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MIXED FINITE ELEMENT METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

J. DOUGLAS, JR.
J. E. ROBERTS
University of Chicago
Department of Mathematics
5734 University Avenue
Chicago, Illinois 60637
USA

ABSTRACT

The mixed finite element method for the Dirichlet problem for the Laplace operator based on the Raviart-Thomas space $V_h \times W_h \subset H(div;\Omega) \times L^2(\Omega)$ *is extended to apply to the elliptic operators*

$$Lp = -div(a \; grad \; p + bp) + cp \qquad (i)$$

and

$$L^*q = -div(a \; grad \; q) + b \cdot grad \; q + cq \qquad (ii)$$

whenever the Dirichlet problems associated with L *and* L^* *are uniquely solvable. Error estimates that are optimal both in order of convergence and in regularity requirements are derived.*

RESUMO

O presente trabalho estende o método dos elementos finitos mistos para problemas de Dirichlet associados a (i) e (ii). Tal extensão, válida quando esses problemas apresentam unicidade, é motivada na aplicação do método ao problema de Dirichlet para o operador de Laplace que utiliza o espaço de Raviart-Thomas $V_h \times W_h \subset H(div;\Omega) \times L^2(\Omega)$. *As estimativas para o erro obtidas são ótimas com relação à ordem de convergência e à regularidade.*

1. INTRODUCTION

Mixed finite element methods have been discussed and analyzed in detail for the Dirichlet problem

$$-\Delta p = f \quad , \quad x \in \Omega \quad , \tag{1a}$$

$$p = 0 \quad , \quad x \in \partial\Omega \quad , \tag{1b}$$

by Brezzi [1], Falk and Osborn [2], Johnson and Thomée [4], Raviart and Thomas [6], Scholz [8], and Thomas [9]. It is easy to extend most of their results to the operator given by -div(a(x)grad p). Joly [5] has treated the more general operator -div(a grad p)+b•grad p+cp under a constraint that insures the coercivity of its associated bilinear form over $H_0^1(\Omega)$; his method is not a straight-forward generalization of the usual mixed method, and the rate of convergence for his method is reduced by a factor of h from the optimal rate.

The objects of this paper are to define mixed methods for elliptic operators in either the divergence form

$$Lp = -\mathrm{div}(a(x)\mathrm{grad}\ p+b(x)p) + c(x)u \tag{2}$$

or the nondivergence form

$$Mq = L^*q = -\mathrm{div}(a\ \mathrm{grad}\ q) + b\cdot\mathrm{grad}\ q + cq \tag{3}$$

and to establish the convergence of these methods under the weaker assumptions that the Dirichlet problems

$$Lp = f \in L^2(\Omega) \quad , \quad x \in \Omega \quad , \tag{4a}$$

$$p = 0 \quad , \quad x \in \partial\Omega \quad , \tag{4b}$$

and

$$Mq = f \quad , \quad x \in \Omega \quad , \tag{5a}$$

$$q = 0 \quad , \quad x \in \partial\Omega \quad , \tag{5b}$$

are uniquely solvable for p and q in $H^2(\Omega) \cap H_0^1(\Omega)$ and that

$$\| p\|_2 + \| q\|_2 \le K\| f\|_0 \quad , \tag{6}$$

where $\|\cdot\|_k$ denotes the norm in $H^k(\Omega)$. No coercivity is required of the bilinear forms associated with L and M over $H_0^1(\Omega)$. The methods will be essentially direct generalizations of the standard mixed method, and optimal order convergence rates will be established under minimal regularity requirements.

The analyses to be presented below are adapted from that of Johnson and Thomée [4], which is analogous to one of Schatz [7] for Galerkin methods under the same conditions on L and M.

2. FORMULATION OF THE MIXED METHODS

Let Ω be a bounded domain in the plane and assume that the coefficients $a(x)$, $b(x)$, and $c(x)$ are smooth (or, at least, sufficiently differentiable that no difficulties arise in the argument below) and that $a(x)$ is bounded below positively. Require that the Dirichlet problems (4) and (5) be uniquely solvable for each $f \in L^2(\Omega)$ and that the regularity bound (6) holds.

