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A parallel iterative procedure applicable to the approximate solution of second order partial differential equations by mixed finite element methods*

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Summary. A parallelizable iterative procedure based on domain decomposition techniques is defined and analyzed for mixed finite element methods for elliptic equations, with the analysis being presented for the decomposition of the domain into the individual elements associated with the mixed method or into larger subdomains. Applications to time-dependent problems are indicated.

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1. Introduction

Our objective is to discuss an iterative procedure related to domain decomposition techniques based on the use of subdomains as small as individual elements for mixed finite element approximations to second order partial differential equations in two or three space variables. Analogous techniques apply in an almost unaltered fashion when larger subdomains are employed; however, the discussion below will be concentrated on the case in which the subdomains are elements. The iterative technique applies directly to coercive elliptic problems and provides a time-stepping procedure for implicit methods for parabolic or hyperbolic equations. The motivation for the procedure is that it can be very naturally and easily implemented on a massively parallel computer by assigning each subdomain (i.e., each element) to its own processor.

Our iterative procedure is very closely related to and based on one introduced by Després [9] for a Helmholz problem and extended to another Helmholz-like problem related to Maxwell's equations by Després et al. [10, 11]. As in these references, we shall make very strong use of the hybridization of mixed finite element methods introduced by Fraeijs de Veubeke [19, 20] more than twenty-five

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years ago and analyzed very carefully by Arnold and Brezzi [1]; see also [3, 5, 6]. The convergence proofs in [9, 10, 11] are given for the differential problems in strong form; only numerical results are presented to validate the iterative procedures for the discrete case in these papers. Another related procedure, applicable to a Helmholz-like problem in elasticity, has been introduced by Feng and Bennethum [18].

The elliptic case will be treated in detail first, since the time-stepping applications are essentially corollaries of the results in the elliptic case. While the practical goal is the treatment of mixed finite element methods for the elliptic problem, the domain decomposition procedure can be considered at the differential level and the iteration applied to a mixed formulation of the differential problem. The convergence proof for the iteration covers the discrete case rigorously; but, since there is a technical difficulty arising from the nonlocal nature of the Sobolev space of order $-\frac{1}{2}$ on the boundary of a subdomain, the proof would be only heuristic for the mixed differential case. Our proof of convergence would also be valid for the strong form of our coercive differential case; however, Després [9] has already indicated this argument. The analysis would also cover a collection of cell-centered finite difference methods and finite volume methods.

Parabolic and hyperbolic problems will be treated after the elliptic problems.

Different domain decomposition procedures for mixed finite element approximations have been considered by Cowser, Ewing, Glowinski, Kinton, Wang, and Wheeler (see [8, 16, 17, 21, 22]).

An outline of the paper is as follows. In Sects. 2 and 3 the domain decomposition is defined and a mixed formulation of the differential problem is recalled; then, the iterative procedure is illustrated for the differential problem. In Sect. 4 the mixed finite element procedure is introduced, the corresponding iteration defined, and a convergence argument given under minimal hypotheses on the partition into subdomains. In Sect. 5 it is shown that the spectral radius of the iterator for the mixed finite element procedure is less than one; in the next section, we show that this spectral radius has a bound of the form $1 - ch$ for quasiregular partitions. If, instead, the decomposition of the domain is fixed and the partition for the finite element procedure is compatible with the decomposition, then this bound is improved to $1 - c\sqrt{h}$. The final section contains a brief treatment of the very effective application of this iterative procedure to time-dependent problems.

2. The domain decomposition

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a bounded domain with a Lipschitz boundary $\partial\Omega$. Let $\{\Omega_j, j = 1, \ldots, M\}$ be a partition of Ω:

$$\bar{\Omega} = \bigcup_{j=1}^{M} \bar{\Omega}_j : \qquad \Omega_j \cap \Omega_k = \emptyset , \quad j \neq k . \tag{2.1}$$

Assume that $\partial\Omega_j$, $j = 1, \ldots, M$, is also Lipschitz and that Ω_j is star-shaped. In practice, with the exception of perhaps a few Ω_j's along $\partial\Omega$, each Ω_j would be convex with a piecewise-smooth boundary. Let

$$\Gamma = \partial\Omega , \qquad \Gamma_j = \Gamma \cap \partial\Omega_j , \qquad \Gamma_{jk} = \Gamma_{kj} = \partial\Omega_j \cap \partial\Omega_k . \tag{2.2}$$

3. The mixed formulation of the differential problem

Consider the Dirichlet problem

(3.1.i) $$-\nabla\cdot(a\nabla u)+cu=f, \quad x\in\Omega,$$

(3.1.ii) $$u=g, \quad x\in\partial\Omega,$$

and assume that the coefficients $a(x)$ and $c(x)$ satisfy the bounds

$$0<a_0\leqq a(x)\leqq a_1<\infty,$$

$$0\leqq c(x)\leqq c_1<\infty,$$

and are sufficiently regular that the existence and uniqueness of a solution of (3.1) lying in $H^s(\Omega)$ for some $s>1$ for reasonable f and g are assured. Let the flux be denoted by

(3.2) $$\boldsymbol{q}=-a\nabla u,$$

and set $\alpha(x)=a(x)^{-1}$. Under reasonable hypotheses, the Dirichlet problem (3.1) is equivalent to its following (global) mixed formulation:

(3.3.i) $$\alpha\boldsymbol{q}+\nabla u=0, \quad x\in\Omega,$$

(3.3.ii) $$\operatorname{div}\boldsymbol{q}+cu=f, \quad x\in\Omega,$$

(3.3.iii) $$u=g, \quad x\in\partial\Omega.$$

The weak formulation of (3.3) is given by seeking $\{\boldsymbol{q},u\}\in H(\operatorname{div},\Omega)\times L^2(\Omega)=V\times W$ such that

