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SECOND-ORDER ABSORBING BOUNDARY CONDITIONS FOR THE WAVE EQUATION: A SOLUTION FOR THE CORNER PROBLEM*

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Abstract. The treatment of domains with corners in the problem of absorbing boundary conditions for the wave equation is very important from a practical point of view. A technical difficulty appears as soon as conditions of order greater than or equal to 2 are considered. A solution is proposed for the two-dimensional case when second-order conditions are used. This solution consists of prescribing an adequate corner condition. The problem thus obtained is analyzed theoretically and the condition is proved to be optimal. The results obtained here are illustrated by numerical simulations. Some extensions to higher-space dimensions and higher-order conditions are proposed.

Key words. wave equations, absorbing boundary conditions, domains with a corner

AMS(MOS) subject classifications. 35B65, 35L20, 65M99, 78A99, 86A99

1. Introduction. In [4] and [5], Engquist and Majda introduced a sequence of absorbing boundary conditions for the two-dimensional wave equation in the half-plane $\{(x_1, x_2), x_2 < 0\}$:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) = 0 \tag{1.1}$$

(the velocity is supposed to be constant and taken equal to 1).

These conditions are given recursively by

$$\begin{aligned} \mathbf{B}_1 u &= \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x_2} = 0 \quad \text{on } \Gamma = \{(x_1, 0)\}, \\ \mathbf{B}_2 u &= \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\,\partial x_2} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 \quad \text{on } \Gamma, \\ \mathbf{B}_{n+1} u &= \frac{\partial}{\partial t}(\mathbf{B}_n u) - \frac{1}{4}\frac{\partial^2}{\partial x_1^2}(\mathbf{B}_{n-1} u) = 0 \quad \text{on } \Gamma. \end{aligned} \tag{1.2}$$

In [5] it is shown that, according to the Kreiss criterion [8], each of these boundary conditions is strongly well posed for the wave equation.

If we assume that the interior equation (1.1) also holds on the boundary we may write (cf. [6])

$$\mathbf{B}_n u = \frac{1}{2^{n-1}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^n u, \qquad n = 1, 2, \cdots, \tag{1.3}$$

and then it is easy to see that each $\mathbf{B}_n$ is transparent for plane-harmonic waves striking

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the boundary at normal incidence. Furthermore, if u is a plane harmonic wave of unit amplitude striking the boundary at an angle of incidence θ from the normal, then the amplitude R_n of the reflected wave (i.e., the reflection coefficient for $\mathbf{B}_n$) is given by

$$R_n=\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^n=o(\theta^{2n}),\qquad n=1,2,\cdots.$$

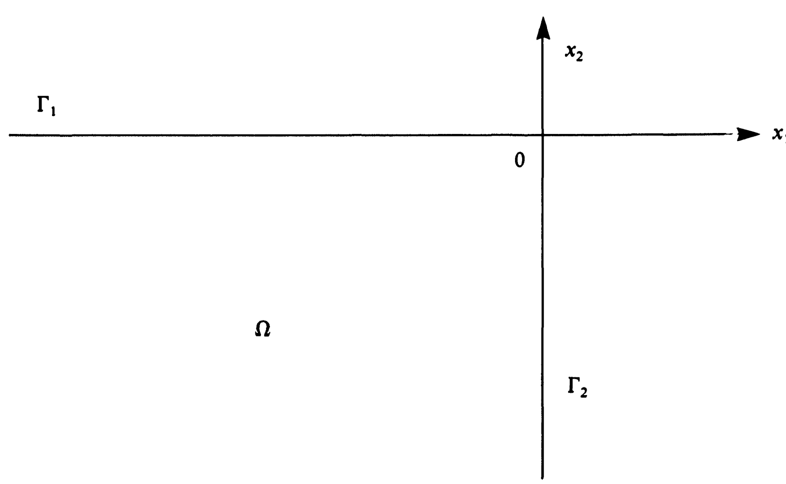
We mention that some extensions of conditions (1.2) and (1.3) have been proposed in [7] and [15].

For many practical problems we are interested in restricting numerical calculations, which should be done in theory on an infinite domain, to a bounded domain. Thus an absorbing condition is needed for the boundary of the bounded domain.

For a domain with a smooth boundary, a solution has been proposed by Engquist and Majda in [5]. This solution, which makes use of the theory of pseudodifferential operators, generalizes conditions (1.2). However, if we use a finite-difference scheme on a uniform grid to compute the numerical solution, it seems more interesting to restrict the effective calculations to a rectangle. For simplicity of exposition, we restrict our attention to the case of the quarter-plane $\Omega=\{x=(x_1,x_2)/x_1<0, x_2<0\}$ and then introduce two absorbing boundaries:

$$\Gamma_1=\{(x_1,0), x_1\leqq 0\},\qquad \Gamma_2=\{(0,x_2), x_2\leqq 0\}.$$

We will denote by zero the corner point $(x_1=0,\ x_2=0)$; see below.



The initial boundary value problem obtained by taking (1.1) in Ω with the first-order boundary conditions

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\Delta u=0 \quad \text{in } \Omega,\\
(1.4)\qquad &\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_2}=0 \quad \text{on } \Gamma_1,\\
&\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_1}=0 \quad \text{on } \Gamma_2,
\end{aligned}$$

and appropriate initial conditions

$$\begin{aligned}
&u(x,0)=u_0(x) \quad \text{in } \Omega,\\
(1.5)\qquad &\frac{\partial u}{\partial t}(x,0)=u_1(x) \quad \text{in } \Omega
\end{aligned}$$

is well posed, as we have the following energy identity:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx\right) + \int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 d\sigma = 0 \tag{1.6}$$

($d\sigma$ denotes the superficial measure on Γ). Thus the presence of the corner does not pose any specific difficulty. However, as indicated by the reflection coefficients and verified by numerical experiments, these first-order conditions give rise to significantly larger reflected waves than do those of second or higher order.

If we look at the second-order conditions written as

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_2\, \partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 \quad \text{on } \Gamma_1, \\ &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1\, \partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} = 0 \quad \text{on } \Gamma_2, \end{aligned} \tag{1.7}$$

it is not at all clear a priori whether the corresponding initial boundary value problem is well posed. Numerically, if we use a classical finite-difference scheme (for example, see [12], [13]), a naive matching of conditions (1.7) at zero is ruled out by the appearance of the second-order spatial derivatives in any expression for $\mathbf{B}_2$. Moreover, it can be shown, as we will see in § 3, that there is no unique solution of the problem in the class of finite energy functions if no condition at the corner is specified. Furthermore, it is not clear what corner condition should be chosen, and as reported in [5], an improper choice of the corner condition may generate instabilities.

Thus, what we propose to do here is to introduce a boundary condition at corner zero and to study this condition from both a mathematical and a numerical point of view.

The outline of this paper is as follows. In § 2 we explain how we construct our corner condition and present different extensions of the method we use. In § 3 we analyse theoretically a family of corner conditions, depending on a parameter, containing our condition and also the one proposed previously in [5]. We try to show in what sense our condition is the best one. In § 4 we present various numerical results illustrating the theoretical results of § 3.

2. Derivation of the boundary condition at the corner. The construction of our corner condition is guided by two quite simple principles:

(i) The corner condition should not introduce a singularity; i.e., regular initial data should yield a regular solution.

(ii) The expression of the corner condition should contain only first-order spatial derivatives in a given direction.

We consider the wave equation (1.1) in Ω with boundary conditions (1.7) and initial conditions (1.5), and we suppose that the initial data functions u_0 and u_1 belong to $C_0^\infty(\Omega)$. The idea is that if u is a C^∞ solution then (1.1) and (1.7) should be satisfied at point zero for all time t:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 && \text{at } x_1 = x_2 = 0 \quad \text{for } t \geqq 0, \\ &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_2\, \partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x_1^2} = 0 && \text{at } x_1 = x_2 = 0 \quad \text{for } t \geqq 0, \\ &\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x_1\, \partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x_2^2} = 0 && \text{at } x_1 = x_2 = 0 \quad \text{for } t \geqq 0. \end{aligned}$$

Adding the second two equations and subtracting $\frac{1}{2}$ times the first equation from the resulting sum, we obtain

$$\frac{3}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x_1\,\partial t}+\frac{\partial^2 u}{\partial x_2\,\partial t}=0 \quad \text{at } x_1=x_2=0 \quad \text{for } t\geqq 0.$$

Then, integrating once with respect to time, we arrive at

$$\frac{3}{2}\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}=0 \quad \text{at } x_1=x_2=0 \quad \text{for } t\geqq 0. \tag{2.1}$$

It is interesting to compare this condition with that proposed by Engquist and Majda in [5]:

$$\sqrt{2}\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}=0 \quad \text{at } x_1=x_2=0 \quad \text{for } t\geqq 0, \tag{2.2}$$

which is transparent for plane-harmonic waves propagating along the diagonal. In practice, conditions (2.1) and (2.2) differ very little as $\sqrt{2}\approx\frac{3}{2}$, although they are obtained from very different considerations.

At this point, as condition (2.1) is necessarily satisfied by any C^2 solution of (1.1) and (1.7), it is reasonable to ask if this condition serves any purpose other than numerical ones, for which it differs negligibly from (2.2). We will see its theoretical utility in § 3.

(i) *Generalization to the n-dimensional case.* Let $n\geqq 2$ and put

$$\Omega_n=\{x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n;\ x_i<0 \text{ for } j=1,2,\cdots,n\}.$$

We consider the wave equation

$$\frac{\partial^2 u}{\partial t^2}-\Delta u=0 \quad \text{in } \Omega \quad \text{for } t>0 \tag{2.3}$$

with initial conditions

$$u(x,0)=u_0(x) \qquad \text{for } x\in\Omega,$$

$$\frac{\partial u}{\partial t}(x,0)=u_1(x) \quad \text{for } x\in\Omega,$$

and the second-order absorbing boundary conditions

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\,\partial x_k}-\frac{1}{2}\sum_{1\leqq j\neq k\leqq n}\frac{\partial^2 u}{\partial x_j} \quad \text{on } \Gamma_k, \tag{2.4}$$

$k=1,\cdots,n$, where Γ_k is the boundary face of Ω whose normal is x_k. We suppose that u_0 and u_1 belong to $C_0^\infty(\Omega_n)$. If u is a C^∞ solution then (2.3) also holds on Γ_k, and the second-order conditions may be written in the form indicated by (1.3):

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_k}\right)^2 u=0 \quad \text{on } \Gamma_k, \tag{2.5}$$

$k=1,\cdots,n.$

For $n>2$ in addition to the condition at the corner itself we need a condition for each lower-dimensional face Γ_J of Ω, where, for $J\subset\{1,2,\cdots,n\}$, we mean by Γ_J

$$\Gamma_J=\{x\in\mathbb{R}^n;\ x_k<0 \text{ for } k\notin J,\ x_k=0 \text{ for } k\in J\}.$$

If u is a C^∞ solution then (2.3) as well as (2.5), for all $k \in J$, holds on Γ_J. If we sum these equations and divide by 2 we obtain

$$\frac{(j+1)}{2}\frac{\partial^2 u}{\partial t^2}+\frac{\partial}{\partial t}\sum_{k\in J}\frac{\partial u}{\partial x_k}-\frac{1}{2}\sum_{k\notin J}\frac{\partial u}{\partial x_k} \quad \text{on } \Gamma_J, \tag{2.6}$$

where j is the number of elements in J.

For the corner itself, where $J=\{1,2,\cdots,n\}$, after integration in time, (2.6) becomes

$$\frac{n+1}{2}\frac{\partial u}{\partial t}+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}=0. \tag{2.7}$$

We note that, by the Engquist–Majda criterion, the corner condition would be

$$\sqrt{n}\frac{\partial u}{\partial t}+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}=0 \tag{2.8}$$

and that, for sufficiently large n, (2.7) is really quite different from (2.8).

(ii) *Generalization to the third-order boundary condition.* Again in dimension 2, we consider the third-order boundary conditions (1.2)

$$\begin{aligned}&\frac{\partial^3 u}{\partial t^3}+\frac{\partial^3 u}{\partial t^2\,\partial x_2}-\frac{3}{4}\frac{\partial^3 u}{\partial t\,\partial x_1^2}-\frac{1}{4}\frac{\partial^3 u}{\partial x_1^2\,\partial x_2}=0 \quad \text{on } \Gamma_1,\\ &\frac{\partial^3 u}{\partial t^3}+\frac{\partial^3 u}{\partial t^2\,\partial x_1}-\frac{3}{4}\frac{\partial^3 u}{\partial t\,\partial x_2^2}-\frac{1}{4}\frac{\partial^3 u}{\partial x_1^2\,\partial x_2}=0 \quad \text{on } \Gamma_2.\end{aligned} \tag{2.9}$$

As before, we assume that u is C^∞ so that (1.1) also holds on Γ_k, $k=1,2$, and we again use the form indicated by (1.3) for the boundary conditions:

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_k}\right)^3 u=0 \quad \text{on } \Gamma_k, \tag{2.10}$$

for $k=1,2$.