Consider first the divergence form (4). The velocity field,

$$u = -(a \operatorname{grad} p + bp) \quad , \tag{7}$$

associated with the solution p is an element of

$$V = H(\operatorname{div};\Omega) = \{v \in L^2(\Omega)^2 : \operatorname{div} v \in L^2(\Omega)\}.$$

Let

$$\alpha(x) = a(x)^{-1} \qquad \text{and} \qquad \beta(x) = \alpha(x)b(x) \quad .$$

Then, a convenient weak form of (7) can be obtained by multiplying (7) by α and testing the result against another element v of V:

$$(\alpha u,v) - (\operatorname{div} v,p) + (\beta p,v) = 0 \quad , \quad v \in V \quad . \tag{8a}$$

Next, let $W=L^2(\Omega)$ and test the equation $\operatorname{div} u+cp=f$ against an element w of W:

$$(\operatorname{div} u,w) + (cp,w) = (f,w) \quad , \quad w \in W \quad . \tag{8b}$$

The pair (8a) and (8b) of equations will be the basis of the mixed finite element approximation to (4).

Let $V_h \times W_h$ be a Raviart-Thomas space [3,4,6,9] contained in $V \times W$; let it be related to a quasi-regular polygonalization of Ω. Let the elements of W_h be piecewise polynomial functions of total degree $k \geq 0$ if triangles are used to partition Ω; if quadrilaterals are used instead, let the restriction of an element to a quadrilateral be the isoparametric image of a tensor product of polynomials of degree k. Let V_h be the related piecewise-polynomial velocity space. The exact forms of these spaces are given in detail in Raviart and Thomas [6] and Thomas [9] for polygonal domains and in Johnson and Thomée [4] and Jensen [3] for domains requiring boundary triangles to have one curvilinear edge.

Our mixed finite element method for approximating the solution of (4) is defined as the determination of a pair $\{u_h,p_h\} \in V_h \times W_h$ such that

$$(\alpha u_h,v) - (\operatorname{div} v,p_h) + (\beta p_h,v) = 0 \quad , \quad v \in V_h \quad , \tag{9a}$$

$$(\operatorname{div} u_h,w) + (cp_h,w) = (f,w) \quad , \quad w \in W_h \quad . \tag{9b}$$

This procedure represents the most direct extension of the mixed method for the Dirichlet problem for the Laplace operator.

Turn now to the nondivergence form (5), let

$$z = -a \operatorname{grad} q \quad , \tag{10}$$

so that the equation $Mq=f$ can be put into the form

$$\operatorname{div} z - \beta\cdot z + cq = f \quad . \tag{11}$$

The relations corresponding to (9) are given by

$$(\alpha z,v) - (\operatorname{div} v,q) = 0 \quad , \quad v \in V \quad , \tag{12a}$$

$$(\operatorname{div} z,w) - (\beta\cdot z,w) + (cq,w) = (f,w) \quad , \quad w \in W \quad . \tag{12b}$$

Consequently, a mixed method approximation to (5) can be obtained by seeking $\{z_h,q_h\} \in V_h \times W_h$ such that

$$(\alpha z_h,v) - (\operatorname{div} v,q_h) = 0 \quad , \quad v \in V_h \quad , \tag{13a}$$

$$(\operatorname{div} z_h,w) - (\beta\cdot z_h,w) + (cq_h,w) = (f,w) \quad , \quad w \in W_h \quad . \tag{13b}$$

Again, this method is the simplest extension of the standard mixed method.