(3.4.i) $$(\alpha\boldsymbol{q},\boldsymbol{v})_\Omega-(u,\operatorname{div}\boldsymbol{v})_\Omega=-\langle g,\boldsymbol{v}\cdot\nu\rangle_\Gamma, \quad \boldsymbol{v}\in V,$$

(3.4.ii) $$(\operatorname{div}\boldsymbol{q},w)_\Omega+(cu,w)_\Omega=(f,w)_\Omega, \quad w\in W.$$

Let us consider decomposing (3.3) or (3.4) over $\{\Omega_j\}$. In addition to requiring $\{\boldsymbol{q}_j,u_j\}$, $j=1,\dots,M$, to satisfy

(3.5.i) $$\alpha\boldsymbol{q}_j+\nabla u_j=0, \quad x\in\Omega_j,$$

(3.5.ii) $$\operatorname{div}\boldsymbol{q}_j+cu_j=f, \quad x\in\Omega_j,$$

(3.5.iii) $$u_j=g, \quad x\in\Gamma_j,$$

it is necessary to impose the consistency conditions

(3.6.i) $$u_j=u_k, \quad x\in\Gamma_{jk},$$

(3.6.ii) $$\boldsymbol{q}_j\cdot\nu_j+\boldsymbol{q}_k\cdot\nu_k=0, \quad x\in\Gamma_{jk},$$

where ν_j is the unit outer normal to Ω_j. It is more convenient [9, 10] to replace (3.6) by the Robin boundary condition

(3.7.i) $$-\beta\boldsymbol{q}_j\cdot\nu_j+u_j=\beta\boldsymbol{q}_k\cdot\nu_k+u_k, \quad x\in\Gamma_{jk}\subset\partial\Omega_j,$$

(3.7.ii) $$-\beta\boldsymbol{q}_k\cdot\nu_k+u_k=\beta\boldsymbol{q}_j\cdot\nu_j+u_j, \quad x\in\Gamma_{jk}\subset\partial\Omega_k,$$

where β is a positive (normally chosen to be a constant) function on $\bigcup \Gamma_{jk}$. Now, move toward a new weak formulation by testing (3.5.i) against a vector $\boldsymbol{v} \in \boldsymbol{V}_j = H(\text{div}, \Omega_j)$:

$$(3.8) \qquad (\alpha \boldsymbol{q}_j, \boldsymbol{v})_{\Omega_j} - (u_j, \text{div} \boldsymbol{v}_j)_{\Omega_j} + \langle u_j, \boldsymbol{v} \cdot \nu \rangle_{\partial \Omega_j} = 0, \quad \boldsymbol{v} \in \boldsymbol{V}_j .$$

Apply (3.5.iii) and (3.7.i) to (3.8) to obtain (3.9.i) below, and test (3.5.ii) against $w \in W_j = L^2(\Omega_j)$ to obtain the second equation in the system below. Thus, the weak mixed formulation of (3.1) over the partition $\{\Omega_j\}$ is given by the seeking of $\{\boldsymbol{q}_j, u_j\} \in \boldsymbol{V}_j \times W_j$, $j = 1, \ldots, M$, such that

$$(3.9.\text{i}) \qquad (\alpha \boldsymbol{q}_j, \boldsymbol{v})_{\Omega_j} - (u_j, \text{div} \boldsymbol{v})_{\Omega_j} + \sum_k \langle \beta(\boldsymbol{q}_j \cdot \nu_j + \boldsymbol{q}_k \cdot \nu_k) + u_k, \boldsymbol{v} \cdot \nu_j \rangle_{\Gamma_{jk}}$$
$$= - \langle g, \boldsymbol{v} \cdot \nu_j \rangle_{\Gamma_j}, \qquad \boldsymbol{v} \in \boldsymbol{V}_j ,$$

$$(3.9.\text{ii}) \qquad (\text{div} \boldsymbol{q}_j, w) + (c u_j, w) = (f, w), \quad w \in W_j .$$

There is a technical difficulty with (3.9.i); if $\boldsymbol{v}_j \in \boldsymbol{V}_j$ and $\boldsymbol{v}_k \in \boldsymbol{V}_k$, it is not necessarily the case that the product of their normal components is integrable on Γ_{jk}. Also, the meaning of the restriction of an L^2-function on Ω_k to Γ_{jk} is not clear. Thus, (3.9) is properly viewed as motivation for the treatment of the discrete case, and the remainder of the remarks in this section must be treated as heuristic.

The objective of a domain decomposition iterative method is to localize the calculations to problems over smaller domains than Ω. Here, it is feasible to localize to each Ω_j by evaluating the quantities in (3.9) related to Ω_j at the new iterate level and those in (3.9) related to neighboring subdomains Ω_k such that $\Gamma_{jk} \neq \emptyset$ at the old level. Specifically, the algorithm in the differential case would be as follows:

$$(3.10) \qquad \text{Select} \{\boldsymbol{q}_j^0, u_j^0\} \in \boldsymbol{V}_j \times W_j, \quad j = 1, \ldots, M, \text{ arbitrarily ;}$$

then recursively compute $\{\boldsymbol{q}_j^n, u_j^n\}$ by solving

$$(3.11.\text{i}) \qquad (\alpha \boldsymbol{q}_j^n, \boldsymbol{v})_{\Omega_j} - (u_j^n, \text{div}\, \boldsymbol{v})_{\Omega_j} + \sum_k \langle \beta \boldsymbol{q}_j^n \cdot \nu_j, \boldsymbol{v} \cdot \nu_j \rangle_{\Gamma_{jk}}$$
$$= - \sum_k \langle \beta \boldsymbol{q}_k^{n-1} \cdot \nu_k$$
$$+ u_k^{n-1}, \boldsymbol{v} \cdot \nu_j \rangle_{jk} - \langle g, \boldsymbol{v} \cdot \nu_j \rangle_{\Gamma_j}, \quad \boldsymbol{v} \in \boldsymbol{V}_j ,$$

$$(3.11.\text{ii}) \qquad (\text{div} \boldsymbol{q}_j^n, w)_{\Omega_j} + (c u_j^n, w)_{\Omega_j} = (f, w)_{\Omega_j}, \quad w \in W_j .$$