We sum the following five equations: (2.10) for $k=1$, (2.10) for $k=2$, 3 times (1.1) differentiated once with respect to time, (1.1) differentiated once with respect to x_1, and (1.1) differentiated once with respect to x_2. Thus we obtain

$$5\frac{\partial^3 u}{\partial t^3}+4\left(\frac{\partial^3 u}{\partial t^2\,\partial x_1}+\frac{\partial^3 u}{\partial t^2\,\partial x_2}\right)-\frac{\partial^3 u}{\partial x_1^2\,\partial x_2}-\frac{\partial^3 u}{\partial x_2^2\,\partial x_1}=0 \quad \text{at } 0. \tag{2.11}$$

The last two terms on the left-hand side need to be eliminated. We differentiate (2.10) on Γ_1 with respect to x_2 and (2.10) on Γ_2 with respect to x_1. Equation (1.1) is differentiated twice, once with respect to each spatial variable. The resulting three equations are summed to obtain an equation that can be integrated in time. We then have

$$\frac{\partial^3 u}{\partial t^2\,\partial x_1}+\frac{\partial^3 u}{\partial t^2\,\partial x_2}+\frac{\partial^3 u}{\partial t\,\partial x_1\,\partial x_2}+3\left(\frac{\partial^3 u}{\partial x_1^2\,\partial x_2}+\frac{\partial^3 u}{\partial x_2^2\,\partial x_1}\right)=0. \tag{2.12}$$

Summing (2.12) with 3 times (2.11) and integrating the result once in time, we arrive at the corner condition

$$15\frac{\partial^2 u}{\partial t^2}+13\left(\frac{\partial^2 u}{\partial t\,\partial x_1}+\frac{\partial^2 u}{\partial t\,\partial x_2}\right)+7\frac{\partial^2 u}{\partial x_1\,\partial x_2}=0 \quad \text{at } 0. \tag{2.13}$$

As in the case of the second-order boundary condition, the corner condition (2.13) can be extended with no specific difficulty to the wave equation in $\mathbb{R}^n$.

It would also be natural to generalize this corner condition to higher-order absorbing boundary conditions (1.2). This problem does not appear to be so easy. For example, as fourth-order spatial derivatives in a given direction appear in the fourth- and fifth-order absorbing boundary conditions (see [5]), our conjecture is that three corner conditions are needed, and more generally that n corner conditions are needed for the $2n$th- and $(2n+1)$st-order conditions.

3. Mathematical analysis of the two-dimensional problem. Our goal in this section is to study from a mathematical point of view the following problem:

$$(\mathrm{P}_\gamma)\quad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}-\Delta u=0 & \text{in } \Omega \quad \text{for } t>0,\\[2mm] \dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial^2 u}{\partial x_2\,\partial t}-\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_1^2}=0 & \text{on } \Gamma_1 \quad \text{for } t>0,\\[2mm] \dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial^2 u}{\partial x_1\,\partial t}-\dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_2^2}=0 & \text{on } \Gamma_2 \quad \text{for } t>0,\\[2mm] \gamma\dfrac{\partial u}{\partial t}+\dfrac{\partial u}{\partial x_1}+\dfrac{\partial u}{\partial x_2}=0 & \text{at } 0 \quad \text{for } t>0,\\[2mm] u(x,0)=u_0(x) & \text{in } \Omega,\\[2mm] \dfrac{\partial u}{\partial t}(x,0)=u_1(x) & \text{in } \Omega. \end{cases}$$

For this problem we have the second-order absorbing boundary conditions:

$$(\mathrm{CL1})\qquad \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x_2\,\partial t}-\frac{1}{2}\frac{\partial^2 u}{\partial x_1^2}=0 \quad \text{on } \Gamma_1 \quad \text{for } t>0,$$

$$(\mathrm{CL2})\qquad \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial x_1\,\partial t}-\frac{1}{2}\frac{\partial^2 u}{\partial x_2^2}=0 \quad \text{on } \Gamma_2 \quad \text{for } t>0,$$

and we consider a general corner condition:

$$(\mathrm{CC}\gamma)\qquad \gamma\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}=0 \quad \text{at } 0 \quad \text{for } t>0,$$

where γ denotes a positive parameter. This generalizes condition (2.1) derived in § 2, and condition (2.2) of Engquist and Majda. We put

$$\gamma^*=\tfrac{3}{2}.$$

It is not obvious how to obtain an existence and uniqueness result for (P_γ) via classical methods based on energy identities and a priori estimates. Moreover, this problem does not fall into the category of corner problems previously treated in the literature (cf. [9], [11], and [14]). Thus we develop an analysis specific to this problem. Nonetheless, we cite the work of Lemrabet [10] on the analogous problem for the Ventcel boundary conditions.

It is clear that (P_γ) can have a $C^\infty(\bar{\Omega}\times\mathbb{R}^+)$ solution only if $\gamma=\gamma^*$, and that in this case condition $(\mathrm{CC}\gamma)$ is superfluous. In § 3.3 we will see that if the initial data functions u_0 and u_1 belong to $C_0^\infty(\Omega)$, then there does indeed exist a solution u of (P_{γ^*}) in $C^\infty(\bar{\Omega}\times\mathbb{R}^+)$. (The corner condition is not used for this construction.)

In § 3.1 the corner condition is used to derive a weak formulation of (P_γ). This weak formulation is used in § 3.2 to obtain a uniqueness result and in § 3.3 to extend the existence result for $\gamma=\gamma^*$ to the case in which the initial data lie in $H^5\times H^4$.

In § 3.4 we obtain an existence result for arbitrary $\gamma>0$ and point out the existence of a singular corner wave when $\gamma\neq\gamma^*$. We then analyze the properties of this corner wave.

3.1. Weak formulation of the problem. Suppose there exists a function u such that

$$\begin{gathered} u\in \mathbf{C}^3(\bar\Omega^*\times\mathbb{R}^+)\cap\mathbf{C}^1(\bar\Omega\times[0,\infty)),\\ \forall t\geqq 0\quad u(t)\text{ has compact support,}\\ u\text{ satisfies the equations of }(\mathrm{P}_\gamma). \end{gathered}\tag{3.1}$$

We differentiate the wave equation (1.1) with respect to t, multiply by a test function $v(x_1,x_2)$ in $C_0^\infty(\bar\Omega)$, and integrate in space to see that such a function u satisfies

$$\frac{d^3}{dt^3}\left(\int_\Omega uv\,dx\right)-\int_\Omega \Delta\frac{\partial u}{\partial t}v\,dx=0.$$

Using Green's formula we obtain

$$\frac{d^3}{dt^3}\left(\int_\Omega u(t)v\,dx\right)+\frac{d}{dt}\left(\int_\Omega \nabla u(t)\nabla v\,dx\right)-\int_\Gamma\frac{\partial^2 u}{\partial n\,\partial t}(t)v\,d\sigma=0.\tag{3.2}$$

Let us transform the boundary integral as follows:

$$-\int_\Gamma\frac{\partial^2 u}{\partial n\,\partial t}(t)v\,d\sigma=-\int_{\Gamma_1}\frac{\partial^2 u}{\partial x_2\,\partial t}v\,dx_1-\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_1\,\partial t}v\,dx_2.$$

Using the boundary conditions on Γ_1 and Γ_2 we see that

$$\begin{aligned} -\int_\Gamma\frac{\partial^2 u}{\partial n\,\partial t}v\,d\sigma&=\frac{d^2}{dt^2}\left(\int_{\Gamma_1}uv\,dx_1\right)-\frac12\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_1^2}v\,dx_1\\ &\quad+\frac{d^2}{dt^2}\left(\int_{\Gamma_2}uv\,dx_2\right)-\frac12\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_2^2}v\,dx_2. \end{aligned}\tag{3.3}$$

A double integration by parts leads to the equality

$$\begin{aligned} -\frac12\int_{\Gamma_1}\frac{\partial^2 u}{\partial x_1^2}v\,dx_1-\frac12\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_2^2}v\,dx_2&=\frac12\int_{\Gamma_1}\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}dx_1+\frac12\int_{\Gamma_2}\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}dx_2\\ &\quad-\frac12\left\{\frac{\partial u}{\partial x_1}(0)+\frac{\partial u}{\partial x_2}(0)\right\}v(0). \end{aligned}\tag{3.4}$$

We finally use the corner equation $(\mathrm{CC}\gamma)$ to obtain

$$\begin{aligned} -\frac12\int_{\Gamma_1}\frac{\partial^2 u}{\partial x_1^2}v\,dx_1-\frac12\int_{\Gamma_2}\frac{\partial^2 u}{\partial x_2^2}v\,dx_2&=\frac12\left\{\int_{\Gamma_1}\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}dx_1+\int_{\Gamma_2}\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}dx_2\right\}\\ &\quad+\frac{\gamma}{2}\frac{d}{dt}(u(0)v(0)). \end{aligned}\tag{3.5}$$

Thus, regrouping (3.2), (3.3), and (3.5), we have proved that u satisfies the equation

$$\begin{aligned} &\frac{d^3}{dt^3}\left(\int_\Omega uv\,dx\right)+\frac{d^2}{dt^2}\left(\int_\Gamma uv\,d\sigma\right)+\frac{d}{dt}\left(\int_\Omega\nabla u\nabla v\,dx+\frac{\gamma}{2}u(0)v(0)\right)\\ &\qquad+\frac12\int_\Gamma\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau}d\sigma=0\quad\text{for all }v\in\mathbf{C}_0^\infty(\bar\Omega). \end{aligned}\tag{3.6}$$

It is clear that (3.6) can have a sense even if u is less regular than required in (3.1). Thus we are led to introduce the functional space

$$(3.7)\qquad V=\{v\in H^1(\Omega);\ v_1(x_1)=v(x_1,0)\in H^1(\Gamma_1),\ v_2(x_2)=v(0,x_2)\in H^1(\Gamma_2),\ v_1(0)=v_2(0)\overset{\text{def}}{=}v(0)\}.$$

Note that V is well defined. Indeed, for v in $H^1(\Omega)$ the trace v_1 (respectively, v_2) belongs to $H^{1/2}(\Gamma_1)$ (respectively, $H^{1/2}(\Gamma_2)$), and thus we can ask if it also belongs to $H^1(\Gamma_1)$ (respectively, $H^1(\Gamma_2)$). Moreover, $v_j(0)$ makes sense as v_j belongs to $H^1((-\infty,0))$. Note also that V contains $\mathbf{C}_0^\infty(\bar\Omega)$. We define the following norm on V:

$$(3.8)\qquad \|v\|_V^2=\int_\Omega |v|^2\,dx+\int_\Omega |\nabla v|^2\,dx+\int_\Gamma \left|\frac{\partial v}{\partial\tau}\right|^2 d\sigma+|v(0)|^2$$

($\partial v/\partial\tau$ is the function on $L^2(\Gamma)$ defined by $\partial v/\partial\tau|\Gamma_j=\partial v_j/\partial x_j$, $j=1,2$). It is easy to show Lemma 3.1.

LEMMA 3.1. *Equipped with the norm* (3.8), *V is a Hilbert space.*

Remark. Note that V is nothing but the closure of $\mathbf{C}_0^\infty(\bar\Omega)$ for the norm $\|\cdot\|_V$.

To complete our mathematical framework, we introduce, for H a Hilbert space (with norm $\|\cdot\|_H$) and σ a positive number, the space

$$(3.9)\qquad L^1_\sigma(\mathbb{R}^+;H)=\left\{v(t):\mathbb{R}^+\to H;\ \int_0^{+\infty}\|v(t)\|_H\,e^{-\sigma t}\,dt<+\infty\right\}.$$

Such a space is of interest because it is possible to define the Laplace transform of any v in V by (see [2], [3])

$$(3.10)\qquad \hat v(p)=\int_0^{+\infty} v(t)\,e^{-pt}\,dt\quad \text{for }\operatorname{Re}(p)>\sigma$$

and to show that the function $p\to\hat v(p)\in H$ is analytic in the half-plane $\operatorname{Re}(p)>\sigma$. In particular we have the property

$$(3.11)\qquad \hat v(p)=0\quad\text{for all } p\in\mathbb{R}\ \text{such that } p>\sigma \text{ implies } v(t)=0\quad \text{for all } t\geqq 0.$$

Note also that $L^1_\sigma(\mathbb{R}^+;V)\subset L^1_{\mathrm{loc}}(\mathbb{R}^+;V)\subset\mathbf{D}'(\mathbb{R}^+;V)$.

Now we can give a definition.