3 ANALYSIS OF THE METHOD FOR THE DIVERGENCE-FORM PROBLEM

Our analysis is based on that given by Johnson and Thomée for the Laplace operator. We shall need a family of projections of V into V_h. First, Raviart and Thomas [6] demonstrated the existence of a projection Π_h: $V \to V_h$ such that, for $v \in V$,

$$(\text{div}(\Pi_h v - v), w) = 0 \quad , \quad w \in W_h \quad , \tag{14a}$$

$$\| \Pi_h v - v \|_0 \leq Q \| v \|_s h^s \quad , \quad 0 \leq s \leq k+1 \quad , \tag{14b}$$

$$\| \text{div}(\Pi_h v - v) \|_0 \leq Q \| \text{div } v \|_s h^s \quad , \quad 0 \leq s \leq k+1 \quad . \tag{14c}$$

We shall use both Π_h and a weighted version of it. Let $\Pi_{a,h}$: $V \to V_h$ be defined by

$$\Pi_{a,h} v = \Pi_h(av) \quad , \quad v \in V \quad . \tag{15}$$

The projection Π_h can be defined by means of moments over the edges and interiors of triangles or quadrilaterals of the partition. It will also be helpful to let P_h: $W \to W_h$ denote $L^2(\Omega)$-projection onto W_h, so that

$$\| P_h w - w \|_0 \leq Q \| w \|_s h^s \quad , \quad 0 \leq s \leq k+1 \quad . \tag{16}$$

Let

$$\xi = u - u_h \quad , \quad \sigma = \Pi_h u - u_h \quad , \tag{17a}$$

$$\eta = p - p_h \quad , \quad \tau = P_h p - p_h \quad . \tag{17b}$$

It follows from (8) and (9) that

$$(\alpha\xi, v) - (\text{div } v, \eta) + (\beta\eta, v) = 0 \quad , \quad v \in V_h \quad , \tag{18a}$$

$$(\text{div } \xi, w) + (c\eta, w) = 0 \quad , \quad w \in W_h \quad , \tag{18b}$$

or, equivalently, that

$$(\alpha\sigma, v) - (\text{div } v, \tau) = (\alpha(\Pi_h u - u), v) - (\text{div } v, P_h p - p) - (\beta\eta, v) \quad , \quad v \in V_h, \tag{19a}$$

$$-(\text{div } \sigma, w) = (c\eta, w) - (\text{div}(P_h u - u), w) \quad , \quad w \in W_h \quad . \tag{19b}$$

Note that the left-hand side of (19) corresponds to the mixed method for the operator $-\nabla \cdot (a\nabla p)$. The argument by Brezzi [1] shows that the mixed method for this operator is invertible over $V_h \times W_h$ with a bound that depends only on the positive upper and lower bounds for $a(x)$; hence,

$$\|\sigma\|_V + \|\tau\|_W \le Q[\|\eta\|_0 + \|\Pi_h u - u\|_V + \|P_h p - p\|_0] \le$$

$$\le Q[\|\eta\|_0 + \|p\|_{s+2} h^s] \quad , \quad s \le k+1 \tag{20}$$

Thus,

$$\|\xi\|_V \le Q[\|\eta\|_0 + \|p\|_{s+2} h^s] \quad , \quad s \le k+1 \tag{21}$$

The preliminary estimate (21) will lead eventually to our final estimate for div ξ; however, duality arguments must be employed to obtain the L^2-estimates for the errors in p and u.

Lemma 3.1. Let $\zeta \in V$, $f \in V'$, and $g \in W'$. If $z \in W_h$ satisfies the equations

$$(\alpha\zeta,v) - (\operatorname{div} v,z) + (\beta z,v) = f(v) \quad , \quad v \in V_h \quad , \tag{22a}$$

$$(\operatorname{div} \zeta,w) + (cz,w) = g(w) \quad , \quad w \in W_h \quad , \tag{22b}$$

then for h sufficiently small

$$\|z\|_0 \le Q[h\|\zeta\|_0 + h^2\|\operatorname{div} \zeta\|_0 + \|f\|_{V'} + \|g\|_{W'}] \quad , \tag{23}$$

provided that $k \ge 1$. For $k=0$, the term $h^2\|\operatorname{div} \zeta\|_0$ should be replaced by $h\|\operatorname{div} \zeta\|_0$.