4. The mixed finite element problem

We shall treat the case in which $\{\Omega_j\}$ is a partition of Ω into individual elements (simplices, rectangles, prisms), though an inspection of the argument would indicate that larger subdomains are permissible. Let $\boldsymbol{V}^h \times W^h$ be a mixed finite element space over $\{\Omega_j\}$; any of the usual choices is acceptable: [3, 5–7, 24–26]. Each of these spaces is defined through local spaces $V_j \times W_j = V(\Omega_j) \times W(\Omega_j)$, and setting

(4.1.i) $$V^h = \{v \in H(\text{div}, \Omega) : v|_{\Omega_j} \in V_j\},$$

(4.1.ii) $$W^h = \{w : w|_{\Omega_j} \in W_j\}.$$

The global mixed finite element approximation to (3.3) is given by restricting (3.4) to the space $V^h \times W^h$; the existence, uniqueness, and convergence properties of the method are very adequately covered in the references cited above, as well as in such papers as [2, 15, 23].

In each space W^h in the various families of mixed elements referenced above, the functions $w \in W^h$ are allowed to be discontinuous across each Γ_{jk}. As a consequence, attempting to impose the consistency conditions (3.7) would force a flux conservation error; i.e., (3.6.ii) would not be satisfied unless the approximate solution $u^h \in W^h$ to the discrete analogue of (3.4) is constant, a totally uninteresting case. So, let us introduce Lagrange multipliers [1, 19, 20] on the edges $\{\Gamma_{jk}\}$. Assume that, when $q_j = q^h|_{\Omega_j}$, $q^h \in V^h$, its normal component $q_j \cdot \nu_j$ on Γ_{jk} is a polynomial of some fixed degree τ, where for simplicity we shall assume τ independent of Γ_{jk} (see [4] if not). Set

(4.2) $$\Lambda^h = \{\lambda : \lambda|_{\Gamma_{jk}} \in P_\tau(\Gamma_{jk}) = \Lambda_{jk}, \Gamma_{jk} \neq \emptyset\};$$

note that there are two copies of P_τ assigned to the set Γ_{jk}: Λ_{jk} and Λ_{kj}. Then, the hybridized mixed finite element method is given by dropping the superscript h and seeking

$$\{q_j \in V_j, u_j \in W_j, \lambda_{jk} \in \Lambda_{jk} : j = 1, \ldots, M;\ k = 1, \ldots, M\}$$

such that

(4.3.i) $$(\alpha q_j, v)_{\Omega_j} - (u_j, \text{div} v)_{\Omega_j} + \sum_k \langle \lambda_{jk}, v \cdot \nu_j \rangle_{\Gamma_{jk}} = -\langle g, v \cdot \nu_j \rangle, \quad v \in V_j,$$

(4.3.ii) $$(\text{div} q_j, w)_{\Omega_j} + (c u_j, w)_{\Omega_j} = (f, w)_{\Omega_j}, \quad w \in W_j,$$

(4.3.iii) $$\langle \mu, q_j \cdot \nu_j + q_k \cdot \nu_k \rangle_{\Gamma_{jk}} = 0, \quad \mu \in \Lambda_{jk}.$$

The constraint (4.3.iii) is equivalent to (3.6.ii), and it follows easily that the pair $\{q, u\}$, where $q|_{\Omega_j} = q_j$ and $u|_{\Omega_j} = u_j$, solves the original discrete problem. In the references cited above, it was assumed that $\lambda_{jk} = \lambda_{kj}$; the limit values of these multipliers resulting from the iteration defined below satisfy this equality.

Let us formulate an iterative version of (4.3). Consider the Lagrange multiplier to be λ_{jk} as seen from Ω_j and λ_{kj} as seen from Ω_k. Then, modify (3.7) to read

(4.4.i) $$-\beta q_j \cdot \nu_j + \lambda_{jk} = \beta q_k \cdot \nu_k + \lambda_{kj}, \quad x \in \Gamma_{jk} \subset \partial\Omega_j,$$

(4.4.ii) $$-\beta q_k \cdot \nu_k + \lambda_{kj} = \beta q_j \cdot \nu_j + \lambda_{jk}, \quad x \in \Gamma_{jk} \subset \partial\Omega_k,$$

so that

$$\langle \lambda_{jk}, v \cdot \nu_j \rangle_{\Gamma_{jk}} = \langle \beta(q_j \cdot \nu_j + q_k \cdot \nu_k) + \lambda_{kj}, v \cdot \nu_j \rangle_{\Gamma_{jk}}.$$

Define the iterative process analogously to (3.10) and (3.11). Let, for all j and k,

(4.5) $$\boldsymbol{q}_j^0 \in \boldsymbol{V}_j, \quad u_j^0 \in W_j, \quad \lambda_{jk}^0 \in \Lambda_{jk}, \quad \lambda_{kj}^0 \in \Lambda_{jk} \quad \text{arbitrarily},$$