DEFINITION 3.1. A function $u(t)$ is a weak solution of problem (P_γ) if and only if there exists $\sigma>0$ such that

(i) $\left(u,\dfrac{du}{dt}\right)\in L^1_\sigma(\mathbb{R}^+;V)\times L^1_\sigma(\mathbb{R}^+;L^2(\Omega))$,

(ii) $u(0)=u_0\in V,\qquad \dfrac{du}{dt}(0)=u_1\in L^2(\Omega)$,

(iii)
$$\frac{d^3}{dt^3}\left(\int_\Omega uv\,dx\right)+\frac{d^2}{dt^2}\left(\int_\Gamma uv\,d\sigma\right)+\frac{d}{dt}\left(\int_\Omega \nabla u\nabla v\,dx+\frac{\gamma}{2}u(0)v(0)\right)+\frac12\int_\Gamma\frac{\partial u}{\partial\tau}\frac{\partial v}{\partial\tau}\,d\sigma=0 \text{ in } \mathbf{D}'(\mathbb{R}^+)\quad \text{for all } v\in V.$$

Remarks. In (i), the derivative du/dt is taken in the sense of distributions with values in $L^2(\Omega)$.

If we choose $v \in \mathbf{D}(\Omega)$, it is easy to see that (iii) can be interpreted as

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in } \mathbf{D}'(\mathbb{R}^+ \times \Omega),$$

which shows, as $\Delta \in \mathscr{L}(H^1(\Omega); H^{-1}(\Omega))$, that d^2u/dt^2 belongs to the space $L^1_\sigma(\mathbb{R}^+; H^{-1}(\Omega))$. Then we deduce that

$$u \in \mathbf{C}^1([0, +\infty); H^{-1}(\Omega)) \cap \mathbf{C}^0([0, +\infty); L^2(\Omega)),$$

which permits us to give a meaning to initial conditions (ii).

3.2. A uniqueness result. Let u be a solution of (P_γ) in the sense of Definition 3.1. Then for $\mathrm{Re}\,(p) > \sigma$ its Laplace transform $\hat{u}(p)$ is, at least formally, a solution of

$$(3.12)\qquad \begin{aligned} &-\Delta \hat{u} + p^2 \hat{u} = u_1 + p u_0 = f && \text{in } \Omega, \\ &-\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial x_1^2} + p\frac{\partial \hat{u}}{\partial x_2} + p^2 \hat{u} = 0 && \text{on } \Gamma_1, \\ &-\frac{1}{2}\frac{\partial^2 \hat{u}}{\partial x_2^2} + p\frac{\partial \hat{u}}{\partial x_1} + p^2 \hat{u} = 0 && \text{on } \Gamma_2, \\ &\frac{\partial \hat{u}}{\partial x_1} + \frac{\partial \hat{u}}{\partial x_2} + \gamma p \hat{u} = 0 && \text{at } 0, \end{aligned}$$

or more exactly, using the weak formulation, of

$$(3.13)\qquad \begin{aligned} &\hat{u}(p) \in V, \\ &\hat{a}(p, \hat{u}(p), \hat{v}) = \langle L(p), \hat{v}\rangle \quad \text{for all } \hat{v} \in V, \end{aligned}$$

where the bilinear form $\hat{a}(p, \cdot, \cdot)$ is defined by

$$(3.14)\qquad \begin{aligned} \hat{a}(p, \hat{u}, \hat{v}) = &\int_\Omega \nabla \hat{u} \cdot \overline{\nabla \hat{v}}\, dx + p^2 \int_\Omega \hat{u}\bar{\hat{v}}\, dx \\ &+ p\int_\Gamma \hat{u}\bar{\hat{v}}\, d\sigma + \frac{1}{2p}\int_\Gamma \frac{\partial \hat{u}}{\partial \tau}\overline{\frac{\partial \hat{v}}{\partial \tau}}\, d\sigma + \frac{\gamma}{2}\,\hat{u}(0)\overline{\hat{v}(0)}, \end{aligned}$$

and the linear form $\langle \hat{L}(p), \cdot\rangle$ is given by

$$(3.15)\qquad \langle \hat{L}(p), \hat{v}\rangle = \int_\Omega u_1 \bar{\hat{v}}\, dx + p\int_\Omega u_0 \bar{\hat{v}}\, dx.$$

We have the following lemma.

LEMMA 3.2. *For $\gamma > 0$, if p is a positive real number, the bilinear form $\hat{a}(p, \hat{u}, \hat{v})$ is V elliptic.*

Proof. It suffices to write

$$\hat{a}(p, \hat{u}, \hat{u}) = \int_\Omega |\nabla \hat{u}|^2\, dx + p^2 \int_\Omega |\hat{u}|^2\, dx + p\int_\Gamma |\hat{u}|^2\, d\sigma + \frac{1}{2p}\int_\Gamma \left|\frac{\partial \hat{u}}{\partial \tau}\right|^2 d\sigma + \frac{\gamma}{2}|\hat{u}(0)|^2$$

to deduce that $\hat{a}(p, \hat{u}, \hat{u}) \geqq \inf(1, 1/2p, p^2, \gamma/2)\|\hat{u}\|_V^2$. $\square$

Then, using the Lax-Milgram lemma, we see that problem (3.13) admits, for (u_0, u_1) in $V \times L^2(\Omega)$, a unique solution $\hat{u}(p)$ in H. In particular, if $u_0 = u_1 = 0$, we necessarily have $\hat{u}(p) = 0$ for all $p > \sigma$. Using property (3.9), we deduce the following theorem.

THEOREM 3.1. *The solution u of P_γ, if it exists, is unique.*

Of course, we could try to develop an existence theory for (P_γ) using the Laplace transform method. We could hope to prove an existence and uniqueness result for problem (3.12) when $p=\eta+i\omega$, $\eta>0$ being fixed and ω varying in $\mathbb{R}$, and to obtain estimates on $\hat{u}(\eta+i\omega)$ (in V) for any ω in $\mathbb{R}$. We did not succeed in obtaining such estimates (we can show that such estimates are available in the region $\omega^2 \leqq 3\eta^2$, but we do not know how to extend these estimates to the whole line $p=\eta+i\omega$).

3.3. The existence of a smooth solution for $\gamma=\gamma^*=\frac{3}{2}$. We assume that the initial data functions u_0 and u_1 belong to $C_0^\infty(\Omega)$. We will construct a solution u of (P_{γ^*}) of class C^∞. Toward this end we remark that if u is such a function then the C^∞ function v defined by

$$\begin{aligned}
&v = L_1 L_2 u,\\
&L_1 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_2\,\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x_1^2},\\
&L_2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1\,\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x_2^2},
\end{aligned} \tag{3.16}$$

satisfies

$$\begin{aligned}
&\frac{\partial^2 v}{\partial t^2} - \Delta v = 0 && \text{in } \Omega \text{ for } t>0,\\
&v_{|\Gamma} = 0 && \text{for } t>0,\\
&v(x,0) = v_0(x) && \text{in } \Omega,\\
&\frac{\partial v}{\partial t}(x,0) = v_1(x) && \text{in } \Omega,
\end{aligned} \tag{3.17}$$

where

$$\begin{aligned}
v_0 &= \frac{1}{2}\Delta^2 u_0 + \frac{1}{4}\frac{\partial^4 u_0}{\partial x_1^2\,\partial x_2^2} + \frac{\partial^2 \Delta u_0}{\partial x_1\,\partial x_2} + \frac{1}{2}\left(\frac{\partial^3 u_1}{\partial x_1^3} + \frac{\partial^3 u_1}{\partial x_2^3}\right) + \frac{\partial^3 u_1}{\partial x_1^2\,\partial x_2} + \frac{\partial^3 u_1}{\partial x_1\,\partial x_2^2},\\
v_1 &= \frac{1}{2}\Delta^2 u_1 + \frac{1}{4}\frac{\partial^4 u_1}{\partial x_1^2\,\partial x_2^2} + \frac{\partial^2 \Delta u_1}{\partial x_1\,\partial x_2} + \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\Delta^2 u_0 - \frac{1}{2}\left(\frac{\partial^3}{\partial x_1^3} + \frac{\partial^3}{\partial x_2^3}\right)\Delta u_0.
\end{aligned} \tag{3.18}$$

Furthermore, as u is a solution of the wave equation, we deduce that

$$v = Lu = \frac{1}{4}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2 u. \tag{3.19}$$

Now, to actually construct a C^∞ solution of (P_{γ^*}), we let v denote the unique solution of (3.17) whose existence is guaranteed by the theory of images. We then let u be the unique solution in the quarter-plane Ω of

$$\begin{aligned}
&Lu = v \quad \text{in } \Omega \quad \text{for } t>0,\\
&u = u_0, \quad \frac{\partial u}{\partial t} = u_1, \quad \frac{\partial^2 u}{\partial t^2} = \Delta u_0, \quad \frac{\partial^3 u}{\partial t^3} = \Delta u_1 \quad \text{in } \Omega \quad \text{for } t=0.
\end{aligned} \tag{3.20}$$

This problem is well posed, as its solution is obtained by integrating in the quarter-plane $\Omega=\{(x_1,x_2);\ x_1<0,\ x_2<0\}$. Moreover, its solution u is C^∞.

THEOREM 3.2. *The function u defined by* (3.17), (3.18), *and* (3.20) *is a classical solution of* (P_{γ^*}).

Proof. To see that u satisfies the wave equation in Ω it suffices to note that $w=\partial^2 u/\partial t^2-\Delta u$ is the unique solution to

$$Lw=0 \quad \text{in } \Omega \quad \text{for } t>0,$$

$$w=\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial t^2}=\frac{\partial^3 w}{\partial t^3}=\frac{\partial^4 w}{\partial t^4}=0 \quad \text{in } \Omega \quad \text{for } t=0.$$

Thus $w=0$ in Ω.

Since u is C^∞, u satisfies the wave equation on Γ. Thus to see that (CL1) holds for u on Γ_1 it is sufficient to show that $w_2=(\partial/\partial t+\partial/\partial x_2)^2 u$ vanishes on Γ_1. For this result we note that $g_1(x_1, t)=w_2(x_1, 0, t)$ is the unique solution of

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x_1}\right)^2 g_1(x_1, t)=0 \qquad \text{for } x_1 \in \Gamma_1 \text{ and } t>0,$$

$$g_1(x_1, 0)=\frac{\partial}{\partial t} g_1(x_1, 0)=0 \qquad \text{for } x_1 \in \Gamma_1,$$

and thus that $g_1(x_1, t)=0$ for $x_1 \in \Gamma_1$.

That (CL2) holds for u on Γ_2 is demonstrated in the same manner. □

For completeness we would like to check that the solution u constructed above belongs to the class of functions for which we obtained the uniqueness result in § 3.2, i.e., we would like to check that u is a weak solution of (P_γ) in the sense of Definition 3.1. (While we know that u is C^∞, we do not yet know that it belongs to the space $L^1_\sigma(\mathbb{R}^+; V)$, for which we must show that the energy does not increase more than exponentially in time.)

In checking the solution we obtain estimates that allow us to extend our existence and uniqueness result to the case in which the initial data (u_0, u_1) lies in $H^5(\Omega)\times H^4(\Omega)$. (This result is probably not optimal, though we have not succeeded in improving it.)