Proof. Let $\psi \in L^2(\Omega)$ and let $\phi \in H^2(\Omega)$ be determined as the solution of the Dirichlet problem

$$L^*\phi = \psi \quad , \quad x \in \Omega \quad ,$$

$$\phi = 0 \quad , \quad x \in \partial\Omega \quad .$$

By (22a),

$$(z,\psi) = (z,L^*\phi) = (z,-\nabla\cdot(a\nabla\phi) + \beta a\cdot\nabla\phi + c\phi) =$$

$$= (z,-\nabla\cdot(\Pi_{a,h}\nabla\phi) + \beta\cdot\Pi_{a,h}\nabla\phi) + (\beta z,a\nabla\phi - \Pi_{a,h}\nabla\phi) + (z,c\phi) =$$

$$= -(\alpha\zeta,\Pi_{a,h}\nabla\phi) + f(\Pi_{a,h}\nabla\phi) + (\beta z,a\nabla\phi - \Pi_{a,h}\nabla\phi) + (cz,\phi)$$

By (22b),

$$(cz,\phi) = (cz+\text{div}\ \zeta,\phi-P_h\phi) + g(P_h\phi) - (\text{div}\ \zeta,\phi) =$$

$$= (cz+\text{div}\ \zeta,\phi-P_h\phi) + g(P_h\phi) + (\zeta,\nabla\phi) .$$

Next, note that

$$(\zeta,\nabla\phi) - (\alpha\zeta,\Pi_{a,h}\nabla\phi) = (\zeta,\nabla\phi) - (\alpha\zeta,a\nabla\phi) + (\alpha\zeta,a\nabla\phi-\Pi_{a,h}\nabla\phi) =$$

$$= (\alpha\zeta,a\nabla\phi-\Pi_{a,h}\nabla\phi) .$$

The three expressions above can be combined to show that

$$(z,\psi) = f(\Pi_{a,h}\nabla\phi) + g(P_h\phi) + (\alpha\zeta+\beta z,a\nabla\phi-\Pi_{a,h}\nabla\phi) +$$

$$+ (\text{div}\ \zeta+cz,\phi-P_h\phi) .$$

Since $\|\phi\|_2 \le Q\|\psi\|_0$ and

$$\|\Pi_{a,h}\nabla\phi\|_V \le Q\|a\nabla\phi\|_V \le Q\|\phi\|_2 ,$$

$$\|P_h\phi\|_W = \|P_h\phi\|_0 \le \|\phi\|_0 ,$$

$$\|a\nabla\phi-\Pi_{a,h}\nabla\phi\|_0 \le Q\|a\nabla\phi\|_1 h \le Q\|\phi\|_2 h ,$$

$$\|\phi-P_h\phi\|_0 \le \begin{cases} Q\|\phi\|_2 h^2 , & \text{if } k\ge 1 , \\ Q\|\phi\|_1 h , & \text{if } k=0 , \end{cases}$$

then

$$|(z,\psi)| \le Q\{\|f\|_{V'}+\|g\|_{W'}+h(\|\zeta\|_0+\|z\|_0)+h^\delta\|\text{div}\ \zeta\|_0\} , \tag{24}$$

where $\delta=2$ for $k\ge 1$ and $\delta=1$ for $k=0$, and (23) follows for small h.

In order to apply Lemma 3.1 rewrite (18) in the form

$$(\alpha\xi,v) - (\text{div}\ v,\tau) + (\beta\tau,v) = (\text{div}\ v-\beta\cdot v,p-P_hp) , \quad v \in V_h , \tag{25a}$$

$$(\text{div}\ \xi,w) + (c\tau,w) = (c(P_hp-p),w) , \quad w \in W_h . \tag{25b}$$

Then, with δ equal to one or two as above,

$$\|\tau\|_0 \le Q[h\|\xi\|_0 + h^\delta\|\operatorname{div}\xi\|_0 + \|p\|_s h^s] \quad , \quad s\le k+1 \tag{26}$$