($\lambda_{jk}^0 = \lambda_{kj}^0$ seems natural) and then compute $\{\boldsymbol{q}_j^n, u_j^n, \lambda_{jk}^n\} \in \boldsymbol{V}_j \times W_j \times \Lambda_{jk}$ recursively as the solution of the equations

(4.6.i) $$(\alpha \boldsymbol{q}_j^n, \boldsymbol{v})_{\Omega_j} - (u_j^n, \operatorname{div}\boldsymbol{v})_{\Omega_j} + \sum_k \langle \beta \boldsymbol{q}_j^n \cdot \nu_j, \boldsymbol{v}\cdot\nu_j \rangle_{\Gamma_{jk}}$$
$$= -\sum_k \langle \beta \boldsymbol{q}_k^{n-1} \cdot \nu_k + \lambda_{kj}^{n-1}, \boldsymbol{v}\cdot\nu_j\rangle_{\Gamma_{jk}} - \langle g, \boldsymbol{v}\cdot\nu_j\rangle_{\Gamma_j}, \quad \boldsymbol{v}\in \boldsymbol{V}_j,$$

(4.6.ii) $$(\operatorname{div}\boldsymbol{q}_j^n, w)_{\Omega_j} + (cu_j^n, w)_{\Omega_j} = (f, w)_{\Omega_j}, \quad w \in W_j,$$

(4.6.iii) $$\lambda_{jk}^n = \beta(\boldsymbol{q}_j^n \cdot \nu_j + \boldsymbol{q}_k^{n-1}\cdot \nu_k) + \lambda_{kj}^{n-1}.$$

Note that (4.6.i) and (4.6.ii) are independent of λ_{jk}^n and determine $\boldsymbol{q}_j^n$ and u_j^n; λ_{jk}^n is then evaluated by (4.6.iii).

Let us demonstrate the convergence of the iteration defined by (4.5)–(4.6). For each of the mixed spaces cited, there exists a solution of the global problem over the decomposition $\{\Omega_j\}$. Set

(4.7) $$\boldsymbol{r}_j^n = \boldsymbol{q}_j - \boldsymbol{q}_j^n, \quad e_j^n = u_j - u_j^n, \quad \mu_{jk}^n = \lambda_{jk} - \lambda_{jk}^n, \quad \mu_{kj}^n = \lambda_{jk} - \lambda_{kj}^n,$$

where $\{\boldsymbol{q}_j, u_j, \lambda_{jk}\}$ is the solution of the global problem on Ω_j; also, interpret the spaces $\boldsymbol{V}_j$, W_j, and Λ_{jk} as is appropriate for each problem.

The error equations can be written in the form

(4.8.i) $$(\alpha \boldsymbol{r}_j^n, \boldsymbol{v})_{\Omega_j} - (e_j^n, \operatorname{div}\boldsymbol{v})_{\Omega_j} + \sum_k \langle \mu_{jk}^n, \boldsymbol{v}\cdot\nu_j\rangle_{\Gamma_{jk}} = 0, \quad \boldsymbol{v}\in \boldsymbol{V}_j,$$

(4.4.ii) $$(\operatorname{div}\boldsymbol{r}_j^n, w)_{\Omega_j} + (ce_j^n, w)_{\Omega_j} = 0, \quad w\in W_j,$$

(4.8.iii) $$\mu_{jk}^n = \beta(\boldsymbol{r}_j^n \cdot \nu_j + \boldsymbol{r}_k^{n-1}\cdot\nu_k) + \mu_{kj}^{n-1}.$$

Choose $\boldsymbol{v} = \boldsymbol{r}_j^n$ in (4.8.i) and $w = u_j^n$ in (4.8.ii) and add the resulting equations; then,

(4.9) $$(\alpha \boldsymbol{r}_j^n, \boldsymbol{r}_j^n)_{\Omega_j} + (ce_j^n, e_j^n)_{\Omega_j} + \sum_k \langle \mu_{jk}^n, \boldsymbol{r}_j^n\cdot\nu_j\rangle_{\Gamma_{jk}} = 0.$$

Let $B_j = \partial\Omega_j \backslash \Gamma_j$, and then note that, by (4.9) and with $|\cdot|_{0,\Gamma_{jk}}$ indicating the L^2-norm on Γ_{jk},

(4.10) $$\sum_k |-\beta \boldsymbol{r}_j^n\cdot\nu_j \pm \mu_{jk}^n|_{0,\Gamma_{jk}}^2$$
$$= \beta^2 |\boldsymbol{r}_j^n\cdot\nu_j|_{0,B_j}^2 + \sum_k |\mu_{jk}^n|_{0,\Gamma_{jk}}^2 \mp 2\beta \sum_k \langle \boldsymbol{r}_j^n\cdot\nu_j, \mu_{jk}^n\rangle_{\Gamma_{jk}}$$
$$= \beta^2 |\boldsymbol{r}_j^n\cdot\nu_j|_{0,B_j}^2 + \sum_k |\mu_{jk}^n|_{0,\Gamma_{jk}}^2 \pm 2\beta\{(\alpha \boldsymbol{r}_j^n, \boldsymbol{r}_j^n)_{\Omega_j} + (ce_j^n, e_j^n)_{\Omega_j}\}.$$

Set

(4.11)
$$E(\{r, e, \mu\}) = \sum_j (\beta^2 |r_j \cdot v_j|^2_{0,B_j} + \sum_k |\mu_{jk}|^2_{0,\Gamma_{jk}})$$
$$+ 2\beta \sum_j \{(\alpha r_j, r_j)_{\Omega_j} + (c e_j, e_j)_{\Omega_j}\} ,$$

and let $E^n = E(\{r^n, e^n, \mu^n\})$. Then,

(4.12)
$$E^n = \sum_j \sum_k | - \beta r_j^n \cdot v_j + \mu_{jk}^n |^2_{0,\Gamma_{jk}}$$
$$= \sum_j \sum_k |\beta r_k^{n-1} \cdot v_k + \mu_{kj}^{n-1}|^2_{0,\Gamma_{jk}}$$
$$= E^{n-1} - 4\beta \sum_j \{(\alpha r_j^{n-1}, r_j^{n-1})_{\Omega_j} + (c e_j^{n-1}, e_j^{n-1})\} .$$