THEOREM 3.3. *The classical solution u of* (P_{γ^*}) *defined by* (3.17), (3.18), *and* (3.20) *is also a weak solution in the sense of Definition* 3.1. *Thus it is unique.*

Proof. Suppose that f and g denote functions regular in Ω such that

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x_1}=g \quad \text{in } \Omega.$$

If we multiply this equation by f, integrate over Ω, and then integrate from zero to t we get

$$\frac{1}{2}\int_\Omega |f|^2\,dx+\int_0^t\int_{\Gamma_2}|f|^2\,dx_2\,ds=\frac{1}{2}\int_\Omega |f_0|^2+\int_0^t\int_\Omega fg\,dx\,ds, \tag{3.21}$$

where $f_0(x)\equiv f(x, 0)$. Then, using Gronwall's Lemma, we obtain

$$\int_\Omega |f|^2\,dx \leqq \left\{\int_\Omega |f_0|^2\,dx+\int_0^t\int_\Omega |g|^2\,dx\,ds\right\}, \tag{3.22}$$

and plugging (3.22) into (3.21) we have

$$\int_0^t\int_{\Gamma_2}|f|^2\,dx_2\,ds \leqq (1+2t)\int_\Omega |f_0|^2\,dx+\int_0^t (1+t-s)\left(\int_\Omega |g|^2\,dx\right)ds. \tag{3.23}$$

Now consider two functions w and g defined in Ω and related by the sequence of equalities (with $w = w_{1122}$)

$$\frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial x_1} = g,$$

$$\frac{\partial w_{12}}{\partial t} + \frac{\partial w_{12}}{\partial x_2} = w_1,$$

$$\frac{\partial w_{112}}{\partial t} + \frac{\partial w_{112}}{\partial x_1} = w_{12},$$

$$\frac{\partial w_{1122}}{\partial t} + \frac{\partial w_{1122}}{\partial x_2} = w_{112}.$$

Then assuming that $\int_\Omega |g|^2\,dx \leqq \|G_0\|^2$ uniformly in time, we get, using (3.22),

$$\int |w_1|^2\,dx \leqq 2(|w_1(0)|^2 + \|G_0\|^2 t),$$

$$\int |w_{12}|^2\,dx \leqq 2\left\{|w_{12}(0)|^2 + 2|w_1(0)|^2 t + \|G_0\|^2 \frac{t^2}{2}\right\},$$

$$\int |w_{112}|^2\,dx \leqq 2\left\{|w_{112}(0)|^2 + 2|w_{12}(0)|^2 t + 2|w_1(0)|^2 t^2 + \| + \|G_0\|^2 \frac{t^3}{3}\right\},$$

and finally, with the aid of (3.22) and (3.23), we see that

$$\text{(3.24)}\quad \begin{aligned} &\int |w|^2\,dx \leqq C(1+t^4)(|w(0)|^2 + |w_{112}(0)|^2 + |w_{12}(0)|^2 + |w_1(0)|^2 + \|G_0\|^2),\\ &\int_0^t \int_{\Gamma_2} |w|^2\,dx_2\,ds \leqq C(1+t^5)(|w(0)|^2 + |w_{112}(0)|^2 + |w_{12}(0)|^2 + |w_1(0)|^2 + \|G_0\|^2). \end{aligned}$$

We can now invert the roles of x_1 and x_2 in the previous estimates. We then introduce w_2, w_{21}, and w_{221} to obtain

$$\text{(3.25)}\quad \begin{aligned} &\int |w|^2\,dx \leqq C(1+t^4)(|w(0)|^2 + |w_{221}(0)|^2 + |w_{21}(0)|^2 + |w_2(0)|^2 + \|G_0\|^2),\\ &\int_0^t \int_{\Gamma_1} |w|^2\,dx_1 \leqq C(1+t^5)(|w(0)|^2 + |w_{221}(0)|^2 + |w_{21}(0)|^2 + |w_2(0)|^2 + \|G_0\|^2). \end{aligned}$$

Now we can apply estimates (3.24) and (3.25) successively to

$$w = \frac{\partial u}{\partial t}, \qquad g = \frac{\partial v}{\partial t},$$

$$w = \frac{\partial u}{\partial x_1}, \qquad g = \frac{\partial v}{\partial x_1},$$

$$w = \frac{\partial u}{\partial x_2}, \qquad g = \frac{\partial v}{\partial v x_2},$$

and we can take $\|G_0\|^2 = \int |v_1|^2\,dx + \int |\nabla v_0|^2\,dx$ since v satisfies a classical energy identity.

Using the relations between (v_0, v_1) and (u_0, u_1) it is easy to get the following estimate:

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx + \int_\Omega |\nabla u|^2\, dx + \int_\Gamma \left|\frac{\partial u}{\partial t}\right|^2 d\sigma + \int_\Gamma \left|\frac{\partial u}{\partial \tau}\right|^2 d\sigma + \int_\Gamma \left|\frac{\partial u}{\partial \nu}\right|^2 d\sigma$$
$$\leqq C(1+t^5)(\|u_0\|^2_{H^5(\Omega)} + \|u_1\|^2_{H^4(\Omega)}),$$

where τ and ν denote unit tangential and normal vector fields, respectively, on Γ. The theorem follows. □

Decomposition of the solution. It is interesting to analyze the structure of the solution we constructed with the help of the theory of images. Indeed, the solution v of (3.20) can be written as

$$v = v_I + v_R^1 + v_R^2 + v_R^0,$$

where $(v_I, v_R^1, v_R^2, v_R^0)$ are, respectively, the restrictions to Ω of the solutions of the wave equation in the whole plane $\mathbb{R}^2$ corresponding to the following initial data ($\tilde{v}_0$, $\tilde{v}_1$ being the extensions of v_0, v_1 by 0 outside Ω):

$$v_I(x_1, x_2, 0) = \tilde{v}_0(x_1, x_2), \qquad \frac{\partial v_I}{\partial t}(x_1, x_2, 0) = \tilde{v}_1(x_1, x_2),$$

$$v_R^1(x_1, x_2, 0) = -\tilde{v}_0(x_1, -x_2), \qquad \frac{\partial v_R^1}{\partial t}(x_1, x_2, 0) = -\tilde{v}_1(x_1, -x_2),$$

$$v_R^2(x_1, x_2, 0) = -\tilde{v}_0(-x_1, x_2), \qquad \frac{\partial v_R^2}{\partial t}(x_1, x_2, 0) = -\tilde{v}_1(-x_1, x_2),$$

$$v_R^0(x_1, x_2, 0) = \tilde{v}_0(-x_1, -x_2), \qquad \frac{\partial v_R^0}{\partial t}(x_1, x_2, 0) = \tilde{v}_1(-x_1, -x_2).$$

Then the solution u constructed in Theorem 3.2 can be equivalently decomposed:

$$u = u_I + u_R^1 + u_R^2 + u_R^0, \tag{3.26}$$

where u_I is the restriction to Ω of the solution $\tilde{u}_I$ of the wave equation without boundary, i.e., with ($\tilde{u}_0$, $\tilde{u}_1$ being defined as were $\tilde{v}_0$, $\tilde{v}_1$),

$$\frac{\partial^2 \tilde{u}_I}{\partial t^2} - \Delta \tilde{u}_I = 0,$$

$$\tilde{u}_I(x, 0) = \tilde{u}_0(x),$$

$$\frac{\partial \tilde{u}_I}{\partial t}(x, 0) = \tilde{u}_1(x),$$

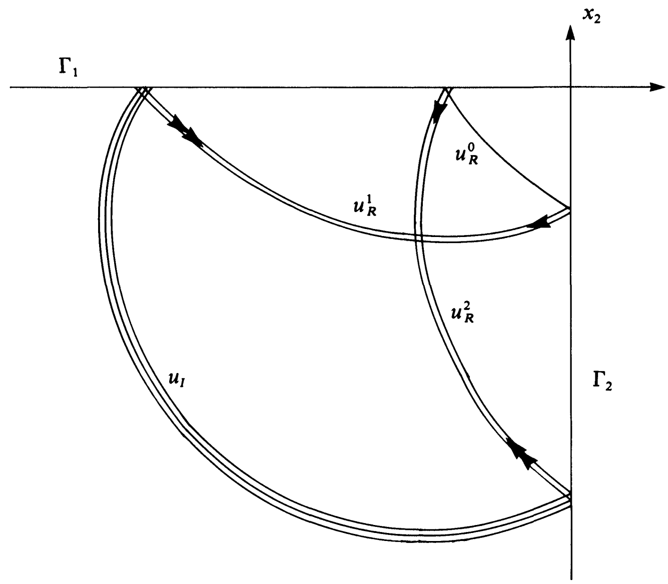
and where u_R^j $(j = 1, 2, 0)$.denotes the restriction to Ω of the solution $\tilde{u}_R^j$ of

$$\mathbf{L}(\tilde{u}_R^j) = \tilde{v}_R^j, \qquad \tilde{u}_R^j(x, 0) = \frac{\partial \tilde{u}_R^j}{\partial t}(x, 0) = \frac{\partial^2 \tilde{u}_R^j}{\partial t^2}(x, 0) = \frac{\partial^3 \tilde{u}_R^j}{\partial t^3}(x, 0) = 0.$$

The decomposition (3.26) can be interpreted as follows:

- u_I is the incident wave.
- u_R^1 is the wave reflected by the absorbing boundary Γ_1.
- u_R^2 is the wave reflected by the absorbing boundary Γ_2.
- u_R^0 is the secondary reflected wave corresponding to the reflection of u_R^1 on Γ_2 and that of u_R^2 on Γ_1.

In the case of a point source, we can represent the different wave fronts corresponding to decomposition (3.26) as shown below. The three curves depicting u_I indicate that the amplitude of u_I is greater than that of u_R^1 and u_R^2, each represented by two curves, which is in turn greater than that of u_R represented by a single curve. Also, along u_R^1 and u_R^2 the arrows point in the direction of decreasing amplitude (cf. numerical results of § 4).



3.4. The case $\gamma \neq \frac{3}{2}$: existence of a corner wave. We return now to the problem (P_γ) for arbitrary $\gamma > 0$. We again assume that all initial data is C^∞ and is compactly supported in Ω. First, suppose that u_γ is a solution to (P_γ) for some given γ. If we denote by u^* the regular solution of P_γ for $\gamma = \frac{3}{2}$ we may define

$$v_\gamma = u_\gamma - u^*.$$

Thus we can decompose u_γ:

$$u_\gamma = u^* + v_\gamma,$$

where u^* is regular everywhere and v_γ is a function that clearly satisfies the following equations (here we omit the index γ for simplicity):

$$(\tilde{P}_\gamma) \quad \begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \Delta v = 0, & x \in \Omega, \quad t > 0, \\[2mm] \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t\, \partial x_2} - \dfrac{1}{2} \dfrac{\partial^2 v}{\partial x_1^2} = 0, & x \in \Gamma_1, \quad t > 0 \quad (\text{CL1}), \\[2mm] \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial^2 v}{\partial t\, \partial x_1} - \dfrac{1}{2} \dfrac{\partial^2 v}{\partial x_2^2} = 0, & x \in \Gamma_2, \quad t > 0 \quad (\text{CL2}), \\[2mm] \gamma \dfrac{\partial^2 v}{\partial t^2} + \dfrac{\partial}{\partial t}\left(\dfrac{\partial v}{\partial x_1} + \dfrac{\partial v}{\partial x_2}\right) = g, & x = 0, \quad t > 0 \quad (\text{CC}\gamma)', \\[2mm] v(x, 0) = \dfrac{\partial v}{\partial t}(x, 0) = 0, & x \in \Omega, \end{cases}$$

with

$$g = g_\gamma \equiv (\gamma - \gamma^*) \frac{\partial^2 u^*}{\partial t^2}. \tag{3.27}$$

(We remark that, in this section, for simplicity of exposition we have chosen to work with the corner condition $(\mathrm{CC}\gamma)'$ instead of $(\mathrm{CC}\gamma)$, since with the zero initial conditions $(\mathrm{CC}\gamma)'$ and $(\mathrm{CC}\gamma)$ are equivalent.)

What we will do in the following is to show that there exists a solution H_γ of $\tilde{P}_\gamma$ with $g = \delta(t)$. Then v_γ will be defined to be $H_\gamma * g_\sigma$ and will thus be a solution to $(\tilde{\mathrm{P}}_\gamma)$ with $g = g_\gamma$. We will show that v_γ has a singularity at the corner and will say that v_γ is the corner wave due to the corner condition $(\mathrm{CC}\gamma)$. Finally, the solution u_γ to $(\tilde{\mathrm{P}}_\gamma)$, with $g = 0$ or equivalently to (P_γ), will be obtained by adding the regular solution u^* of (P_{γ^*}) to v_γ: $u_\gamma = v_\gamma + u^*$.

To construct the solution H_γ to $(\tilde{\mathrm{P}}_\gamma)$ we will proceed much as for the construction in § 3.3 of the regular solution u^* to (P_{γ^*}), except that here we need to introduce a singularity at the corner point. Thus we will start with the elementary solution G to the wave equation in $\mathbb{R}^2$ at the point $(0, 0)$. Then, as before, we will integrate G along the characteristics of (3.19) to obtain $\tilde{G}$, which should satisfy the wave equation as well as the zero initial conditions of $(\tilde{\mathrm{P}}_\gamma)$. However, as G does not vanish on the boundary Γ, there is no reason why $\tilde{G}$ should satisfy the boundary conditions (CL1) and (CL2) of $(\tilde{\mathrm{P}}_\gamma)$. However, we observe that $\partial^2 G/\partial x_1\, \partial x_2$, which will play the same role here as v in § 3.3, does vanish on Γ. Therefore, taking the corresponding derivative of $\tilde{G}$, $H = \partial^2 G/\partial x_1\, \partial x_2$, we will obtain a function H that we will demonstrate satisfies all of the equations of $(\tilde{\mathrm{P}}_\gamma)$ except the corner condition. Moreover, we will show that a constant multiple $H_\gamma = a_\gamma H$ of H does indeed satisfy the condition $(\mathrm{CC}\gamma)'$.