Since

$$\|\eta\|_0 \le \|\tau\|_0 + \|p - P_h p\|_0 \quad ,$$

(21) and (26) imply that

$$\|\eta\|_0 \le Q\|p\|_{s+1} h^s \quad , \quad 1\le s\le k+1 \tag{27}$$

This is again a preliminary estimate; however, it and (21) do give the final estimate for the error in the divergence of u:

$$\|\operatorname{div}\xi\|_0 \le Q\|p\|_{s+2} h^s \quad , \quad 1\le s\le k+1 . \tag{28}$$

The not particularly useful case s=0 in (28) can also be shown to result from (21) and (26), since it follows from them that $\|\eta\|_0 \le Q\|p\|_2$. Note that (28) is optimal both as to rate of convergence and to the regularity required of the solution of the differential problem.

Optimal order estimates (i.e., $O(h^s)$ for $s\le k+1$) for $\|\eta\|_0$ and $\|\xi\|_0$ have also been obtained in (27) and (21); however, both estimates require greater than minimal regularity. Now (14a) and (18b) imply that

$$(\operatorname{div}\Pi_h\xi, w) = (\operatorname{div}\xi, w) = -(c\eta, w) \quad , \quad w \in W_h .$$

Since $\operatorname{div} V_h \subset W_h$,

$$\|\operatorname{div}(\Pi_h u - u_h)\|_0 = \|\operatorname{div}\Pi_h\xi\|_0 \le Q\|\eta\|_0 \le Q\|p\|_{s+1} h^s, \quad 1\le s\le k+1 .$$

Next, take $v=\sigma=\Pi_h u-u$ in (19a):

$$(\alpha\sigma,\sigma) = (\operatorname{div}\sigma,\eta) + (\alpha(\Pi_h u-u) - \beta\eta,\sigma) \quad ,$$

from which it follows that

$$\|\sigma\|_0 \le Q\|p\|_{s+1} h^s \quad , \quad 1\le s\le k+1 . \tag{29}$$

Thus,

$$\|\xi\|_0 \le Q\|p\|_{s+1} h^s \quad , \quad 1\le s\le k+1 \quad , \tag{30}$$

which is optimal with respect to regularity. Finally, (30) and (28) can be applied to (26) to see that, for $k \geq 1$,

$$\|\eta\|_0 \leq Q\|p\|_s h^s \quad , \quad 2 \leq s \leq k+1 \quad , \tag{31}$$

which is the final estimate for η.

Throughout the argument to estimate the error $\{\xi,\eta\}$, it has been tacitly assumed that (9) is solvable. The existence and uniqueness of a solution of (9) can be established easily from the bounds derived above. As usual, it suffices to demonstrate uniqueness. Momentarily, interpret u_h and p_h in (9) as a solution of the homogeneous problem. The argument leading from (18) to (21) implies that

$$\|u_h\|_V \leq Q\|p_h\|_0 \quad . \tag{32}$$

Then, Lemma 3.1, with $f=0$ and $g=0$, shows that

$$\|p_h\|_0 \leq Qh\|u_h\|_V \quad , \tag{33}$$

so that u_h and p_h vanish for small h.

The results derived above can be collected as follows.

Theorem 3.2. Assume that the Dirichlet problem (4) has a unique solution for every $f \in L^2(\Omega)$. Then, for h sufficiently small, the mixed method (9) has a unique solution. Moreover, if the index of the Raviart-Thomas space $V_h x W_h$ is the nonnegative integer k, then

(a) $\|\mathrm{div}(u-u_h)\|_0 \leq Q\|p\|_{s+2} h^s \quad , \quad 0 \leq s \leq k+1 \quad ,$

(b) $\|u-u_h\|_0 \leq Q\|p\|_{s+1} h^s \quad , \quad 1 \leq s \leq k+1 \quad ,$

and

(c) $$\|p-p_h\|_0 \leq \begin{cases} Q\|p\|_s h^s \quad , \quad 2 \leq s \leq k+1 \quad \text{if} \quad k \geq 1 \; , \\ Q\|p\|_2 h \quad , \quad \text{if} \quad k=0 \quad . \end{cases}$$

The bound (21) simplified the derivation of the final error estimates; however, a slightly more convoluted application of Lemma 3.1 and the argument leading to (29) can lead to the same error bounds.