Since $\{E^n\}$ is a decreasing sequence of nonnegative numbers,

(4.13)
$$\sum_{n=1}^{\infty} \sum_j (\alpha r_j^n, r_j^n)_{\Omega_j} = \sum_{n=1}^{\infty} (\alpha r^n, r^n)_{\Omega} < \infty,$$

so that

(4.14)
$$r^n \to 0 \quad \text{in } L^2(\Omega) \quad \text{as } n \to \infty .$$

If the function $c(x) \geqq c_0 > 0$ on Ω, then it would follow also that $e^n \to 0$ in $L^2(\Omega)$ as $n \to \infty$. However, we did not assume c_0 to be positive, just nonnegative.

If ρ_j is the ratio of the diameter of Ω_j to the diameter of its inscribed sphere, then for each of the mixed finite element spaces we have referenced, it is known that

(4.15)
$$|r_j \cdot v_j|_{0,\partial\Omega_j} \leqq M_1(\rho_j, \operatorname{diam} \Omega_j) \|r_j\|_{0,\Omega_j} .$$

(If the partition is quasiregular, $M_1(\rho_j, \operatorname{diam} \Omega_j) \leqq Ch^{-1}$; this fact and some other consequences of quasiregularity will be used in Sect. 6 to derive a rate of convergence; here, we obtain convergence at an unspecified rate under weaker hypotheses.) Thus,

(4.16)
$$|r_j^n \cdot v_j|_{0,\partial\Omega_j} \to 0, \quad j = 1, \ldots, M ,$$

so that, in particular,

(4.17)
$$r_j^n \cdot v_j + r_k^{n-1} \cdot v_k \to 0 \quad \text{in } L^2(\Gamma_{jk}) .$$

This is not enough, but we can begin with a boundary element Ω_j (i.e., an element with one face, flat or curved, contained in Γ). For each of the families of mixed finite element spaces referenced (for the Raviart-Thomas-Nedelec elements, see [15]; for the others, see the original references for the spaces), it is shown that a feasible set of degrees of freedom for V_j can include

(4.18.i)
$$v \cdot v_j, \quad x \in \Gamma_{jk}, \quad (\text{thus}, x \in \partial\Omega_j \backslash \Gamma_j) ,$$

(4.18.ii)
$$\operatorname{div} v, \quad x \in \Omega_j .$$

Moreover, these degrees of freedom can be supplemented, if necessary, in such a way that

(4.19) $$\|v\|_{0,\Omega_j} \leqq M(\rho_j, \operatorname{diam}\Omega_j)(\|\operatorname{div} v\|_{0,\Omega_j} + \sum |v \cdot \nu_j|_{0,\Gamma_{jk}}) .$$

Now, choose $v = v^n \in V_j$ on the boundary element Ω_j such that

$$\operatorname{div} v^n = e_j^n \quad \text{on } \Omega_j \qquad \text{and} \qquad v^n \cdot \nu_j = 0 \quad \text{on } \partial\Omega_j \backslash \Gamma_j.$$

Then, (4.8.i) implies that

$$\|e_j^n\|_{0,\Omega_j}^2 = (\alpha r_j^n, v^n)_{\Omega_j} \leqq M_j \|r_j^n\|_{0,\Omega_j} \|e_j^n\|_{0,\Omega_j}.$$

Hence,

(4.20) $$\|e_j^n\|_{0,\Omega_j} \to 0 \quad \text{if } \Gamma_j \neq \varnothing .$$

If, instead, we choose $v \in V_j$ such that

$$\operatorname{div} v = 0 \quad \text{on } \Omega_j; \qquad v \cdot \nu_j = \begin{cases} \mu_{jk}^n & \text{on } \Gamma_{jk}, \\ 0 & \text{on } \Gamma_{jl}, l \neq k , \end{cases}$$

then

$$|\mu_{jk}^n|_{0,\Gamma_{jk}}^2 = -(\alpha r_j^n, v^n) \leqq M_j \|r_j^n\|_{0,\Omega_j} |\mu_{jk}^n|_{0,\Gamma_{jk}} ,$$

and

(4.21) $$|\mu_{jk}^n|_{0,\Gamma_{jk}} \to 0 \quad \text{if } \Gamma_j \neq \varnothing .$$

By (4.17),

(4.22) $$|\mu_{kj}^n|_{0,\Gamma_{jk}} \to 0 \quad \text{if } \Gamma_j \neq \varnothing .$$

Thus, we have proved convergence of q_j^n, u_j^n, λ_{jk}^n, and λ_{kj}^n on boundary elements. Consider an element having a common face Γ_{jk^*} with one of the boundary elements. Use $\operatorname{div} v$ and $v \cdot \nu_j$, $j \neq k^*$, as degrees of freedom and repeat the above argument. Since the same scaling argument that gives (4.19) would also show for this interior element (and, thus, with flat faces only) that

(4.23) $$|v \cdot \nu_j|_{0,\Gamma_{jk^*}} \leqq M_j(\|\operatorname{div} v\|_{0,\Omega_j} + \sum_{k \neq k^*} |v \cdot \nu_j|_{0,\Gamma_{jk}}) ,$$

the only new term arising is

$$|\langle \mu_{jk^*}^n, v \cdot \nu_j \rangle_{\Gamma_{jk^*}}| \leqq |\mu_{jk^*}^n|_{0,\Gamma_{jk^*}} |v \cdot \nu_j|_{0,\Gamma_{jk^*}} ,$$

and convergence takes place for u_j^n, λ_{jk}^n, and λ_{kj}^n on these elements, as well. The argument can be repeated until the domain is exhausted.