To be more precise, we let G denote the solution to

$$\frac{\partial^2 G}{\partial t^2} - \Delta G = \delta(x) \times \delta(t), \quad x \in \mathbb{R}^2, \quad t > 0,$$

$$G(x, 0) = \frac{\partial G}{\partial t}(x, 0) = 0, \qquad x \in \mathbb{R}^2.$$

Then G is given explicitly by

$$G(x, t) = \begin{cases} \frac{1}{2}\pi (t^2 - x_1^2 - x_2^2)^{-1/2}, & x_1^2 + x_2^2 < t^2, \\ 0 & \text{otherwise.} \end{cases}$$

Next we integrate G twice in the direction $(t + x_1)$ and twice in the direction $(t + x_2)$ to obtain the solution $\tilde{G}$ to the problem

$$\begin{aligned} L\tilde{G} &= \frac{1}{4}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}\right)^2 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_2}\right)^2 \tilde{G} = G, \qquad x \in \mathbb{R}^2, \quad t > 0, \\ \tilde{G}(x, 0) &= \frac{\partial \tilde{G}}{\partial t}(x, 0) = \frac{\partial^2 \tilde{G}}{\partial t^2}(x, 0) = \frac{\partial^3 \tilde{G}}{\partial t^3}(x, 0) = 0, \quad x \in \mathbb{R}^2. \end{aligned} \tag{3.28}$$

An explicit expression for $\tilde{G}$ may be obtained by straightforward calculation:

$$\tilde{G} = \begin{cases} \frac{1}{12}(t - x_1 - x_2)^3 F(b(x, t)), & x_1^2 + x_2^2 < t^2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.29}$$

where

$$(3.30)\qquad \begin{aligned} F(b) &= \int_0^b \int_0^s \frac{\sigma^{3/2}}{(1+\sigma)^2}\, d\sigma\, ds, \\ b(x, t) &= \frac{t^2 - x_1^2 - x_2^2}{(t - x_1 - x_2)^2}. \end{aligned}$$

Remark. We may further calculate that

$$(3.31)\qquad F(b) = \left(5 + \frac{4}{3} b\right)\sqrt{b} - (5 + 3b) \arctan \sqrt{b},$$

with b given by (3.30); however, as we are interested in the derivatives of $\tilde{G}$, the form (3.30) is easier to work with.

Concerning the regularity of $\tilde{G}$, we note that $\tilde{G}$ is bounded but singular along the plane $x_1 + x_2 = t$. However, the intersection of this plane with the domain that interests us, $\bar{\Omega} \times \mathbb{R}^+$, is the point $x = 0$, $t = 0$.

Next we introduce the function H defined by

$$H(x, t) = \frac{\partial^2 \tilde{G}}{\partial x_1\, \partial x_2}(x, t).$$

We can show the following result.

THEOREM 3.4. (i) *H vanishes in the part of $\bar{\Omega} \times \mathbb{R}^+$ outside the closure of the cone $x_1^2 + x_2^2 < t^2$.*

(ii) *H is C^∞ in the part of $\bar{\Omega} \times \mathbb{R}^+$.*

(iii) *H is C^1 in $\bar{\Omega} \times \mathbb{R}^+ \backslash \{(0, 0, 0)\}$.*

(iv) *The second derivatives of H are singular along $x_1^2 + x_2^2 = t^2$.*

(v) *H satisfies the wave equation in the distributional sense in $\mathbb{R}^2 \times \mathbb{R}^+_*$ and in the classical sense in $\bar{\Omega} \times \mathbb{R}^+_*$ except along $x_1^2 + x_2^2 = t^2$. In particular*

$$\frac{\partial^2 H}{\partial t^2} - \Delta H = 0, \qquad (x, t) \in \bar{\Omega} \times \mathbb{R}^+ \cap \{(x, t) : x_1^2 + x_2^2 < t\}.$$

H also satisfies the initial conditions

$$H(x, 0) = \frac{\partial H}{\partial t}(x, 0) = 0, \qquad x \in \Omega.$$

(vi) *H satisfies the boundary conditions* (CL1) *and* (CL2).

(vii) *H satisfies* $(\mathrm{CC}_{\gamma^*})'$ *with $g = 0$ for $t > 0$.*

Proof. See Theorem 3.3, Corollary 3.1, and Lemmas 3.2 and 3.3 of [1] for the details. The idea is the same as that for the smooth solution given in Theorem 3.2, the difference being that here we have introduced a singularity via the elementary solution G. □

We point out that while H satisfies $(\mathrm{CC}_{\gamma^*})'$ with $g = 0$ for $t > 0$, it does not satisfy (CC_{γ^*}). In fact we establish the following important result.

LEMMA 3.3. *The first derivatives of H in time and in space at the corner $x_1 = x_2 = 0$, for $t > 0$, are constant in time and positive, i.e.,*

$$\frac{\partial H}{\partial t}(0, t) = c_1 > 0 \quad \textit{for } t > 0,$$

$$\frac{\partial H}{\partial x_1}(0, t) = \frac{\partial H}{\partial x_2}(0, t) = c_2 > 0 \quad \textit{for } t > 0.$$

Thus for $\gamma > 0$, we have along the axis $(0, 0, t)$, $t > 0$,

$$\frac{d}{dt}\left(\gamma\frac{\partial H}{\partial t}+\frac{\partial H}{\partial x_1}+\frac{\partial H}{\partial x_2}\right)=\frac{1}{a_\gamma}\,\delta(0),$$

where the amplitude a_γ is defined by

$$\frac{1}{a_\gamma}=\gamma c_1+2c_2>0.$$

Proof. We consider first the time derivative. From (3.29), (3.30) we calculate

$$(3.32)\qquad \begin{aligned} H=\frac{\partial^2}{\partial x_1\,\partial x_2}\tilde{G}&=\frac{1}{2}(t-x_1-x_2)F(b)-\frac{1}{4}(t-x_1-x_2)^2F'(b)\left(\frac{\partial b}{\partial x_1}+\frac{\partial b}{\partial x_2}\right)\\ &\quad+\frac{1}{12}(t-x_1-x_2)^3\left(F''(b)\frac{\partial b}{\partial x_1}\frac{\partial b}{\partial x_2}+F'(b)\frac{\partial^2 b}{\partial x_1\,\partial x_2}\right).\end{aligned}$$

Now for $x_1=x_2=0$ and $t>0$, $b=1$ while $\partial b/\partial x_1=\partial b/\partial x_2=t/2$ and $\partial^2 b/\partial x_1\,\partial x_2=6/t^2$. Thus

$$H(0,t)=\frac{t}{4}\left(2F(1)-2F'(1)+\frac{4}{3}F''(1)\right),\qquad t>0,$$

and we conclude that $\partial H/\partial t\,(0,t)$ is constant, $t>0$. More precisely, we have

$$F(1)-F'(1)=-\int_0^1\frac{\sigma^{5/2}}{(1+\sigma)^2}\,d\sigma=\frac{23}{6}-\frac{5\pi}{4},\qquad F''(1)=\frac{1}{4},$$

from which we deduce that

$$\frac{\partial H}{\partial t}(0,t)=2-\frac{5\pi}{8}>0.$$

As H is symmetric in the variables x_1, x_2, we clearly have $\partial H/\partial x_1=\partial H/\partial x_2$ at $x=0$. We can obtain an expression for $\partial H/\partial x_1$ from (3.29) by straightforward calculation that permits us to determine that $\partial H/\partial x_1$ is constant in time at $x_1=x_2=0$, $t>0$, but that is not very amenable to determining that the constant is positive. Thus it is simpler to proceed in an indirect fashion.

As H satisfies the wave equation in Ω, $t>0$, in the distributional sense, on integrating over Ω and differentiating in time we obtain

$$\frac{d^3}{dt^3}\int_\Omega H\,dx_1\,dx_2-\frac{d}{dt}\int_\Gamma\frac{\partial H}{\partial \nu}\,d\gamma=0,\qquad t>0.$$

Integrating conditions (CL1) over Γ_1 and (CL2) over Γ_2 and adding, we have

$$\frac{d^2}{dt^2}\int_\Gamma H\,d\gamma+\frac{d}{dt}\int_\Gamma\frac{\partial H}{\partial\nu}\,d\gamma=\frac{1}{2}\left[\frac{\partial H}{\partial x_1}(0,0,t)+\frac{\partial H}{\partial x_2}(0,0,t)\right].$$

We then combine these two equations to arrive at

$$\frac{d^3}{dt^3}\int_\Omega H\,dx_1\,dx_2+\frac{d^2}{dt^2}\int_\Gamma H\,d\gamma=\frac{1}{2}\left[\frac{\partial H}{\partial x_1}(0,0,t)+\frac{\partial H}{\partial x_2}(0,0,t)\right].$$

The relation between H and $\tilde{G}$, $H=\partial^2\tilde{G}/\partial x_1\,\partial x_2$, gives us

$$\frac{\partial^3}{\partial t^3}\tilde{G}(0,0,t)+\frac{\partial^2}{\partial t^2}\left[\frac{\partial\tilde{G}}{\partial x_1}(0,0,t)+\frac{\partial\tilde{G}}{\partial x_2}(0,0,t)\right]=\frac{1}{2}\left[\frac{\partial H}{\partial x_1}(0,0,t)+\frac{\partial H}{\partial x_2}(0,0,t)\right].$$

We calculate from (3.28), (3.29)

$$\tilde{G}(0,0,t)=\frac{1}{12}t^3F(1),$$

$$\frac{\partial \tilde{G}}{\partial x_1}(0,0,t)=\frac{\partial \tilde{G}}{\partial x_2}(0,0,t)=\frac{t^2}{4}\left[-F(1)+\frac{2}{3}F'(1)\right].$$

Thus we have

$$\begin{aligned}\frac{\partial H}{\partial x_1}(0,0,t)&=\frac{\partial H}{\partial x_2}(0,0,t)\\&=-\frac{1}{2}F(1)+\frac{2}{3}F'(1)\\&=\frac{1}{2}\int_0^1\frac{(\sigma+\frac{1}{3})\sigma^{3/2}}{(1+\sigma)^2}\,d\sigma\\&=\frac{\pi-3}{2}>0.\end{aligned}$$

□

It is now natural to define

(3.33) $$H_\gamma=a_\gamma H,$$

and it follows that

$$\frac{d}{dt}\left(\gamma\frac{\partial H_\gamma}{\partial t}+\frac{\partial H_\gamma}{\partial x_1}+\frac{\partial H_\gamma}{\partial x_2}\right)=\delta(0)$$

along the half-line (0, 0, t), $t>0$. Thus H_γ is a solution to $\tilde{P}_\gamma$ with $g=\delta(t)$. We next put

(3.34) $$v_\gamma=H_\gamma * g_\gamma,$$

where g_γ is given by (3.27), and finally define

(3.35) $$u_\gamma=v_\gamma+u^*.$$

THEOREM 3.5. *The function u_γ given by* (3.35) *is the unique weak solution of P_γ. u^* is the regular part of the solution and v_γ is the singular corner wave, whose properties are*:

(i) *v_γ is of finite energy for any $t>0$;*

(ii) *v_γ is of class C^1 in $\bar{\Omega}\times\mathbb{R}^+$;*

(iii) *v_γ has singular second-order spatial derivatives along* (a) *the line $x=0$, and* (b) *the cone $|x|=t$.*

Proof. See [1] for the proof.

We conclude this section with three important remarks.

Remark 3.1. For each positive γ, u_γ is proportional to u^* along the axis $x_1=x_2=0$. Thus u_γ is C^∞ in time along this axis.

Remark 3.2. For each positive γ, u_γ is a solution of (1.1), (1.5), and (1.7) in the class of finite energy functions. We thus have a counterexample to the uniqueness of the solution of the problem without a corner condition.

Remark 3.3. It is interesting to note that for each positive γ, the corner wave v_γ is proportional to $H*\partial^2u^*/\partial t^2$:

$$v_\gamma=\frac{\gamma-\gamma^*}{\gamma c_1+2c_2}H*\frac{\partial^2u^*}{\partial t^2},$$

and that the constant of proportionality remains bounded between $-(\gamma^*/2c_2)$ and $1/c_1$ as γ ranges between 0 and $+\infty$, the constant being negative for $\gamma<\gamma^*$, positive for $\gamma>\gamma^*$, and vanishing for $\gamma=\gamma^*$. In fact we see that v_γ is well defined even for $\gamma\leqq 0$ except for $\gamma=-2(c_2/c_1)$.

4. Numerical results.

4.1. Numerical scheme. Here we exhibit the results of numerical experiments confirming the conclusions obtained in the previous section. These results were obtained using a variational scheme, which we apply not to the wave equation (1.1) itself but to its time derivative so we can use the boundary equations after the standard integration by parts in the variational formulation. The discretization in time is obtained using an explicit scheme incorporating the four time levels $n-2$, $n-1$, n, and $n+1$, as a third-order time derivative appears in the time-differentiated wave equation. We remark that it is not the numerical scheme per se that interests us here, and that another scheme—the finite difference scheme proposed by Engquist and Majda in [5], for example—could just as well have been used.

We let Ω denote the unit square $(0,1)\times(0,1)$ with boundary Γ written as the disjoint union of Γ_1 the upper edge, Γ_2 the right-hand edge, C the upper right-hand corner, and Γ_R the left and lower edges with the three remaining corners (see Fig. 1).

