf{v}^n, \\ \|\text{grad } p^n\|_{\mathcal{L}^2(\Omega)_v^2}^2 &\leq C \|\mathbf{u}_n - \mathbf{v}^n - \text{grad } \phi_n\|_{\mathcal{L}^2(\Omega)_v^2}^2 \end{aligned}$$

---

<sup>3</sup> $\text{Div}_{v,h_n}$  is extended to an operator on  $\mathcal{L}^2(\Omega)$  in the obvious way; cf. (3.4).

First,  $q^n$  converges strongly to  $q$  in  $\mathcal{H}^1(\Omega)$  and we can take a sequence  $\{q_n\}_{n=1}^\infty$  with  $q_n \in \mathcal{P}_{h_n}$ , converging strongly to  $q$  in  $\mathcal{H}^1(\Omega)$ . Then we have  $\|q^n - q_n\|$  converges to 0 since  $q^n$  and  $q_n$  converge strongly to  $q$  in  $\mathcal{H}^1(\Omega)$ .

Second, to approach  $\mathbf{u}_n - \mathbf{z}^n$  we write it as the sum  $\text{grad}(p^n + q^n)$  to which we apply the following lemma of Kikuchi [17, Lem. 4].

LEMMA A.1. *Suppose that  $\rho \in \mathcal{H}^1(\Omega)$  and that, for each  $K \in \mathcal{T}_h$ ,  $\rho|_K \in \mathcal{H}^{1+\sigma}(K)$ . Then  $\rho \in \mathcal{C}(\bar{\Omega})$  and*

$$\|\text{grad}(\pi_h \rho - \rho)\|_{\mathcal{L}^2(\Omega)^2} \leq C h^\sigma \left\{ \sum_{K \in \mathcal{T}_h} \|\text{grad} \rho\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2},$$

where  $\pi_h$  is the interpolation operator from  $\mathcal{C}(\bar{\Omega})$  onto  $\mathcal{P}_h$  which assigns to a function in  $\mathcal{C}(\bar{\Omega})$  the function in  $\mathcal{P}_h$  which agrees with it on the vertices of the elements of  $\mathcal{T}_h$ .

To see that  $\rho = p^n + q^n$  satisfies the hypothesis of Lemma A.1 it is sufficient to check that  $\text{grad}(p^n + q^n)|_K \in \mathcal{H}^\sigma(K)^2$  for each  $K \in \mathcal{T}_h$  since  $p^n + q^n \in \mathcal{H}^1(\Omega)$ . But,  $\text{grad}(p^n + q^n) = \mathbf{u}_n - \mathbf{z}^n$ , and, for each  $K$ ,  $\mathbf{z}^n \in \mathcal{H}^\sigma(K)^2$  by the regularity hypothesis (HR) and  $\mathbf{u}_n \in \mathcal{H}^\sigma(K)^2$  because it is a polynomial.

We obtain

$$\|\mathbf{u}_n - \mathbf{z}^n - \text{grad} \pi_h(p^n + q^n)\| \leq C h^\sigma \left\{ \sum_K \|\mathbf{u}_n - \mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2}.$$

Then taking

$$\phi_n = \pi_h(p^n + q^n) - q_n,$$

we have

$$\|\text{grad} p^n\|^2 \leq C \left\{ h^\sigma \left\{ \sum_K \|\mathbf{u}_n - \mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2} + \|\text{grad}(q^n - q_n)\| \right\}.$$

As noted above,  $\|\text{grad}(q^n - q_n)\|$  converges to 0. Thus to conclude we need to show that

$$\left\{ \sum_K \|\mathbf{u}_n - \mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2}$$

remains bounded as  $n$  increases.

For each  $K \in \mathcal{T}_h$ ,

$$\|\mathbf{u}_n - \mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \leq 2\|\mathbf{u}_n\|_{\mathcal{H}^\sigma(K)^2}^2 + 2\|\mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2.$$

Moreover, using (HR) and an additivity property of the  $\mathcal{H}^\sigma$ -norm (which can be proved by interpolation), we obtain

$$\left\{ \sum_K \|\mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2} \leq \|\mathbf{z}^n\|_{\mathcal{H}^\sigma(\Omega)^2} \leq C \|\mathbf{v}^n\|_{\mathcal{H}(\text{curl}, \Omega)} \leq C \|\mathbf{u}_n\|_{\mathcal{H}(\text{curl}, \Omega)}.$$

For the term in  $\mathbf{u}_n$ , we need another lemma of Kikuchi [17, Lem. 5].

LEMMA A.2. Let  $\chi \in \mathcal{R}_h$  and let  $K \in \mathcal{T}_h$ . Then  $\chi|_K$  belongs to  $\mathcal{H}^1(K)^2 \subset \mathcal{H}^\sigma(K)^2$ ,  $0 \leq \sigma \leq 1$ , as well as to  $\mathcal{H}(\text{curl}, K)$  and satisfies

$$\begin{aligned} \|\text{grad } \chi|_K\|_{\mathcal{L}^2(\Omega)^4} &= 2^{-1/2} \|\text{curl } \chi|_K\|_{\mathcal{L}^2(\Omega)}, \\ \|\chi|_K\|_{\mathcal{H}^\sigma(K)^2} &\leq C \|\chi|_K\|_{\mathcal{H}^1(K)^2}. \end{aligned}$$

Thus we have

$$\begin{aligned} \|\mathbf{u}_n\|_{\mathcal{H}^\sigma(K)^2}^2 &\leq \|\mathbf{u}_n\|_{\mathcal{H}^1(K)^2}^2 \\ &\leq C (\|\mathbf{u}_n\|_{\mathcal{L}^2(K)^2}^2 + \|\text{curl } \mathbf{u}_n\|_{\mathcal{L}^2(K)}^2) \end{aligned}$$

and

$$\begin{aligned} \left\{ \sum_K \|\mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2 \right\}^{1/2} &\leq C (\|\mathbf{u}_n\|_{\mathcal{L}^2(\Omega)^2}^2 + \|\text{curl } \mathbf{u}_n\|_{\mathcal{L}^2(\Omega)}^2) \\ &\leq C \|\mathbf{u}_n\|_{\mathcal{H}(\text{curl}, \Omega)}^2. \end{aligned}$$

Thus, since the sequence  $\{\mathbf{u}_n\}_{n=1}^\infty$  is a bounded sequence in  $\mathcal{H}_0(\text{curl}, \Omega)$ ,  $\{\sum_K \|\mathbf{u}_n - \mathbf{z}^n\|_{\mathcal{H}^\sigma(K)^2}^2\}^{1/2}$  remains bounded as  $n$  increases and  $\|\text{grad } p^n\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . The proof is now complete.  $\square$

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