We have finished the proof of the convergence of (4.6) in the discrete case, as stated in the following theorem.

Theorem 4.1. *The iterates* $\{q_j^n, u_j^n, \lambda_{jk}^n\} \in V_j \times W_j \times \Lambda_{jk}$ *converge to the solution* $\{q_j, u_j, \lambda_{jk}\}$ *of the global hybridized mixed finite element procedure* (4.3) *in the following senses:*

(i) $q_j^n \to q_j = q^*|_{\Omega_j}$ in $L^2(\Omega_j)$,
(ii) $u_j^n \to u_j = u^*|_{\Omega_j}$ in $L^2(\Omega_j)$,
(iii) λ_{jk}^n and $\lambda_{kj}^n \to \lambda_{jk}$ in $L^2(\Gamma_{jk})$,

where $\{q^*, u^*\} \in V_h \times W_h$ *is the solution of the global mixed finite element method.*

5. Spectral radius of the iterator without quasiregularity assumptions

The iterative procedure described in Sect. 4 is actually a simple iterative method to approximate the fixed point of an appropriately defined operator. Let us recall that in the definition of $\Lambda^h = \{\Lambda_{jk}: j, k = 1, \ldots, M\}$ it is not assumed that $\Lambda_{jk} = \Lambda_{kj}$. Now, let $T_{f,g}$ be the affine mapping from $V^h \times W^h \times \Lambda^h$ to itself such that, for any $(s, p, \theta) \in V^h \times W^h \times \Lambda^h$, $(r, e, \mu) \equiv T_{f,g}(s, p, \theta)$ is the solution for the following equations:

(5.1.i)
$$(\alpha r_j, v)_{\Omega_j} - (e_j, div v)_{\Omega_j} + \sum_k \langle \beta r_j \cdot v_j, v \cdot v_j \rangle_{\Gamma_{jk}}$$
$$= -\sum_k \langle \beta s_k \cdot v_k + \theta_{kj}, v \cdot v_j \rangle_{\Gamma_{jk}} - \langle g, v \cdot v_j \rangle_{\Gamma_j}, \quad v \in V_j,$$

(5.1.ii)
$$(\operatorname{div} r_j, w)_{\Omega_j} + (c e_j, w)_{\Omega_j} = (f, w)_{\Omega_j}, \quad w \in W_j,$$

(5.1.iii)
$$\mu_{jk} = \beta(r_j \cdot v_j + s_k \cdot v_k) + \theta_{kj}.$$

Lemma 5.1. *The triple $(q, u, \lambda) \in V^h \times W^h \times \Lambda^h$ is the solution of the discrete problem* (4.3) *if and only if it is a fixed point for the operator $T_{f,g}$. If (q, u, λ) is a fixed point of $T_{f,g}$, then $\lambda_{jk} = \lambda_{kj}$ for all j and k.*

Proof. Let (q, u, λ) be a fixed point of the operator for $T_{f,g}$. Substituting the equality (5.1.iii) into (5.1.i) yields (4.3.i). Then, (4.3.ii) is trivially verified by (5.1.ii). It follows from (5.1.iii), with (s, θ) and (r, μ) replaced by (q, λ), that

$$\lambda_{jk} = \beta(q_j \cdot v_j + q_k \cdot v_k) + \lambda_{kj},$$
$$\lambda_{kj} = \beta(q_k \cdot v_k + q_j \cdot v_j) + \lambda_{jk}.$$

Summing the above equations implies that

$$q_j \cdot v_j + q_k \cdot v_k = 0,$$

which is equivalent to (4.3.iii). This shows that $\lambda_{jk} = \lambda_{kj}$ and that any fixed point of $T_{f,g}$ is a solution of the problem (4.3). It is trivial to see that any solution of (4.3) is a fixed point of the operator $T_{f,g}$, and the lemma has been proved.

Let $T_0 = T_{0,0}$, and let $\tilde{f} = T_{f,g}(0, 0, 0)$. Then,

(5.2)
$$T_{f,g}(s, p, \theta) = T_0(s, p, \theta) + \tilde{f}.$$

The fixed point of the operator $T_{f,g}$ is characterized as a solution of the equation

(5.3)
$$(I - T_0)(q, u, \lambda) = \tilde{f}. \quad \square$$

Lemma 5.2. *If $\rho(T_0)$ is the spectral radius of T_0, then*

(5.4)
$$\rho(T_0) < 1.$$

Thus, the iterative procedure (4.6) *is convergent.*

Proof. Let γ be an eigenvalue of T_0 and (q, u, λ) be the corresponding eigenvector, so that

$$T_0(q, u, \lambda) = \gamma(q, u, \lambda).$$

Our objective is to show that $|\gamma| < 1$. It follows from (4.11) that

$$E(T_0\{\boldsymbol{q}, u, \lambda\}) = |\gamma|^2 E(\{\boldsymbol{q}, u, \lambda\}) \,. \tag{5.5}$$

Also, by (4.12),

$$E(T_0\{\boldsymbol{q}, u, \lambda\}) = E(\{\boldsymbol{q}, u, \lambda\}) - 4\beta\sum_j\{(\alpha\boldsymbol{q}_j, \boldsymbol{q}_j)_{\Omega_j} + (cu_j, u_j)_{0,\Omega_j}\} \,.$$