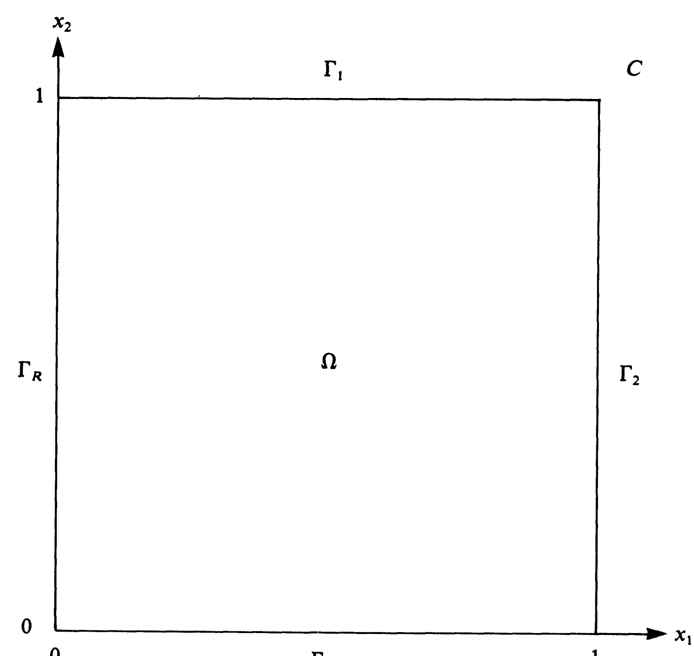


FIG. 1. *Domain* Ω *and boundary* $\Gamma=\Gamma_1\cup\Gamma_2\cup C\cup\Gamma_R$.

The problem we will solve numerically is the following:

$$\frac{\partial^3 u}{\partial t^3}-\frac{\partial \Delta u}{\partial t}=\frac{\partial f}{\partial t}\quad \text{in } \Omega, \tag{4.1}$$

with boundary condition

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\,\partial x_2}-\frac{1}{2}\frac{\partial^2 u}{\partial x_1^2}=0\quad \text{in } \Gamma_1, \tag*{(4.2)$_1$}$$

$$(4.2)_2 \qquad \frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\,\partial x_1}-\frac{1}{2}\frac{\partial^2 u}{\partial x_2^2}=0 \quad \text{in } \Gamma_2,$$

$$(4.3) \qquad \gamma\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}=0 \quad \text{at } C,$$

$$(4.4) \qquad u=0 \quad \text{on } \Gamma_R,$$

and initial conditions

$$(4.5) \qquad \begin{aligned} u&=u_0 \quad &&\text{at } t=0,\\ \frac{\partial u}{\partial t}&=u_1 \quad &&\text{at } t=0, \end{aligned}$$

where $f\in L^2(\Omega)$ is of compact support.

The solution u will be sought in the space W of functions in $H^1(\Omega)$ having trace in $H_0^1(\Gamma)$, where by $H_0^1(\Gamma)$ we will denote the Hilbert space of functions in $L^2(\Gamma)$ whose restrictions to Γ_1 and to Γ_2 are H^1 functions determining the same value at the corner C and whose restriction to Γ_R is identically zero (see § 3.1):

$$\begin{aligned} W&=\{v\in H^1(\Omega)\colon \text{trace}\,(u)\in H_0^1(\Gamma)\},\\ H_0^1(\Gamma)&=\{\phi\in L^2(\Gamma)\colon \phi_1=\phi_{|\Gamma_1}\in H^1(\Gamma_1);\ \phi_2=\phi_{|\Gamma_2}\in H^1(\Gamma_2);\\ &\qquad\qquad \phi_1(C)=\phi_2(C);\ \text{and } \phi_{|\Gamma_R}\equiv 0\}. \end{aligned}$$

Thus, to obtain a variational form of (4.1), we multiply by a test function $v\in W$ and integrate over Ω:

$$\left(\frac{\partial^3 u}{\partial t^3}-\frac{\partial\Delta u}{\partial t}, v\right)=\left(\frac{\partial f}{\partial t}, v\right) \quad \text{for all } v\in W,$$

where $(,)$ denotes the inner product in $L^2(\Omega)$.

Integrating by parts, we have

$$(4.6) \qquad \frac{\partial^3}{\partial t^3}(u,v)+\frac{\partial}{\partial t}(\nabla u,\nabla v)-\left\langle\frac{\partial^2 u}{\partial t\,\partial\nu}, v\right\rangle_\Gamma=\frac{\partial}{\partial t}(f,v) \quad \text{for all } v\in V,$$

where $\langle,\rangle_\Gamma$ denotes the inner product in $L^2(\Gamma)$ and ν is the outward pointing unit normal vector on Γ. The boundary conditions (4.2) and (4.4) imply

$$(4.7) \qquad -\left\langle\frac{\partial^2 u}{\partial t\,\partial\nu}, v\right\rangle_\Gamma=\frac{\partial^2}{\partial t^2}\langle u,v\rangle_\Gamma-\frac{1}{2}\left\langle\frac{\partial^2 u}{\partial x_1^2}, v\right\rangle_{\Gamma_1}-\frac{1}{2}\left\langle\frac{\partial^2 u}{\partial x_2^2}, v\right\rangle_{\Gamma_2},$$

where $\langle,\rangle_{\Gamma_1}$ and $\langle,\rangle_{\Gamma_2}$ have the obvious meanings, and integration by parts yields

$$(4.8) \qquad \begin{aligned} -\left\langle\frac{\partial^2 u}{\partial t\,\partial\nu}, v\right\rangle_\Gamma&=\frac{\partial^2}{\partial t^2}\langle u,v\rangle_\Gamma+\frac{1}{2}\left\langle\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\right\rangle_{\Gamma_1}+\frac{1}{2}\left\langle\frac{\partial u}{\partial x_2},\frac{\partial v}{\partial x_2}\right\rangle_{\Gamma_2}\\ &\quad-\frac{1}{2}\left(\frac{\partial u}{\partial x_1}(1,1)+\frac{\partial u}{\partial x_2}(1,1)\right)v(1,1). \end{aligned}$$

Plugging (4.8) into (4.6) and applying the corner condition (4.3), we arrive at the variational problem on which our numerical scheme is based: Find $u \in W$ such that

$$
\begin{aligned}
(4.9)\quad & \frac{\partial^3}{\partial t^3}(u, v)+\frac{\partial}{\partial t}(\nabla u, \nabla v)+\frac{\partial^2}{\partial t^2}\langle u, v\rangle_\Gamma+\frac{1}{2}\left\langle\frac{\partial u}{\partial x_1}, \frac{\partial v}{\partial x_1}\right\rangle_{\Gamma_1}+\frac{1}{2}\left\langle\frac{\partial u}{\partial x_2}, \frac{\partial v}{\partial x_2}\right\rangle_{\Gamma_2} \\
& +\frac{1}{2}\gamma\frac{\partial}{\partial t}u(1,1)v(1,1)=\frac{\partial}{\partial t}(f, v) \quad \forall v \in W.
\end{aligned}
$$

To discretize Ω we use a uniform mesh of triangles obtained by cutting diagonally, from the upper left to the lower right, each of the squares of a uniform mesh of squares.

The approximation u_h of u will be sought in the space W_h of functions in W whose restriction to each triangle of the mesh is linear. Thus the degrees of freedom are the values at the vertices. Mass lumping is used for the integrals not involving spatial derivatives. Other integrals are computed exactly.

The time discretization is an explicit four-level scheme involving times m, $m+1$, $m-1$, and $m-2$ and is centered at time $m-\frac{1}{2}$. For more details see [1].

4.2. Description of experiments. In these experiments we are concerned with the reflection by the corner. We study the corner condition (4.3) for several choices of the constant γ, $\gamma=1.5$, $\gamma=0.1$, and $\gamma=3.0$.

To generate strong reflections at the corner, the source was placed near the upper right-hand corner $C=(1,1)$ of the domain $\Omega=[0,1]\times[0,1]$ with center at $(.89, .89)$ and radius .04 (see Fig. 2). The source was introduced as a right-hand side, $f(x_1, x_2, t)=g(x_1, x_2)h(t)$:

$$
g(x_1, x_2)=\begin{cases}10{,}000\times(1-r/.04) & \text{if } r<.04, \\ 0 & \text{otherwise,}\end{cases}
$$

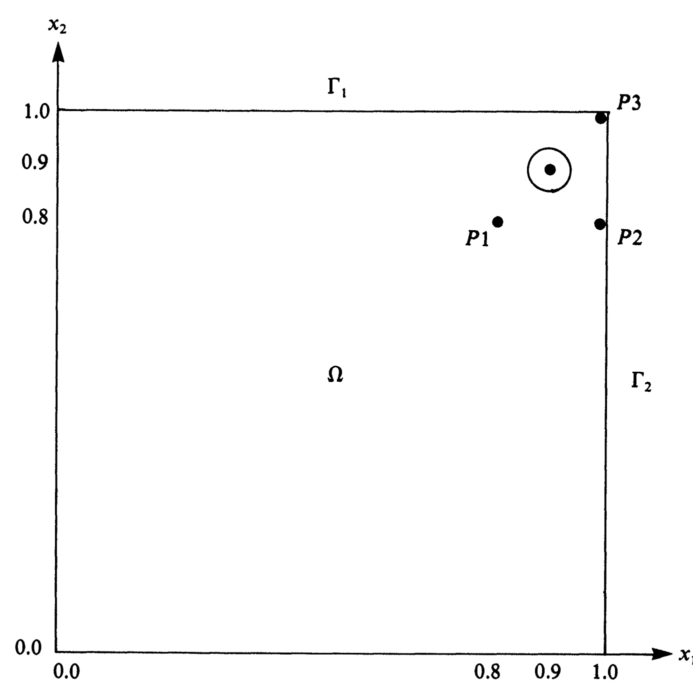


FIG. 2. *Source and location of seismograms.*

where r is the distance of the point (x_1, x_2) from the center of the source $(.89, .89)$,

$$h(t) = \begin{cases} e^{-10}(1-t/.05)^2 & \text{if } t < .1, \\ 0 & \text{otherwise.} \end{cases}$$

The calculations were done with meshsize $\Delta x_1 = \Delta x_2 = .01$ and timestep $\Delta t = .005$.

The results of the experiments are presented first in the form of "snapshots" representing the displacement u as a function of (x_1, x_2) at the times $t = .3$ and $t = .5$, and then as seismograms representing the displacement u as a function of time t, $0 \leqq t \leqq 1$, at the points $P1 = (.79, .79)$ $P2 = (.99, .79)$, and $P3 = (.99, .99)$ (see Fig. 2).

The solutions obtained with the various choices of γ are compared with the so-called "exact" solution, obtained by doing the calculations on the larger rectangle $[0, 2] \times [0, 2]$ so that the wave has not reached any edge and thus there is no reflection by the observation times $t = .3$ and $t = .5$, and so that the reflection has not reached any of the observation points P1, P2, or P3 by the end time of the experiment, $t = 1$. We have also presented as seismograms results obtained by using the first-order absorbing boundary condition for the boundaries Γ_1 and Γ_2. To obtain these results, $(4.2)_{1,2}$ are replaced by

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\, \partial x_2} = 0 \quad \text{on } \Gamma_1,$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t\, \partial x_1} = 0 \quad \text{on } \Gamma_2,$$

so that (4.7) becomes

$$-\left\langle \frac{\partial^2 u}{\partial t\, \partial \nu}, v \right\rangle_\Gamma = \frac{\partial^2}{\partial t^2} \langle u, v \rangle_\Gamma,$$

and (4.9) is replaced by

$$\frac{\partial^3}{\partial t^3}(u, v) + \frac{\partial}{\partial t}(\nabla u, \nabla v) + \frac{\partial^2}{\partial t^2}\langle u, v\rangle_\Gamma = \frac{\partial}{\partial t}(f, v) \quad \text{for all } v \in W.$$

4.3. Experimental results. In Figs. 3–10 we see snapshots representing the displacement u as a function of (x_1, x_2) at times $t = .3$ and $t = .5$. Figures 3 and 4 give the exact solution. In Figs. 5–10, to see the reflected wave better, we have pictured the difference between the solution obtained with the second-order absorbing boundary condition for the sides $x_1 = 1$ and $x_2 = 1$ and the corner condition for the various choices of γ, and the exact solution.

In Figs. 5, 7, and 9, for $t = .3$ we see at a distance of about $\frac{1}{3}$ from the corner C near both boundaries $x_1 = 1$ and $x_2 = 1$, a reflection that is due to the second-order absorbing boundary condition. The amplitude of this reflection is .14, or 16% of the amplitude .81 of the initial wave at time $t = .3$. In Figs. 7 and 9 there appears much nearer the corner a reflection due to the corner that is the γ-wave. In Fig. 7, for $\gamma = 0.1$, γ too small, the γ-wave is positive of amplitude .25, or 31 percent of the amplitude of the initial wave, whereas in Fig. 9, for $\gamma = 3.0$, γ too large, the γ-wave is negative of amplitude .15, or 19 percent of the amplitude of the original wave.