4. ANALYSIS OF THE METHOD FOR THE PROBLEM IN NON-DIVERGENCE FORM

If $\xi=z-z_h$, $\sigma=\Pi_h z-z_h$, $\eta=q-q_h$, and $\tau=P_h q-q_h$, the error equations take the form

$$(\alpha\xi,v) - (\text{div } v,\eta) = 0 \quad , \quad v \in V_h \quad , \tag{34a}$$

$$(\text{div } \xi,w) - (\beta\cdot\xi,w) + (c\eta,w) = 0 \quad , \quad w \in W_h \tag{34b}$$

The convergence argument is more conveniently presented in this case without the use of the analogue of (21). We begin with a result corresponding to Lemma 3.1.

<u>Lemma 4.1</u>. Let $\zeta \in V$, $f \in V'$, and $g \in W'$. If $y \in W_h$ satisfies

$$(\alpha\zeta,v) - (\text{div } v,y) = f(v) \quad , \quad v \in V_h \quad , \tag{35a}$$

$$(\text{div } \zeta,w) - (\beta\cdot\zeta,w) + (cy,w) = g(w) \quad , \quad w \in W_h \quad , \tag{35b}$$

then

$$\| y\|_0 \leq Q[h\| \zeta\|_0 + h^\delta \| \text{div } \zeta\|_0 + \| f\|_{V'} + \| g\|_{W'}] \quad , \tag{36}$$

where $\delta=2$ for $k\geq 1$ and $\delta=1$ for $k=0$.

<u>Proof</u>. Let $\psi \in L^2(\Omega)$ and determine $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ as the solution of the Dirichlet problem given by $L\phi=\psi$. Then,

$$\begin{aligned}(y,\psi) &= (y,-\nabla\cdot(a\nabla\phi+b\phi)+c\phi) = (y,-\nabla\cdot(\Pi_{a,h}(\nabla\phi+\beta\phi))+c\phi) = \\ &= -(\alpha\zeta,\Pi_{a,h}(\nabla\phi+\beta\phi)) + f(\Pi_{a,h}(\nabla\phi+\beta\phi)) + (cy,\phi) = \\ &= -(\zeta,\nabla\phi+\beta\phi) + (\alpha\zeta,a\nabla\phi+b\phi-\Pi_{a,h}(\nabla\phi+\beta\phi)) + \\ &\quad + f(\Pi_{a,h}(\nabla\phi+\beta\phi)) + (cy,\phi) \quad .\end{aligned}$$

Now,

$$(cy,\phi) - (\zeta,\nabla\phi+\beta\phi) = (\nabla\cdot\zeta-\beta\cdot\zeta+cy,\phi-P_h\phi) + g(P_h\phi) \quad ,$$

and the inequality (36) follows as in the proof of Lemma 3.1.

Rewrite (34) as

$$(\alpha\xi,v) - (\operatorname{div} v,\tau) = (\operatorname{div} v,q-P_hq) \quad , \quad v \in V_h \quad , \tag{37a}$$

$$(\operatorname{div} \xi-\beta\cdot\xi,w) + (c\tau,w) = (c(P_hq-q),w) \quad , \quad w \in W_h \quad , \tag{37b}$$

and note that $(\operatorname{div} v,q-P_hq)=0$, since $\operatorname{div} V_h \subset W_h$. Lemma 4.1 implies that

$$\|\tau\|_0 \le Q[h\|\xi\|_0+h^\delta\|\operatorname{div} \xi\|_0+\|q\|_s h^s] \quad , \quad s\le k+1 \quad , \tag{38}$$

so that

$$\|\eta\|_0 \le Q[h\|\xi\|_0+h^\delta\|\operatorname{div} \xi\|_0+\|q\|_s h^s] \quad , \quad s\le k+1 \quad . \tag{39}$$