Combining the above and (5.5) yields

$$|\gamma|^2 = \left(E(\{\boldsymbol{q}, u, \lambda\}) - 4\beta\sum_j\{(\alpha\boldsymbol{q}_j, \boldsymbol{q}_j)_{\Omega_j} + (cu_j, u_j)_{0,\Omega_j}\}\right)/E(\{\boldsymbol{q}, u, \lambda\}) \,, \tag{5.6}$$

which implies that $|\gamma| \leqq 1$. Equality holds if and only if

$$(\alpha\boldsymbol{q}_j, \boldsymbol{q}_j)_{\Omega_j} + (cu_j, u_j)_{0,\Omega_j} = 0, \quad j = 1, \ldots, M \,. \tag{5.7}$$

Suppose $|\gamma| = 1$. We would like to derive a contradiction by showing that the eigenvector is trivial. First, it follows from (5.7) that $\boldsymbol{q} = 0$. Then, (5.1.iii) implies that

$$\lambda_{jk} = \gamma\lambda_{kj} \,. \tag{5.8}$$

Now, choose $\boldsymbol{v} \in \boldsymbol{V}_j$ on a boundary subdomain Ω_j such that

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{on } \Omega_j \,,$$

$$\boldsymbol{v}\cdot\nu_j = \lambda_{jk} \quad \text{on } \partial\Omega_j\backslash\Gamma_j \,.$$

Then, (5.1.i) implies that

$$|\lambda_{jk}|^2_{0,\Gamma_{jk}} = 0, \quad \text{for all } k \,.$$

This indicates that $\lambda_{jk} = 0$ for all the boundary subdomain Ω_j. By (5.8), $\lambda_{jk^*} = \gamma\lambda_{k^*j} = 0$ if Γ_{jk^*} is a common face with one of the boundary subdomains. By induction, it is easy to show that $\lambda_{jk} = 0$ for all j and k. Finally, the Eqs. (5.1.i) and (5.1.iii), together with the fact that $\boldsymbol{q} = 0$ and $\lambda = 0$, implies that

$$(u_j, \operatorname{div}\boldsymbol{v})_{\Omega_j} = 0, \quad \boldsymbol{v} \in \boldsymbol{V}_j \,.$$

It follows that $u_j = 0$ for all j, so that $|\gamma| < 1$ and the iterative procedure converges. □

6. Spectral radius of the iterator with quasiregularity

Assume that there exists a constant Q such that

$$E(\{\boldsymbol{q}, u, \lambda\}) \leqq 4Q\beta\sum_j\{(\alpha\boldsymbol{q}_j, \boldsymbol{q}_j)_{\Omega_j} + (cu_j, u_j)_{0,\Omega_j}\} \tag{6.1}$$

for any eigenvector of T_0. If so, (5.6) implies that, for the corresponding eigenvalue γ,

$$|\gamma|^2 \leqq 1 - 1/Q \,, \tag{6.2}$$

and an estimate for the convergence rate for (4.6) would follow. Thus, it suffices to derive the inequality (6.1).

In order to find an estimate of Q in terms of h, assume that the partition $\{\Omega_j\}$ is quasiregular. Then, there exists a constant C such that

$$|\boldsymbol{q}_j \cdot \nu_j|_{0,B_j}^2 \leqq Ch^{-1} \|\boldsymbol{q}_j\|_{0,\Omega_j}^2 . \tag{6.3}$$

It is easy to see from (5.1) that the eigenvector $(\boldsymbol{q}, u, \lambda)$ satisfies the equation

$$(\alpha \boldsymbol{q}_j, \boldsymbol{v})_{\Omega_j} - (u_j, \operatorname{div} \boldsymbol{v})_{\Omega_j} + \sum_k \langle \lambda_{jk}, \boldsymbol{v} \cdot \nu_j \rangle_{\Gamma_{jk}} = 0, \quad \boldsymbol{v} \in \boldsymbol{V}_j . \tag{6.4}$$

Now, choose $\boldsymbol{v} \in \boldsymbol{V}_j$ such that

$$\boldsymbol{v} \cdot \nu_j = -\lambda_{jk}, \qquad \operatorname{div} \boldsymbol{v} = \tilde{g}_j = -\frac{1}{|\Omega_j|} \sum_k \int_{\Gamma_{jk}} \lambda_{jk} , \tag{6.5}$$

$$\|\boldsymbol{v}\|_{0,\Omega_j}^2 \leqq C_1 |\lambda_j|_{0,B_j}^2 .$$

It follows from (6.4) that

$$\begin{aligned} \sum_k |\lambda_{jk}|_{0,\Gamma_{jk}}^2 &= (\alpha \boldsymbol{q}_j, \boldsymbol{v}) - (u_j, \operatorname{div} \boldsymbol{v}) \\ &= (\alpha \boldsymbol{q}_j, \boldsymbol{v}) + |\tilde{g}_j|(u_j, 1) \\ &\leqq C \|\boldsymbol{q}_j\|_{0,\Omega_j} \|\boldsymbol{v}\|_{0,\Omega_j} + |\tilde{g}_j| |\Omega_j|^{1/2} \|u_j\|_{0,\Omega_j} . \end{aligned} \tag{6.6}$$

By the Schwarz inequality, we have

$$|\tilde{g}_j| \leqq \frac{|\partial\Omega_j|^{1/2}}{|\Omega_j|} \left(\sum_k |\lambda_{jk}|_{0,\Gamma_{jk}}^2 \right)^{1/2} .$$

Now, substituting the above inequality and (6.5) into (6.6), we obtain the bound

$$|\lambda_j|_{B_j}^2 = \sum_k |\lambda_{jk}|_{0,\Gamma_{jk}}^2 \leqq C(C_1^{1/2} + \sqrt{|\partial\Omega_j|/|\Omega_j|}\,) |\lambda_j|_{0,B_j} ,$$