In Figs. 6, 8, and 10, for $t = 5$ we see the reflection due to the second-order boundary condition now at a distance about $\frac{1}{2}$ from the corner. The amplitude of the reflection is now .23, or 38 percent of the amplitude .61 of the initial wave at time $t = .5$. (This increase in the amplitude of the reflection is expected, since at time $t = .5$

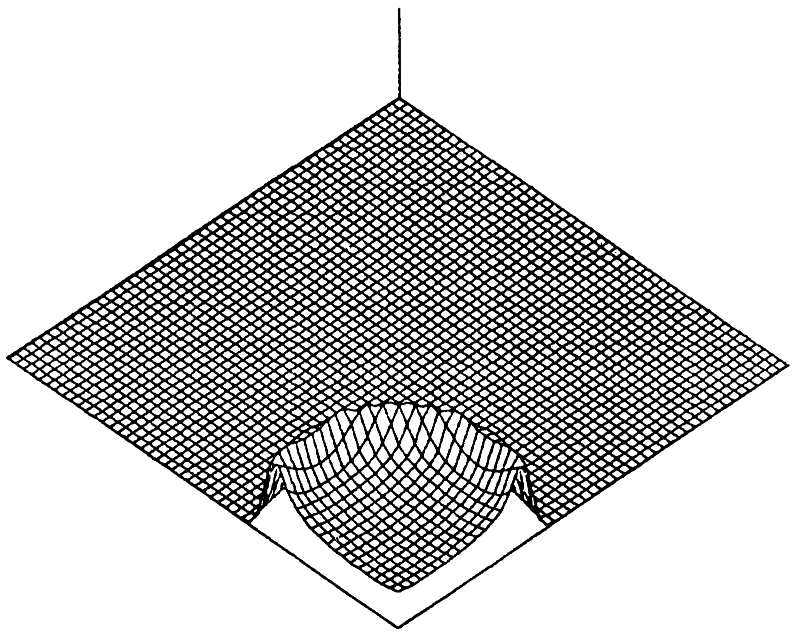


FIG. 3. *Exact solution. Time* $t = .3$; *minimum* $= 0.00$; *maximum* $= 0.81$.

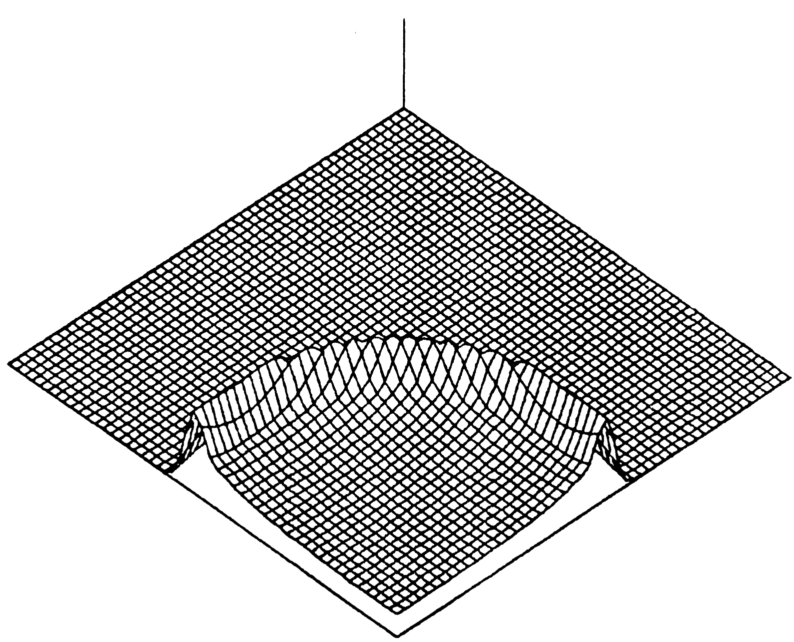


FIG. 4. *Exact solution. Time* $t = .5$; *minimum* $= 0.00$; *maximum* $= 0.61$.

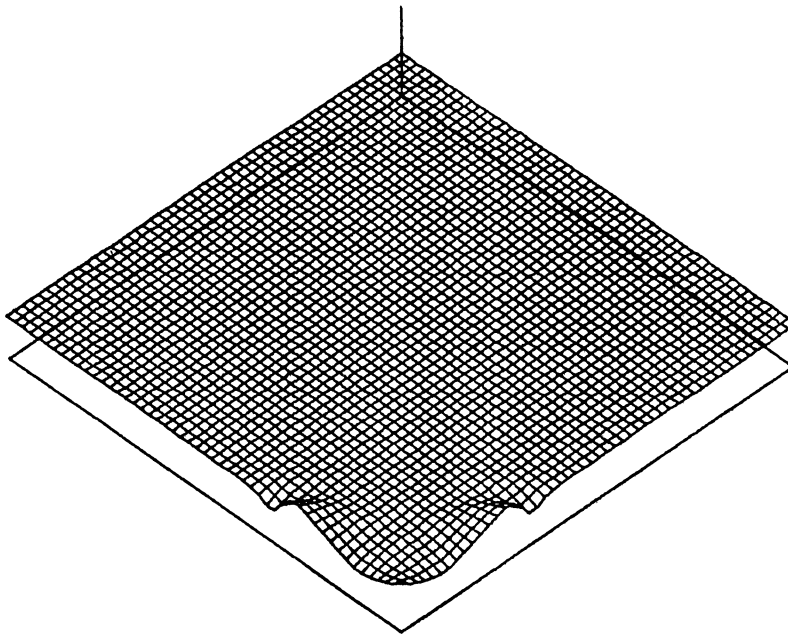


FIG. 5. *Difference between solution calculated with* $\gamma = 1.5$ *and exact solution. Time* $t = .3$; *minimum* $= -0.06$; *maximum* $= 0.08$.

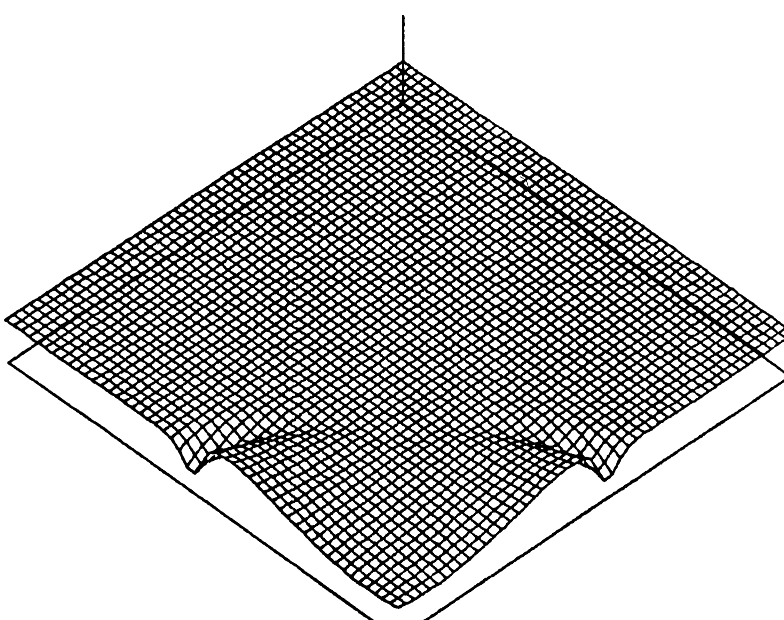


FIG. 6. *Difference between solution calculated with* $\gamma = 1.5$ *and exact solution. Time* $t = .5$; *minimum* $= -0.13$; *maximum* $= 0.10$.

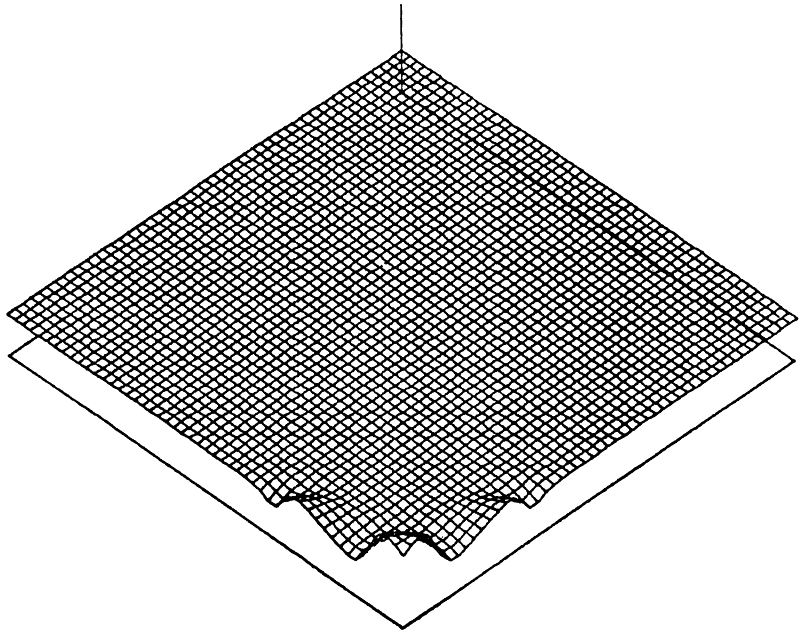


FIG. 7. *Difference between solution calculated with* $\gamma = 0.1$ *and exact solution. Time* $t = .3$; *minimum* $= -0.05$; *maximum* $= 0.24$.

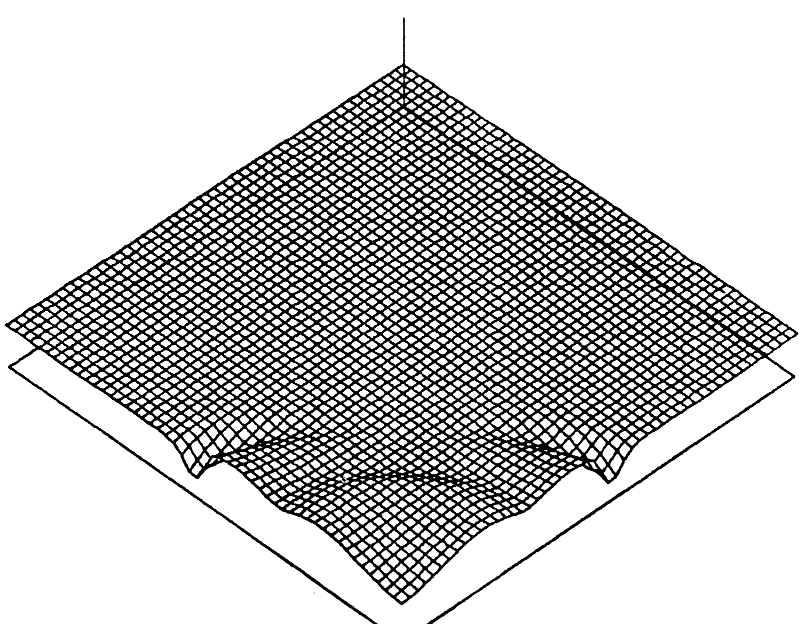


FIG. 8. *Difference between solution calculated with* $\gamma = 0.1$ *and exact solution. Time* $t = .5$; *minimum* $= 0.13$; *maximum* $= 0.10$.

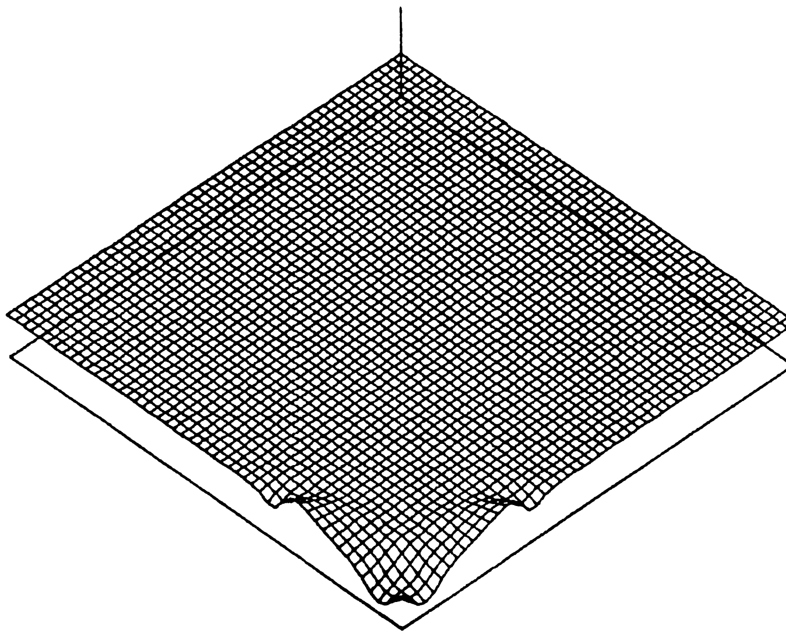


FIG. 9. *Difference between solution calculated with* $\gamma = 3.0$ *and exact solution. Time* $t = .3$; *minimum* $= -0.18$; *maximum* $= 0.08$.