Next, take the test functions $v=\sigma$ and $w=\tau$ in (34) and note that

$$(\operatorname{div} \xi,w) = (\operatorname{div} \Pi_h\xi,w) = (\operatorname{div} \sigma,w) \quad , \quad w \in W_h \quad , \tag{40a}$$

$$(\operatorname{div} \sigma,\eta) = (\operatorname{div} \sigma,P_h\eta) = (\operatorname{div} \sigma,\tau) \ ; \tag{40b}$$

then,

$$(\alpha\xi,\sigma) - (\operatorname{div} \sigma,\tau) = 0 \quad ,$$

$$(\operatorname{div} \sigma-\beta\cdot\xi,\tau) + (c\eta,\tau) = 0 \quad .$$

Add these two relations to see that

$$(\alpha\xi,\sigma) - (\beta\cdot\xi-c\eta,\tau) = 0 \quad .$$

Thus,

$$(\alpha\sigma,\sigma) = (\alpha(\Pi_hz-z),\sigma) + (\beta\cdot\sigma+c\eta,\tau) + (\beta\cdot(z-\Pi_hz),\tau) \quad ,$$

and it follows that

$$\|\sigma\|_0 \le Q[\|\eta\|_0+\|q\|_{s+1} h^s] \quad , \quad s\le k+1 \quad . \tag{41}$$

Hence, (39) and (41) imply that, for h sufficiently small,

$$\|\xi\|_0 \le Q[h^\delta\|\operatorname{div} \xi\|_0+\|q\|_{s+1} h^s] \quad , \quad s\le k+1 \quad . \tag{42}$$

It follows from (34b) and (40a) that

$$(\mathrm{div}\ \sigma, w) = (\beta\cdot\xi - c\eta, w) \quad , \quad w \in W_h \quad ;$$

consequently,

$$\|\mathrm{div}\ \xi\|_0 \le \|\mathrm{div}\ \sigma\|_0 + Q\|q\|_{s+2}h^s \le Q[\|\xi\|_0 + \|\eta\|_0 + \|q\|_{s+2}h^s] \le$$

$$\le Q[h^\delta\|\mathrm{div}\ \xi\|_0 + \|q\|_{s+2}h^s] \quad ,$$

and

$$\|\mathrm{div}\ \xi\|_0 \le Q\|q\|_{s+2}h^s \quad , \quad s\le k+1 \quad , \tag{43}$$

for small h. Then, (42) and (39) can be used to show that

$$\|\xi\|_0 \le Q\|q\|_{s+1}h^s \quad , \quad 1\le s\le k+1 \quad , \tag{44a}$$

$$\|\eta\|_0 \le \begin{cases} Q\|q\|_s h^s \quad , \quad 2\le s\le k+1 \quad \text{if} \quad k\ge 1 \quad , \\ Q\|q\|_2 h \quad , \quad \text{if } k=0 \end{cases} \tag{44b}$$

Again an argument to demonstrate existence and uniqueness of the solution of (13) for sufficiently small h can be extracted from the development above, and the following theorem has been proved.

Theorem 4.2. Assume that the Dirichlet problem (5) has a unique solution for every $f \in L^2(\Omega)$. Then, for h sufficiently small, there exists a unique solution $\{z_h, q_h\} \in V_h \times W_h$, the Raviart-Thomas space of index k, to the mixed method (13). Moreover,

(a) $\|\mathrm{div}(z-z_h)\|_0 \le Q\|q\|_{s+2}h^s \quad , \quad 0\le s\le k+1 \quad ,$

(b) $\|z-z_h\|_0 \le Q\|q\|_{s+1}h^s \quad , \quad 1\le s\le k+1 \quad ,$

(c) $\|q-q_h\|_0 \le \begin{cases} Q\|q\|_s h^s \quad , \quad 2\le s\le k+1 \quad \text{for} \quad k\ge 1 \quad , \\ Q\|q\|_2 h \quad , \quad \text{for} \quad k=0 \end{cases}$

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