By first eliminating $|\lambda_j|_{0,B_j}$ and then squaring both sides, we find that

$$|\lambda_j|_{B_j}^2 \leqq C(C_1 \|\boldsymbol{q}_j\|_{0,\Omega_j}^2 + |\partial\Omega_j|/|\Omega_j| \|u_j\|_{0,\Omega_j}^2) . \tag{6.7}$$

Assume that the coefficient $c(x)$ is bounded below by a positive number: $c(x) \geqq c_0 > 0$. It then follows from (6.3) and (6.7) that there exists a constant C, independent of h, such that

$$\sum_j (\beta^2 |\boldsymbol{q}_j \cdot \nu_j|_{0,B_j}^2 + \sum_k |\lambda_{jk}|_{0,\Gamma_{jk}}^2) \leqq C(C_1 \beta^{-1} + \beta h^{-1} + \rho \beta^{-1} c_0^{-1}) \cdot$$
$$\cdot 4\beta \sum_j (\|\boldsymbol{q}_j\|_{0,\Omega_j}^2 + c_0 \|u_j\|_{0,\Omega_j}^2) , \tag{6.8}$$

where

$$\rho = \max_j |\partial\Omega_j|/|\Omega_j| . \tag{6.9}$$

The estimate (6.8) indicates that (6.1) is verified with

$$Q = C(\beta^{-1} C_1 + \beta h^{-1} + \rho \beta^{-1} c_0^{-1}) .$$

If constant Q is considered as a function of β, then it is minimized by taking $\beta = \sqrt{h(C_1 + \rho c_0^{-1})}$. Such a choice of β yields the estimate

(6.10) $$Q = 2C\sqrt{\frac{C_1 + \rho c_0^{-1}}{h}} .$$

Combining the above estimate with (6.2) yields the following result.

Theorem 6.1. *Let $c(x) \geqq c_0 > 0$ and ρ be as above. Assume that the parameter β in the iterative procedure (4.6) satisfies $\beta = \sqrt{h(C_1 + \rho c_0^{-1})}$. Then, the spectral radius $r(T_0)$ of the operator T_0 is bounded as follows:*

$$r(T_0) \leqq 1 - \frac{\sqrt{h}}{2C\sqrt{C_1 + \rho c_0^{-1}}} \equiv \gamma_0 ,$$

and the iteration (4.6) converges with an error at the n^{th} iteration bounded asymptotically by $O(\gamma_0^n)$.

Now, we consider two particular examples for the domain decomposition method (4.6). First, assume the triangulation $\{\Omega_j\}$ of Ω into elements to be quasiregular and that the subdomains in the domain decomposition to coincide with $\{\Omega_j\}$. In addition, assume $c_0 = O(1)$. Then, by a scaling argument, we see that

$$C_1 = O(h), \qquad \rho c_0^{-1} = O(h^{-1}) .$$

By choosing the parameter $\beta = \sqrt{h(C_1 + \rho c_0^{-1})} = O(1)$, it follows that (4.6) converges with rate bounded by

$$\gamma_0 = 1 - ch .$$

Numerical experiments have confirmed that the $(1 - ch)$-rate is a correct estimate for the choice of β given above.

In our second example, we consider $c_0 = O(1)$ and subdomains so that $|\Omega_j| = O(1)$. For reasonably shaped subdomains, $\rho = O(1)$, so that the choice

$$\beta = O(\sqrt{h}) ,$$

leads to the estimate

$$\gamma_0 = 1 - c\sqrt{h} ,$$

for the spectral radius of the iterator in (4.6). This rate of convergence can be seen to be optimal for the case when Ω is a rectangle divided into two equal parts.

7. Time-dependent problems

Consider as a typical example the heat conduction problem given by finding $u: \Omega \times J \to \mathbb{R}$, where $J = (0, T]$, such that

(7.1) $$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(a\nabla u) &= f, \quad x\in\Omega,\ t\in J , \\ u &= g, \quad x\in\partial\Omega,\ t\in J , \\ u &= u_0, \quad x\in\Omega,\ t = 0 . \end{aligned}$$

Approximate (7.1) implicitly by backwards differencing in time to obtain the system

$$(7.2)\qquad \begin{aligned} &\frac{u^l - u^{l-1}}{\Delta t} - \operatorname{div}(a^l \nabla u^l) = f^l,\ x \in \Omega,\ t^l = l\Delta t \in J\,, \\ &u^l = g^l,\ x \in \partial\Omega,\ t^l \in J\,, \\ &u^0 = u_0,\quad x \in \Omega\,. \end{aligned}$$

At each time level t^l (7.1) represents an elliptic problem for u^l; if a mixed method is employed to approximate its solution, then the resulting algebraic equations are of exactly the form treated in Sect. 4. Quite good initial guesses can be computed by extrapolation from the values obtained at previous time levels (see, e.g., [13, 14]), where both rules for obtaining initial guesses and for stopping iteration processes are discussed. Note that

$$c_0 \sim \frac{1}{\Delta t}\,;$$

thus, the choice

$$\beta = \sqrt{h^2 + \Delta t}$$

is indicated for a decomposition into individual elements. In this case, $r(T_0) \leqq 1 - c$ for some $c < 1$ if the time step is required to satisfy the natural relation $\Delta t = O(h^2)$. For the incomplete iteration methods described first in [12] and refined in [13] and [14], this bound on $r(T_0)$ implies that only some fixed number of iterations are required for each time step.

Similarly, second order hyperbolic problems can be treated by mixed methods and the iterative procedure of Sect. 4 applied.

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