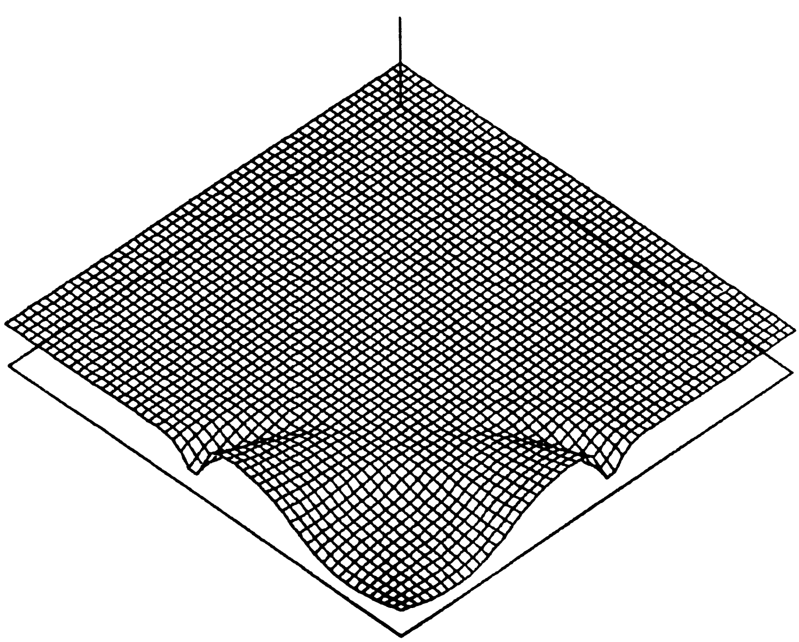


FIG. 10. *Difference between solution calculated with* $\gamma = 3.0$ *and exact solution. Time* $t = .5$; *minimum* $= -0.15$; *maximum* $= 0.10$.

the incident wave is striking the boundary at an angle from the normal much larger than at time $t = .3$.) In Fig. 8 the γ-wave for small γ is now of amplitude .15, or 25 percent of the amplitude of the incident wave. In Fig. 10 the γ-wave for large γ is of amplitude .10, or 16 percent of the amplitude of the incident wave.

Thus at the earlier time we see that the reflection due to the corner, when γ is improperly chosen, is as large as or larger than that due to the second-order condition itself. At the later time, however, it has diminished, whereas the reflection due to the second-order condition has increased. It is important to keep in mind, though, that the large amplitude reflection due to the second-order boundary condition is caused by waves striking the boundary at large angles from the normal. Thus the large reflection propagates at angles close to the tangential and does not travel very rapidly into the interior of the domain, while the reflection due to the corner travels directly toward the center of the domain.

We remark that in Figs. 5 and 6, where the "good" γ, $\gamma = 1.5$, has been used, there seems also to be a slight reflection from the corner, though it is much smaller than the γ-waves appearing in Figs. 7–10. This is, however, only the effect of the second-order absorbing boundary condition.

The last six figures, Figs. 11–16[1], are the seismograms giving the response at three points at equal distances from the source, P1 = (.79, .79) in the interior, P2 = (.99, .79)

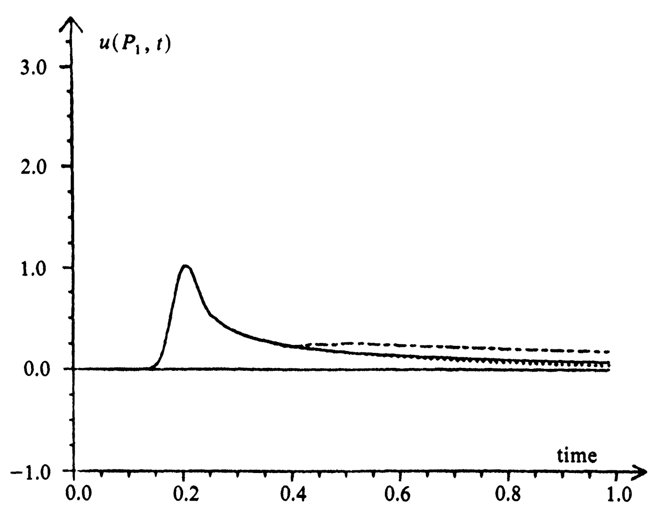


FIG. 11. *Response at* $P_1 = (.79, .79)$. *Exact solution*; *second-order* ABC ($\gamma = 1.5$); *first-order* ABC.

near the boundary, and P3 = (.99, .99) near the corner. For the odd-numbered figures the solid line represents the exact solution, the dotted line the solution obtained with the second-order condition with $\gamma = 1.5$, and the dashed line the solution obtained with the first-order condition. At all of the points, even at P1 which is farthest from the boundary, the better behavior of the second-order condition is quite pronounced.

For the even-numbered figures the three curves all represent solutions obtained with the second-order condition but with different choices of the constant γ. For the solution represented by the solid line we have $\gamma = 1.5$, by the dotted line $\gamma = 0.1$, and

[1] In Figs. 11-16, "absorbing boundary condition" is abbreviated "ABC."

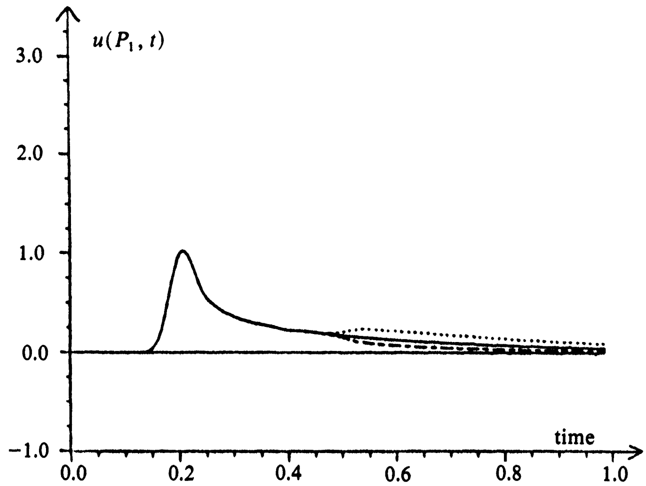


FIG. 12. *Response at* $P_1 = (.79, .79)$. *Second-order* ABC $\gamma = 1.5$; *second-order* ABC $\gamma = 0.1$; *second-order* ABC $\gamma = 3.0$.

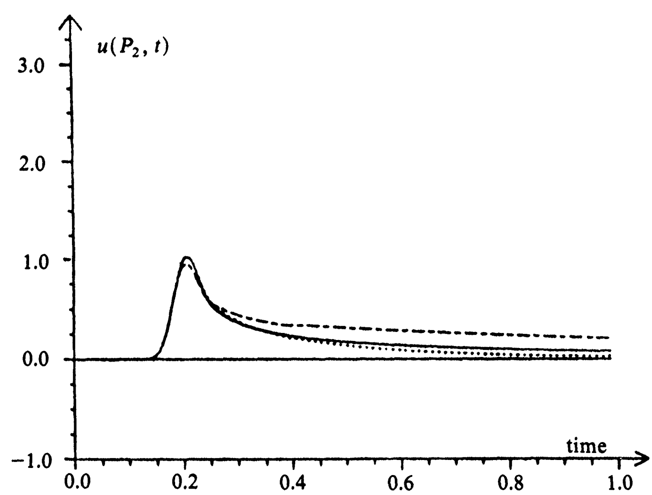


FIG. 13. *Response at* $P_2 = (.99, .79)$. *Exact solution*; *second-order* ABC ($\gamma = 1.5$); *first-order* ABC.

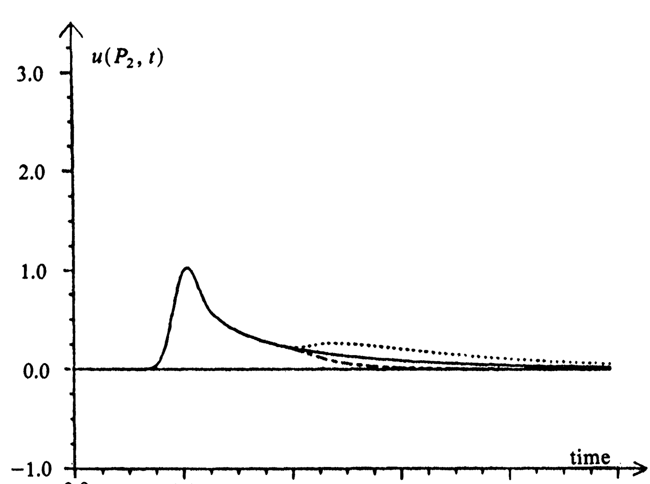


FIG. 14. *Response at* $P_2 = (.99, .79)$. *Second-order* ABC $\gamma = 1.5$; *second-order* ABC $\gamma = 0.1$; *second-order* ABC $\gamma = 3.0$.

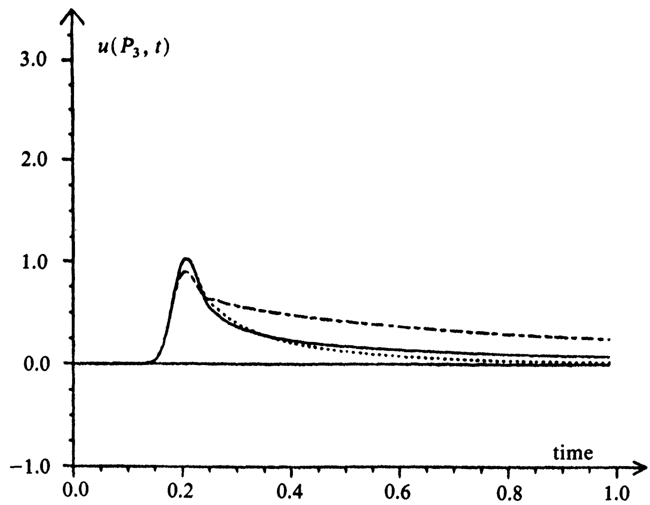


FIG. 15. *Response at* $P_3 = (.99, .99)$. *Exact solution*; *second-order* ABC ($\gamma = 1.5$); *first-order* ABC.

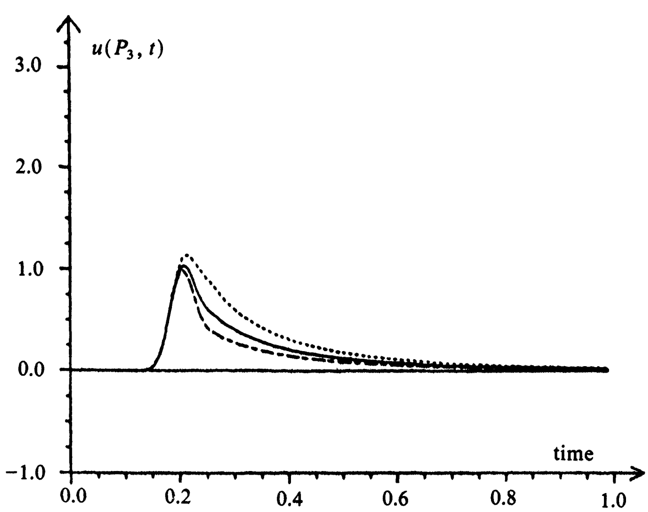


FIG. 16. *Response at* $P_3 = (.99, .99)$. *Second-order* ABC $\gamma = 1.5$; *second-order* ABC $\gamma = 0.1$; *second-order* ABC $\gamma = 3.0$.

by the dashed line $\gamma = 3.0$. Again at each point it is quite clear, especially on comparison with the exact solution in the corresponding odd-numbered figure, that the better solution is obtained with $\gamma = 1.5$. We do remark that by the final time of the observation all of the solutions with the second-order condition have nearly converged to the exact solution, while the solution with the first-order condition remains at some distance.

As a final remark, we point out that we have not presented results with the value $\gamma = \sqrt{2}$ as proposed by Engquist and Majda [5], since with the scale we used we could hardly see the difference between results obtained with $\gamma = \sqrt{2}$ and those obtained with $\gamma = 1.5$. In fact, for the two-dimensional case with the second-order absorbing boundary condition, the solution we have presented differs little from that proposed in [5], though it was obtained from a very different point of view. We have seen that the second-order condition is much better than the first-order condition even in the presence of a corner, and that the condition chosen for the corner is significant. Also we have shown theoretically that our approach to the corner problem yields a solution to the problem in three dimensions and can be used to obtain a solution for the corner problem when higher-order boundary conditions are used.

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